Optimal stopping problems with irregular payoff functions

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One dimensional diffusions

- Consider a one dimensional diffusion

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t,$$

with $b : I \to \mathbb{R}$, $\sigma : I \to \mathbb{R}$ where $I = (l, r)$ is an open interval.

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- We assume

$$\forall x \in I, \quad \sigma^2(x) > 0,$$
$$\forall x \in I, \quad \exists \varepsilon > 0, \quad \int_{x-\varepsilon}^{x+\varepsilon} \frac{1 + |b(y)|}{\sigma^2(y)} dy < \infty.$$

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- Under these conditions, we have existence and uniqueness in law of a weak solution, subject to $X_0 = x$, $x \in I$. We also assume no explosion

- Let $\mathcal{L}_0$ be the infinitesimal generator of the diffusion

$$\mathcal{L}_0\, u(x) = \frac{\sigma^2(x)}{2} u''(x) + b(x) u'(x), \quad x \in I,$$

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- Define the scale function

$$p(x) = \int_c^x e^{-\int_c^y \frac{2b(z)}{\sigma^2(z)} dz} dy, \quad x \in I.$$

Note that $\mathcal{L}_0 \, p = 0$.

- Define the speed measure

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- No explosion if and only if

$$\lim_{x\to l} v(x) = \lim_{x\to r} v(x) = +\infty,$$

where

$$v(x) = \int_c^x (p(x) - p(y))m(dy).$$

Infinite horizon

Given a bounded nonnegative Borel function $f : I \to \mathbb{R}$, and a locally bounded Borel function $r : I \to \mathbb{R}$, with $\inf_I r > 0$, define

$$v_f(x) = \sup_{\tau \in \mathcal{T}^0} \mathbb{E}_x \left(e^{-\Lambda_\tau} f(X_\tau) \right), \quad x \in I,$$

where $\mathcal{T}^0$ is the set of all stopping times with respect to the natural filtration of X, and $\Lambda_t = \int_0^t r(X_s)ds$.

S. Dayanik and I. Karatzas (2003) characterize v_f as the smallest p-*concave* majorant of f.

Denote by $\hat{f}$ the upper semicontinuous envelope of f:

$$\hat{f}(x) = \limsup_{y \to x} f(y), \quad x \in I.$$

Theorem 1 *The function v_f is the only continuous and bounded function on I, such that v_f is the difference of two convex functions and solves the variational inequality*

$$\begin{cases} v \geq \hat{f}, \quad \mathcal{L}_0 v - rv \leq 0 \\ (v - \hat{f})\,(\mathcal{L}_0 v - rv) = 0 \end{cases}$$

Note that $\mathcal{L}_0 v$ is a measure. We also have $v_f = v_{\hat{f}}$.

Finite horizon

Denote by $\mathcal{T}_t^0$ (resp. $\bar{\mathcal{T}}_t^0$) the set of all stopping times with respect to the (right continuous) natural filtration of X, with values in the interval $[0,t)$ (resp. $[0,t]$). Consider the functions u_f and v_f defined on $(0,+\infty) \times I$ as follows:

$$u_f(t,x) = \sup_{\tau \in \mathcal{T}_t^0} \mathbb{E}_x \left[e^{-\Lambda_\tau} f(X_\tau) \right], \quad (1)$$

$$v_f(t,x) = \sup_{\tau \in \bar{\mathcal{T}}_t^0} \mathbb{E}_x \left[e^{-\Lambda_\tau} f(X_\tau) \right], \quad (2)$$

Recall $\Lambda_t = \int_0^t r(X_s)\, ds$. We have $u_f \le v_f$, and $u_f = u_{\hat{f}}$.

Theorem 2 *We have* $u_f = v_f$ *and the function* v_f *is jointly continuous on* $(0, +\infty) \times I$*.*

The equality $u_f = v_f$ is an easy consequence of the following Proposition.

