Optimal investment and consumption in a Black-Scholes market with Lévy-driven stochastic coefficients

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Delong, L. and Klüppelberg, C. (2007)
Optimal investment and consumption in a Black-Scholes market with stochastic coefficients driven by a non-Gaussian Ornstein-Uhlenbeck process.
Submitted for publication.

Benth, F.E., Karlsen, K.H. and Reikvam, K. (2003)
Merton's portfolio optimization problem in a Black and Scholes Market with non-Gaussian stochastic volatility of Ornstein-Uhlenbeck type.
Mathematical Finance **13**, 215-244.

The financial market

$(\Omega, \mathcal{F}, (\mathcal{F})_{0 \le t \le T}, P)$ filtered probability space satisfying the usual conditions.

$$\begin{aligned} \frac{dB(t)}{B(t)} &= r(Y(t-))dt \quad B(0) = 1 \\ \frac{dS(t)}{S(t)} &= \mu(Y(t-))dt + \sigma(Y(t-))dW(t) \quad S(0) = s > 0 \end{aligned}$$

where $(W(t))_{0 \le t \le T}$ is an adapted Brownian motion,
independent of the positive OU process

$$dY(t) = -\lambda Y(t-)dt + dL(\lambda t) \quad Y(0) = y > 0$$

L a subordinator with Laplace exponent $\psi(w) < \infty$ for $w < w_0 \in (0, \infty]$.

Assumptions

(A1) $r, \mu : (0, \infty) \to [0, \infty)$ and $\sigma : (0, \infty) \to (0, \infty)$ satisfy for nonnegative constants,

$$r(y) \leq A_r + B_r y, \quad \mu(y) \leq A_\mu + B_\mu y, \quad \sigma^2(y) \leq A_\sigma + B_\sigma y, \quad y > 0$$

(A2) r', μ', σ' are continuous and satisfy analogous linear growth conditions,

(A3) $\inf_{y \in \mathcal{D}_2} \sigma(y) > 0$, where the set $\mathcal{D}_2$ will be obvious later.

Example [Barndorff-Nielsen & Shephard model]

$$\begin{aligned} \frac{dB(t)}{B(t)} &= r\,dt \\ \frac{dS(t)}{S(t)} &= (\mu + \beta Y(t-))dt + \sqrt{Y(t-)}dW(t). \end{aligned}$$

The optimization problem

The wealth process

$X^{c,\pi} := (X^{c,\pi}(t))_{0\le t\le T}$ depends on

- $\pi(t)$ fraction of the wealth invested in the risky asset

- $c(t)$ consumption

$$\begin{aligned} dX^{c,\pi}(t) &= \pi(t)X^{c,\pi}(t)\big(\mu(Y(t-))dt + \sigma(Y(t-))dW(t)\big) \\ &\quad + \big(1-\pi(t)\big)X^{c,\pi}(t)r(Y(t-)dt - c(t)dt, \end{aligned}$$

$$\begin{aligned} dX^{c,\pi}(t) &= X^{c,\pi}(t)\big(\pi(t)(\mu(Y(t-)) - r(Y(t-)))dt + r(Y(t-))dt \\ &\quad +\pi(t)\sigma(Y(t-))dW(t)\big) - c(t)dt. \end{aligned} \tag{1}$$

The optimization problem

We invoke a power utility function:

$$\sup_{c,\pi} \mathbb{E}[\int_0^T \big(c(s)\big)^\gamma ds + \big(X^{c,\pi}(T)\big)^\gamma \mid X(0) = x, Y(0) = y]$$

with corresponding optimal value function

$$V(t,x,y) = \sup_{(c,\pi)\in\mathcal{A}} \mathbb{E}[\int_t^T \big(c(s)\big)^\gamma ds + \big(X^{c,\pi}(T)\big)^\gamma \mid X(t) = x, Y(t) = y].$$

Definition A strategy $(c,\pi) := (c(t),\pi(t))_{0\le t\le T}$ is admissible ($(c,\pi)\in\mathcal{A}$), if

(1) $(c,\pi) : (0,T]\times\Omega\to[0,\infty)\times[0,1]$ is $(\mathcal{F}_t)_{0\le t\le T}$-progressively measurable,

