True upper bounds for Bermudan products via non-nested Monte Carlo

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Bermudan options

- In the background: An arbitrage free market of tradable assets on a filtered probability space $(\Omega, \mathcal{F}_t, \mathcal{F}, Q)$, where Q is a fixed *pricing measure* (*equivalent martingale measure*) connected to a numeraire $\mathcal{N}$.

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Definition

A *Bermudan option* consists of a discrete set of time points $\mathcal{E} = \{T_0, \ldots, T_{\mathcal{I}}\}$ and an $\mathcal{F}_t$-adapted process $\mathcal{Z}(t)$.

- Interpretation: The holder of the Bermudan option can choose one time point out of the set $\mathcal{E}$, at which she exercises the cash-flow $\mathcal{Z}$, i.e. she receives e.g. $\mathcal{Z}(T_i)$.

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- Question: How to compute the fair price of a Bermudan option numerically?

Connection to optimal stopping

- Consider the discounted cashflow $Z(i) = \mathcal{Z}(T_i)/\mathcal{N}(T_i)$.
- Assume w.l.o.g. $\mathcal{N}(0) = 1$.
- By no-arbitrage arguments the fair price of the Bermudan option (relative to the pricing measure Q and the numeraire $\mathcal{N}$) is given by

$$\sup_{\tau \in \mathcal{T}_{0,\mathcal{I}}} E^Q[Z(\tau)]$$

  where $\mathcal{T}_{0,\mathcal{I}}$ is the set of $\{0, \ldots, \mathcal{I}\}$-valued stopping times.
- From now on: All (conditional) expectations are taken under Q.

Backward dynamic programming

- Idea: Find an optimal stopping time $\tau^*(i)$ provided the option has not been exercised before time i.
- At terminal time:

$$\tau^*(\mathcal{I}) = \mathcal{I}$$

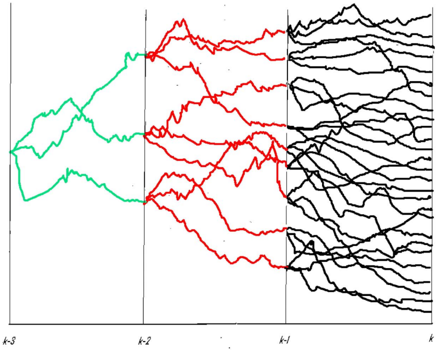
At time i:

$$\tau^*(i) = \begin{cases} i, & Z(i) \geq E[Z(\tau^*(i+1))|\mathcal{F}_i] \\ \tau^*(i+1), & \text{otherwise} \end{cases}$$

- Then $\tau^*(0)$ is an optimal stopping time and $E[Z(\tau^*(0))]$ is the fair price of the Bermudan option.

Nested conditional expectations

- Problem: How to approximate the nested conditional expectations in the backward dynamic program?
- Naive approach: Average over simulated paths (plain Monte Carlo) as suggested by the Law of Large Numbers.



Nested conditional expectations

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- Naive approach: Average over simulated paths (plain Monte Carlo) as suggested by the Law of Large Numbers.

Infeasible: Computational cost explodes rapidly with the number of exercise dates.

Lower bounds

- Trivial: Any stopping time σ induces a lower bound by

$$E[Z(\sigma)].$$

- Many algorithms have been proposed to find a 'good' approximative strategy σ.

- Longstaff/Schwartz: Estimate all conditional expectations in the backward dynamic program by least squares Monte Carlo.
$\rightarrow$ Estimators $(\hat{\tau}^*(i), i = 0, \ldots, \mathcal{I})$ of $(\tau^*(i), i = 0, \ldots, \mathcal{I})$

Conditional expectations via least squares Monte Carlo

- Markovian setting: $\mathbb{R}^D$-valued Markov process $X(i)$ such that $Z(i) = h(i, X(i))$.
- Then: $E[f(X(j))|\mathcal{F}_i] = E[f(X(j))|X(i)] = u(X(i))$ is a regression.
- Aim: Estimate the regression function u as a linear combination of basis functions with the coefficients estimated by simulation.

