Impulse control problem on finite horizon with intervention lag and execution delay

Benjamin BRUDER
Laboratoire de Probabilités et
Modèles Aléatoires, CNRS, UMR 7599
Université Paris 7
and SGAM, Société Générale
bruder@math.jussieu.fr

Huyên PHAM
Laboratoire de Probabilités et
Modèles Aléatoires, CNRS, UMR 7599
Université Paris 7
and Institut Universitaire de France
pham@math.jussieu.fr

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Abstract

We consider impulse control problems in finite horizon for diffusions with decision lag and execution delay. The new feature is that our general framework deals with the important case in applications when several consecutive orders may be decided before the effective execution of the first one. We show that the value functions for such control problems satisfy a suitable version of dynamic programming principle in finite dimension, which takes into account the past dependence of state process through the pending orders. The corresponding Bellman partial differential equations (PDE) system is derived, and exhibit some peculiarities on the coupled equations, domains and boundary conditions. We prove a unique characterization of the value functions to this nonstandard PDE system by means of viscosity solutions. We then provide an algorithm to find the value functions and the optimal control. This easily implementable algorithm involves backward and forward iterations on the domains and the value functions, which appear in turn as original arguments in the proofs for the boundary conditions and uniqueness results.

Key words : Impulse control, execution delay, diffusion processes, dynamic programming, viscosity solutions.

MSC Classification (2000): 93E20, 62L15, 49L20, 49L25.

1 Introduction

In this paper, we consider a general impulse control problem in finite horizon of a diffusion process X, with intervention lag and execution delay. This means that we may intervene on the diffusion system at any times τ_i separated at least by some fixed positive lag h, by giving some impulse ξ_i based on the information at τ_i. However, the execution of the impulse decided at τ_i is carried out with delay mh, $m \geq 1$, i.e. it is implemented at time $\tau_i + mh$, moving the system from $X_{(\tau_i+mh)^-}$ to $\Gamma(X_{(\tau_i+mh)^-}, \xi_i)$. The objective is to maximize over impulse controls $(\tau_i, \xi_i)_i$ the expected total profit on finite horizon T, of the form

$$\mathbb{E}\Big[\int_0^T f(X_t)dt + g(X_T) + \sum_{\tau_i+mh\leq T} c(X_{(\tau_i+mh)^-}, \xi_i)\Big].$$

Such formulations appear naturally in decision-making problems in economics and finance. In many situations, firms or investors face regulatory delays (delivery lag), which may be significant, and thus need to be taken into account when management strategies are decided in an uncertain environment. Problems where firm's investment are subject to delivery lag can be found in the real options literature, for example in [2] and [1]. In financial market context, execution delay is related to liquidity risk (see [9]), and occurs with transaction, which requires heavy preparatory work as for hedge funds.

From a mathematical viewpoint, it is well-known that impulse control problems without delay, i.e. $m = 0$, lead to variational partial differential equations (PDE), see e.g. the books [5] and [7]. Impulse control problems in the presence of delay were studied in [8] and [6] for $m = 1$, that is when no more than one pending order is allowed at any time. In this case, it is shown that the delay problem may be transformed into a no-delay impulse control problem. The paper [4] also considers the case $m = 1$, but when the value of the impulse is chosen at the time of execution, and on infinite horizon, and these two conditions are crucial in the proposed probabilistic resolution. We mention also the work [3], which studies impulse problems in infinite horizon with arbitrary number of pending orders, but for a linear diffusion model, and with additive orders, so that the problem is easily reduced to a finite-dimensional one where the value functions with pending orders are directly related to the value function without order.

The main contribution of this paper is to provide a study of impulse control problems on finite horizon in a fairly general diffusion framework that deals with the important case in applications when the number of pending orders is finite, but not restricted to one, i.e. $m \geq 1$. Our chief goal is to obtain a unique tractable PDE characterization of the value functions for such problems. As usual in stochastic control problems, the first step is the derivation of a dynamic programming principle (DPP). We show a suitable version of DPP, which takes into account the past dependence of the controlled diffusion via the finite number of pending orders. The corresponding Bellman PDE system reveals some nonstandard features both on the form of the differential operators and their domains, and on the boundary conditions. Following the modern approach to stochastic control, we prove that the value functions are viscosity solutions to this Bellman PDE system, and we also state comparison principles, which allows to obtain a unique PDE characterization.

From this PDE representation, we provide an easily implemented algorithm to compute the value functions, and so as byproducts the optimal impulse control. This algorithm involves forward and backward iterations on the value functions and on the domains, and appear actually as original arguments in the proofs for the boundary conditions and comparison principles.

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