(2) $\int_0^T c(s)ds<\infty$ $\mathbb{P}$-a.s.,

(3) the SDE (1) has a unique, positive solution $X^{c,\pi}$ on $[0,T]$. □

The HJB equation

$$\sup_{(c,\pi)\in[0,\infty)\times[0,1]}\{c^\gamma+\frac{\partial v}{\partial t}(t,x,y)+\frac{\partial v}{\partial x}(t,x,y)\Big(\pi x(\mu(y)-r(y))+xr(y)-c\Big)$$
$$+\frac{1}{2}\frac{\partial^2 v}{\partial x^2}(t,x,y)\pi^2x^2\sigma^2(y)-\lambda\frac{\partial v}{\partial y}(t,x,y)y \tag{2}$$
$$+\lambda\int_{z>0}\Big(v(t,x,y+z)-v(t,x,y)\Big)\nu(dz)\}=0,\quad v(T,x,y)=x^\gamma.$$

Natural ansatz:

$$v(t,x,y) = x^\gamma f(t,y)$$

which implies

$$\begin{aligned} \widehat{c} &= x f(t,y)^{-\frac{1}{1-\gamma}}, &(3)\\ \widehat{\pi} &= \arg\max_{\pi\in[0,1]} \{\pi(\mu(y)-r(y)) - \frac{1}{2}\pi^2(1-\gamma)\sigma^2(y)\}. &(4)\end{aligned}$$

Define

$$\begin{aligned} \mathcal{D}_1 &= \{y>0, \mu(y)-r(y)<0\},\\ \mathcal{D}_2 &= \{y>0, \mu(y)-r(y)>0, (1-\gamma)\sigma^2(y)>\mu(y)-r(y)\},\\ \mathcal{D}_3 &= \{y>0, \mu(y)-r(y)>0, (1-\gamma)\sigma^2(y)<\mu(y)-r(y)\}.\end{aligned}$$

The strategy $\widehat{\pi}$ is given by

$$\widehat{\pi} = \begin{cases} 0 & y \in \mathcal{D}_1 \\ \dfrac{\mu(y) - r(y)}{(1-\gamma)\sigma^2(y)} & y \in \mathcal{D}_2 \\ 1 & y \in \mathcal{D}_3. \end{cases}$$

Lemma Define the function

$$
\begin{aligned}
Q(y) &= \max_{\pi\in[0,1]} \{\pi(\mu(y) - r(y)) - \frac{1}{2}\pi^2(1-\gamma)\sigma^2(y)\} + r(y) \qquad (5)\\
&= \begin{cases} r(y) & y \in \mathcal{D}_1 \\ \dfrac{(\mu(y) - r(y))^2}{2(1-\gamma)\sigma^2(y)} + r(y) & y \in \mathcal{D}_2 \\ \mu(y) - \frac{1}{2}(1-\gamma)\sigma^2(y) & y \in \mathcal{D}_3. \end{cases}
\end{aligned}
$$

Then for $A, B \geq 0$ and $C, D \geq 0$

$$
\begin{aligned}
0 \leq r(y) \leq Q(y) &\leq A + By \quad y > 0 \\
|\frac{dQ}{dy}(y)| &\leq C + Dy \quad y > 0.
\end{aligned}
$$

□

By substituting (3) and (4) into the HJB equation (2) we arrive at the PIDE

$$\begin{aligned} 0 \;=\; & \frac{\partial f}{\partial t}(t,y) - \lambda \frac{\partial f}{\partial y}(t,y)y + \lambda \int_{z>0} \big(f(t,y+z) - f(t,y) \big) \nu(dz) \\ & + \gamma f(t,y) Q(y) + (1-\gamma) f(t,y)^{-\frac{\gamma}{1-\gamma}}, \quad f(T,y) = 1. \end{aligned}$$

Task: show existence and uniqueness of a classical solution.