Conditional expectations via least squares Monte Carlo

Pseudo-Algorithm:

1. Choose a vector of basis functions

$$\psi(i,x) = (\psi_1(i,x), \dots, \psi_K(i,x));\ x \in \mathbb{R}^D;$$

2. Simulate L independent copies $X_\lambda(i),\ \lambda = 1, \dots, L$ of X;
3. Solve the least squares problem

$$\begin{aligned} a(i,j;f) &= \arg\min_{a \in \mathbb{R}^K} \frac{1}{L} \sum_{\lambda=1}^{L} (f(X_\lambda(j)) - \psi(i, X_\lambda(i))a)^2 \\ &\approx \arg\min_{a \in \mathbb{R}^K} E\left[(f(X(j)) - \psi(i, X(i))a)^2\right]; \end{aligned}$$

4. Define, as estimator for $E[f(X(j))|\mathcal{F}_i] = E[f(X(j))|X(i)]$,

$$\hat{E}[f(X(j))|X(i)] = \psi(i, X(i))a(i,j;f).$$

Dual upper bounds

- The value process of the optimal stopping problem

$$Y^*(i) = E[Z(\tau^*(i))|\mathcal{F}_i] \qquad \text{Snell envelope}$$

  is the smallest supermartingale which dominates Z.

- Suppose Y is some supermartingale dominating Z.

  $\Rightarrow$ $Y(0)$ is an upper bound for the Bermudan price.

Constructing dominating supermartingales

Rogers and Haugh & Kogan suggest:

- Start with some martingale M such that $M(0) = 0$.
- Define

$$Y_{up}(i; M) = M(i) + E[\max_{i \le j \le \mathcal{I}} (Z(j) - M(j)) \,|\mathcal{F}_i].$$

  Then Y is a dominating supermartingale.
- Simulate the upper bound $Y_{up}(0; M)$ by plain Monte Carlo

$$Y_{up}(0; M) \approx \frac{1}{L} \sum_{\lambda=1}^{L} \max_{0 \le j \le \mathcal{I}} ({}_{\lambda}Z(j) - {}_{\lambda}M(j))$$

  to get an estimator which is biased high.

Question: How to choose the martingale?

Upper bounds from lower bounds

- Given a stopping family $\tau = (\tau(0), \dots, \tau(\mathcal{I}))$ define

$$Y_{low}(i;\tau) = E[Z(\tau(i))|\mathcal{F}_i].$$

(Expected gain when employing strategy τ)

- Consider the martingale part from the Doob-decomposition,

$$M(i+1;\tau) - M(i;\tau) = Y_{low}(i+1;\tau) - E[Y_{low}(i+1;\tau)|\mathcal{F}_i].$$

- The duality gap of the strategy τ is

$$\Delta\tau = Y_{up}(0; M(\cdot,\tau)) - Y_{low}(0;\tau).$$

- For the optimal strategy τ^* we have (Rogers; Haugh & Kogan)

$$\Delta\tau^* = 0.$$

Estimating the Doob martingale: problems

- Procedure requires to estimate

$$M(i+1;\tau) - M(i;\tau) = Y_{low}(i+1;\tau) - E[Y_{low}(i+1;\tau)|\mathcal{F}_i].$$

- Estimating the conditional expectation on the right hand side typically destroys the martingale property of the estimator $\hat{M}(\cdot;\tau)$.

- Hence, $Y_{up}(0;\hat{M}(\cdot;\tau))$ may fail to be an upper bound.

The Andersen-Broadie algorithm

Plain Monte Carlo: Estimate the conditional expectations by averaging over sets of inner samples.

Advantages:

- Y_{up}^{AB} is biased high, (although ${}_{\lambda}\hat{M}(i;\hat{\tau})$ fail to be martingales in general).

  Reason: Use of plain Monte Carlo and convexity of the max-operator.

- Converges to $Y_{up}(0; M(\cdot;\hat{\tau}))$ as the number of inner and outer simulations increases.

Disadvantage:

- One layer of nested simulation is required.