Existence of the solution

For Q as in (5) define operator $\mathcal{L}$

$$(\mathcal{L}f)(t,y) = \mathbb{E}^{t,y}\Big[(1-\gamma)\int_t^T e^{\gamma\int_t^s Q(Y(u))du} f(s,Y(s))^{-\frac{\gamma}{1-\gamma}}ds + e^{\gamma\int_t^T Q(Y(s))ds}\Big]$$

Applying (heuristically) the Feynman-Kac formula to the above PIDE:

$$(\mathcal{L}f)(t,y) = f(t,y) \quad (t,y) \in [0,T]\times(0,\infty). \tag{6}$$

Lower bound for $\mathcal{L}$

$$V(t,x,y) \quad \geq \quad x^{\gamma}\mathbb{E}[e^{\gamma \int_t^T r(Y(s))ds}] \quad (t,x,y) \in [0,T] \times (0,\infty) \times (0,\infty),$$

This implies for the solution to (6)

$$\begin{aligned} f(t,y) \quad &\geq \quad \mathbb{E}[e^{\gamma \int_t^T r(Y(s))ds}] \geq 1 \qquad (7) \\ \Longrightarrow \qquad (\mathcal{L}f)(t,y) \quad &\geq \quad \mathbb{E}[e^{\gamma \int_t^T r(Y(s))ds}] \geq 1 \end{aligned}$$

Upper bound for $\mathcal{L}$

If (7) holds, then $f(t,y)^{-\frac{\gamma}{1-\gamma}} \leq 1$. We obtain

$$(\mathcal{L}f)(t,y) \quad \leq \quad (1 + \frac{1-\gamma}{A'})e^{A'(T-t)+B'y},$$

where $A' = \gamma A + \lambda\psi(\frac{\gamma B}{\lambda}) > 0$ and $B' = \frac{\gamma B}{\lambda} \geq 0$, which is also a condition on the existence of the Laplace exponent of L.

Denote by $\mathcal{C}_e([0,T] \times (0,\infty))$ the continuous functions f on $[0,T] \times (0,\infty)$ satisfying

$$1 \leq f(t,y) \leq \left(1 + \frac{1-\gamma}{A'}\right) e^{A'(T-t)+B'y}$$

and define a norm on this space by

$$||f|| = \sup_{(t,y)\in[0,T]\times(0,\infty)} |e^{-\alpha(T-t)-B'y} f(t,y)|,$$

for some $\alpha > A'$. This space is a complete, metric space.

The mapping $\mathcal{L}$ is a contraction in this space for $\alpha > A' + \gamma$.

Then we can apply **Banach's fixed point theorem** and conclude that there exists a unique $\widehat{f} \in \mathcal{C}_e([0,T] \times (0,\infty))$ satisfying

$$(\mathcal{L}\widehat{f})(t,y) = \widehat{f}(t,y)$$

Differentiability of the solution

Differentiability in space.

Starting with a function f_1 with certain properties (differentiability, bounded derivative) we use the fixed point iteration $f_{n+1} = \mathcal{L} f_n$.
Then all f_n have these properties with the same bounds for the derivatives.
Then we conclude that also $\widehat{f}$ shares these properties.
This proves that $\widehat{f}$ is continuously differentiable in the space variable.

Differentiability in time.

First note that by the above continuously differentiability we can apply Itô's formula:

$$\lim_{s\to 0}\frac{\mathbb{E}^{0,y}[\widehat{f}(t,Y(s))]-\widehat{f}(t,y)}{s}$$

$$= -\lambda\frac{\partial\widehat{f}}{\partial y}(t,y)y+\lambda\int_{z>0}\big(\widehat{f}(t,y+z)-\widehat{f}(t,y)\big)\nu(dz),$$

since exponential moments exist.

Proposition $\widehat{f}$ satisfies the PIDE

$$0 = \frac{\partial\widehat{f}}{\partial t}(t,y)-\lambda\frac{\partial\widehat{f}}{\partial y}(t,y)y+\lambda\int_{z>0}\big(\widehat{f}(t,y+z)-\widehat{f}(t,y)\big)\nu(dz)$$
$$+\gamma\widehat{f}(t,y)Q(y)+(1-\gamma)(\widehat{f}(t,y))^{-\frac{\gamma}{1-\gamma}},\quad \widehat{f}(T,y)=1,$$

Moreover, $\widehat{f}\in\mathcal{C}^{1,1}([0,T)\times(0,\infty))$. □

Optimality of the solution

We apply a localizing sequence as in Benth et al. (2003).