Fast upper bounds: Idea

Aim: Find an estimator $\hat{M}$ for the martingale

$$M(i+1;\tau) - M(i;\tau) = Y_{low}(i+1;\tau) - E[Y_{low}(i+1;\tau)|\mathcal{F}_i].$$

such that

1. $\hat{M}$ is a martingale;
2. No need for nested simulations;

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Framework: $Z(i) = h(T_i, X(T_i))$, where

$$dX(t) = a(t, X(t))dt + b(t, X(t))dW(t), \quad X_0 = x,$$

- W is a $\mathcal{D}$-dim. Brownian motion on $[0, T]$;
- the coefficient functions a, b are Lipschitz in space and $1/2$-Hölder in time;
- X is D-dimensional.

Fast upper bounds: Idea

Idea:

- Thanks to the martingale representation theorem there is an adapted process U such that

$$M(i+1;\tau) - M(i;\tau) = \int_{T_i}^{T_{i+1}} U(s)dW(s).$$

- Given a partition $\pi \subset \mathcal{E}$ of $[0, T]$, find a non-anticipating estimator U^π for U, and consider the martingale

$$M^\pi(i) = \sum_{t_j \in \pi;\ t_j < T_i} U^\pi(t_j)(W(t_{j+1}) - W(t_j)).$$

Estimating the integrand

- Candidate for the dth component of U^π:

$$U^\pi_d(t_j) = E\left[\frac{W_d(t_{j+1}) - W_d(t_j)}{t_{j+1} - t_j} M^\pi(i) \middle| \mathcal{F}_{t_j} \right]; \quad T_{i-1} \leq t_j < T_i.$$

- By the tower property of the conditional expectation the r.h.s. becomes

$$E\left[\frac{W_d(t_{j+1}) - W_d(t_j)}{t_{j+1} - t_j} h(\tau(i), X(\tau(i))) \middle| X(t_j) \right]; \quad T_{i-1} \leq t_j < T_i.$$

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- Choose the stopping times $\hat{\tau}(i)$ from the LS-algorithm as input and estimate the conditional expectations above by least squares regression:
$\Rightarrow$ estimator $\hat{U}_d^\pi$ for U_d^π $\quad \Rightarrow$ martingale $\hat{M}^\pi$ based on $\hat{U}_d^\pi$.
- Calculate an upper biased estimator for the Bermudan price based on $\hat{M}^\pi$.

Example: Bermudan max-call on several stocks

- A Bermudan max-call on n assets is the right, but not the obligation, to buy at one time point from a specified set the most expensive of the n stocks for a pre-specified price K, the strike price.
- Setting: Stocks are modeled by n independent, identically distributed Black-Scholes model with dividend yield δ, drift r and volatility σ under Q;

  numeraire is given by a bank account, continuously compounded with rate r.

  $\longrightarrow$ Benchmark problem for high-dimensional early exercise options.

Numerical results for the Bermudan max-call

- n stocks, interest rate $r = 5\%$, volatility $\sigma = 20\%$, dividend yield $\delta = 10\%$, initial price x_0
- 9 exercise dates equally spaced on 3 years.
- Strike (exercise price): $K = 100$
- Basis: uses information about the prices and deltas of the underlying European options.

D	x_0	Lower Bound Y_0	Upper Bound $Y_0^{up}(\widehat{M}^\pi)$	95% CI A&B
2	90	8.0242±0.075	8.0891±0.068	[8.053, 8.082]
	100	13.859±0.094	13.958±0.085	[13.892, 13.934]
	110	21.330±0.109	21.459±0.097	[21.316, 21.359]
5	90	16.575±0.072	16.681±0.070	[16.602, 16.655]
	100	26.104±0.085	26.273±0.079	[26.109, 26.292]
	110	36.701±0.098	36.902±0.091	[36.704, 36.832]

Discussion of the algorithm

Advantages

- Fast and easy to implement;
- Converges to $Y_{up}(0; M(\cdot; \hat{\tau}))$, when the mesh of the time grid decreases, the basis exhausts a complete system and the simulated paths tend to infinity.

  (BSDE techniques applied for the proof).

Disadvantages

- The interplay of several error sources is difficult to handle;
- Quality of the upper bounds heavily depends on the choice of basis.

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Thankyou for your attention

This talk was based on joint work with D. Belomestny and J. Schoenmakers (WIAS Berlin):

- Belomestny, B., Schoenmakers: True upper bounds for Bermudan products via non-nested Monte Carlo, *Math. Finance*, under revision.