Theorem Let $v \in \mathcal{C}^{1,2,1}([0,T) \times (0,\infty) \times (0,\infty)) \cap \mathcal{C}([0,T] \times (0,\infty) \times (0,\infty))$ satisfy for every $(c,\pi) \in \mathcal{A}$

$$\begin{aligned} 0 \quad \geq \quad & c^\gamma + \frac{\partial v}{\partial t}(t,x,y) + \frac{\partial v}{\partial x}(t,x,y)\left(\pi x(\mu(y) - r(y)) + xr(y) - c\right) \\ & + \frac{1}{2}\frac{\partial^2 v}{\partial x^2}(t,x,y)\pi^2 x^2 \sigma^2(y) - \frac{\partial v}{\partial y}(t,x,y)\lambda y \\ & + \lambda \int_{z>0} \big(v(t,x,y+z) - v(t,x,y)\big)\nu(dz), \end{aligned}$$

for all $(t,x,y) \in [0,T] \times (0,\infty) \times (0,\infty)$, with

$$v(T,x,y) = x^\gamma, \quad (x,y) \in (0,\infty) \times (0,\infty).$$

Assume also that

$$\mathbb{E}^{0,x,y}\Big[\int_0^T \int_{z>0} |v(t, X^{c,\pi}(t), Y(t-) + z) - v(t, X^{c,\pi}(t), Y(t-))|^2 \nu(dz)dt\Big] < \infty,$$

and that $\{v^-(\tau, X^{c,\pi}(\tau), Y(\tau))\}_{0<\tau\le T}$ is uniformly integrable for all $(\mathcal{F}_t)_{0\le t\le T}$-stopping times τ. Then

$$v(t,x,y) \ge V(t,x,y), \quad (t,x,y) \in [0,T] \times (0,\infty) \times (0,\infty).$$

Moreover, if there exists an admissible control $(\widehat{c}, \widehat{\pi})$ such that

$$\begin{aligned} 0 &= \widehat{c}^{\gamma} + \frac{\partial v}{\partial t}(t,x,y) + \frac{\partial v}{\partial x}(t,x,y)\left(\widehat{\pi}x(\mu(y)-r(y)) + xr(y) - \widehat{c}\right) \\ &\quad + \frac{1}{2}\frac{\partial^2 v}{\partial x^2}(t,x,y)\widehat{\pi}^2 x^2 \sigma^2(y) - \frac{\partial v}{\partial y}(t,x,y)\lambda y \\ &\quad + \lambda \int_{z>0} \big(v(t,x,y+z) - v(t,x,y)\big)\nu(dz)\}, \end{aligned}$$

for all $(t,x,y) \in [0,T] \times (0,\infty) \times (0,\infty)$, and that $\{v(\tau, X^{\widehat{c},\widehat{\pi}}(\tau), Y(\tau))\}_{0<\tau\leq T}$ is uniformly integrable for all $(\mathcal{F}_t)_{0\leq t\leq T}$-stopping times τ, then

$$v(t,x,y) = V(t,x,y), \quad (t,x,y) \in [0,T] \times (0,\infty) \times (0,\infty),$$

and $(\widehat{c}, \widehat{\pi})$ is the optimal strategy for (2). □

Theorem Assume that $(A1)-(A3)$ hold and $\psi(w)<\infty$ for $w<w_0$ and $w_0>0$ large enough. Define the investment strategy

$$\widehat{\pi}(t)=\arg\max_{\pi\in[0,1]}\{\pi\big(\mu(Y(t-))-r(Y(t-))\big)-\frac{1}{2}\pi^2(1-\gamma)\sigma^2(Y(t-))\},$$

and the consumption

$$\widehat{c}(t)=X^{\widehat{c},\widehat{\pi}}(t)(\widehat{f}(t,Y(t-)))^{-\frac{1}{1-\gamma}}$$

where $\widehat{f}$ is the unique solution of the fixed point equation in $\mathcal{C}^{1,1}([0,T)\times(0,\infty))\cap\mathcal{C}_e([0,T]\times(0,\infty))$:

$$f(t,y)\quad=\quad\mathbb{E}^{t,y}\big[(1-\gamma)\int_t^T e^{\gamma\int_t^s Q(Y(u))du}f(s,Y(s))^{-\frac{\gamma}{1-\gamma}}ds+e^{\gamma\int_t^T Q(Y(s))ds}\big],$$

and $X^{\widehat{c},\widehat{\pi}}$ is the wealth process under $(\widehat{c},\widehat{\pi})$, defined as

$$X^{\widehat{c},\widehat{\pi}}(t) = x\, e^{\int_0^t \left(\widehat{\pi}(s)(\mu(Y(s-))-r(Y(s-)))+r(Y(s-))-(\widehat{f}(s,Y(s-)))^{-\frac{1}{1-\gamma}}\right)ds} \\ e^{-\frac{1}{2}\int_0^t(\widehat{\pi}(s)\sigma(Y(s-)))^2 ds+\int_0^t \widehat{\pi}(s)\sigma(Y(s-))dW(s)}.$$

Then the pair $(\widehat{c},\widehat{\pi})$ is the optimal strategy for the consumption and investment problem (2). $\square$

Example [Barndorff-Nielsen & Shephard model]

Model parameters:

$T = 1$, $\gamma = 0.75$, $\lambda = 1/6$, $\mu(y) = 0.1 - 0.167y$, $\sigma^2(y) = y$, $r(y) = 0$.

L compound Poisson with jump intensity 0.5 and expo$(1/15)$-distributed jumps.

$Y(0) = 0.2$ (equal to the long-term vola)

PIDE is solved numerically with finite difference method.

Theoretical optimal investment strategy:

$$\widehat{\pi}(y) = \begin{cases} 1 & y < 0.24, \\ \dfrac{0.1 - 0.167y}{0.25y} & y \in [0.24, 0.6), \\ 0 & y \geq 0.6, \end{cases}$$

□

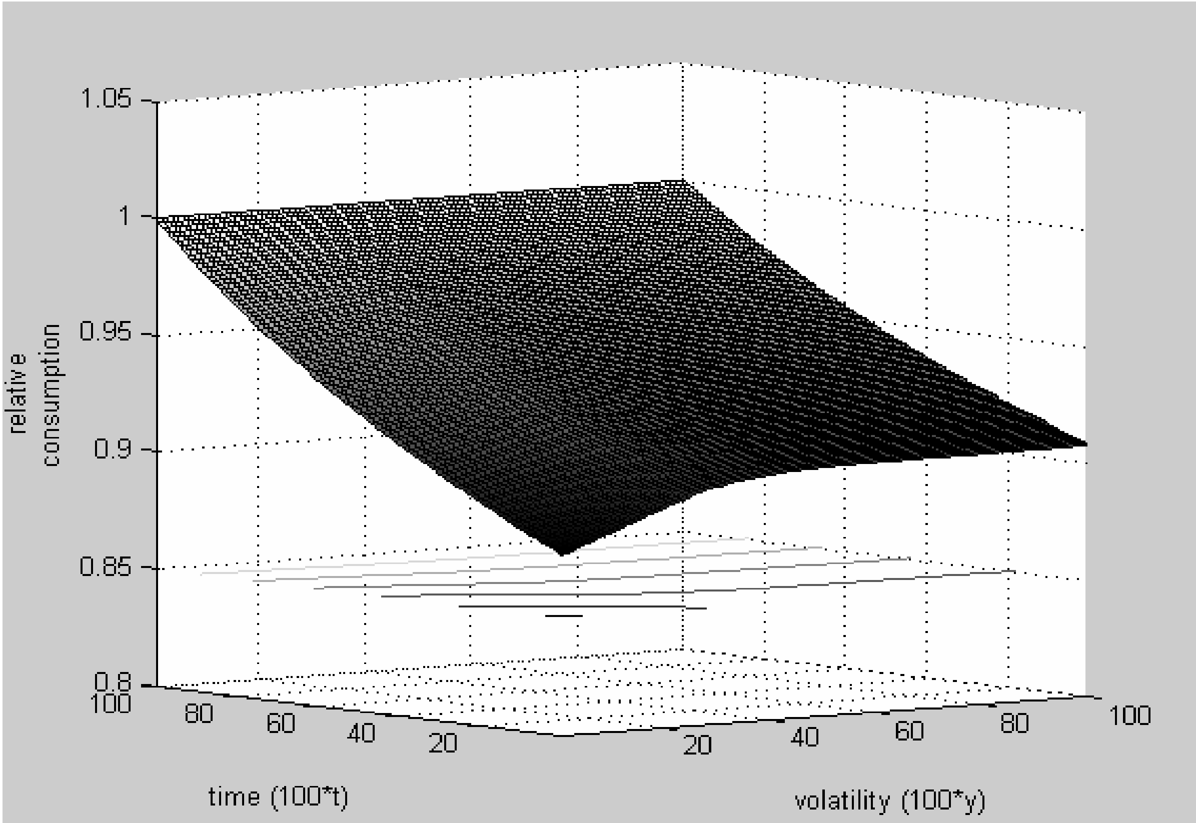

Figure 1: The optimal consumption rate as a function of time to maturity and the volatility level.

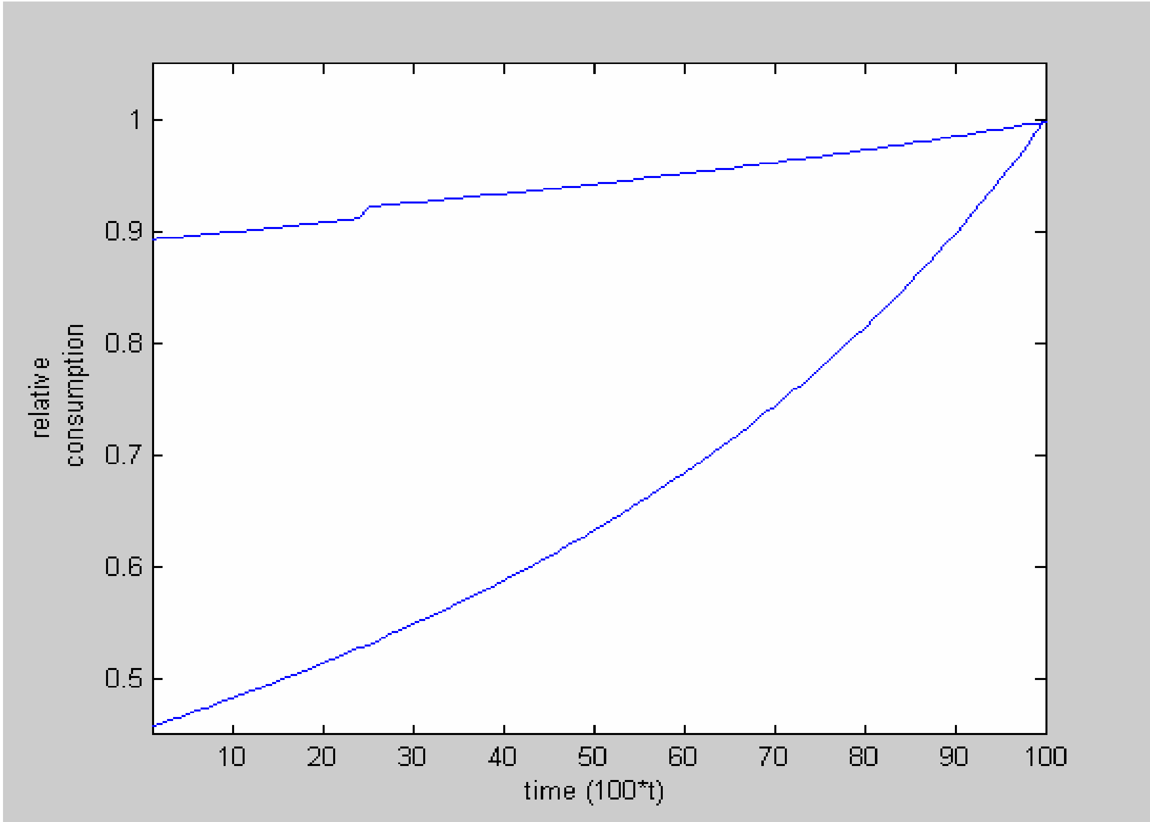

Figure 2: The optimal consumption rate in the stochastic volatility model (upper curve) and in the model with constant volatility (lower curve).