Proposition 3 *Let* τ *be a stopping time with values in* $[0, t]$*. We have*

$$\mathbb{E}_x \left[e^{-\Lambda_\tau} f(X_\tau) \right] = \lim_{s \to t, s < t} \mathbb{E}_x \left[e^{-\Lambda_{\tau \wedge s}} f(X_{\tau \wedge s}) \right]$$

The variational inequality satisfied by the value function should involve the operator

$$-\frac{\partial}{\partial t} + \mathcal{L},$$

where the operator $\mathcal{L}$ is defined by

$$\begin{aligned} \mathcal{L}u(t,x) &= \mathcal{L}_0 u(t,x) - r(x)u(t,x) \\ &= \frac{\sigma^2(x)}{2}\frac{\partial^2 u}{\partial x^2}(t,x) + b(x)\frac{\partial u}{\partial x^2}(t,x) - r(x)u(t,x) \\ &\quad (t,x) \in (0,+\infty) \times I. \end{aligned}$$

- For a smooth function u, we have

$$\mathcal{L}_0 u(t,x) = \frac{\sigma^2(x)}{2}\left(\frac{\partial^2 u}{\partial x^2}(t,x) + \frac{2b(x)}{\sigma^2(x)}\frac{\partial u}{\partial x}(t,x)\right)$$

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- The scale function p satisfies $\frac{d}{dx}\left(\frac{1}{p'}\right) = \frac{2b}{\sigma^2}\frac{1}{p'}$.

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- Hence

$$\begin{aligned}\mathcal{L}_0 u &= \frac{\sigma^2 p'}{2}\left(\frac{1}{p'}\frac{\partial^2 u}{\partial x^2} + \frac{2b}{\sigma^2}\frac{1}{p'}\frac{\partial u}{\partial x}\right)\\ &= \frac{\sigma^2 p'}{2}\frac{\partial}{\partial x}\left(\frac{1}{p'}\frac{\partial u}{\partial x}\right).\end{aligned}$$

- We now have

$$
\begin{aligned}
-\frac{\partial u}{\partial t} + \mathcal{L}u &= -\frac{\partial u}{\partial t} + \mathcal{L}_0 u - ru \\
&= -\frac{\partial u}{\partial t} + \frac{\sigma^2 p'}{2}\frac{\partial}{\partial x}\left(\frac{1}{p'}\frac{\partial u}{\partial x}\right) - ru
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- For a smooth test function Φ with compact support in $(0,+\infty)\times I$, we have

$$
\int\int \mathcal{A}u\Phi dt dx = \int\int u\left(\frac{\partial \Phi}{\partial t} + \mathcal{L}\Phi\right) dt m(dx)
$$

Theorem 4 *The value function v_f is the only continuous and bounded function on $(0, +\infty) \times I$ satisfying the following conditions*

1. $v \geq f$, $\mathcal{A}v \leq 0$ *on* $(0, +\infty) \times I$,
2. $\mathcal{A}v = 0$ *on the open set* $U := \{(t, x) \in (0, +\infty) \times I \mid v(t, x) > \hat{f}(x)\}$,
3. *For every* $x \in I$, $\lim_{t \to 0} v(t, x) = \hat{f}(x)$.

Density estimates

- We want to prove that if τ is a stopping time with values in $[0,t]$. We have

$$\mathbb{E}_x\left[e^{-\Lambda_\tau} f(X_\tau)\right] = \lim_{s\to t, s<t} \mathbb{E}_x\left[e^{-\Lambda_{\tau\wedge s}} f(X_{\tau\wedge s})\right]$$

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- Write

$$\mathbb{E}_x\left[e^{-\Lambda_{\tau\wedge s}} f(X_{\tau\wedge s})\right] = \mathbb{E}_x\left[e^{-\Lambda_\tau} f(X_\tau)\mathbf{1}_{\{\tau<s\}}\right] + \mathbb{E}_x\left[e^{-\Lambda_s} f(X_s)\mathbf{1}_{\{\tau\geq s\}}\right].$$

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- By dominated convergence,

$$\lim_{s\to t, s<t}\mathbb{E}_x\left[e^{-\Lambda_\tau}f(X_\tau)\mathbf{1}_{\{\tau<s\}}\right] = \mathbb{E}_x\left[e^{-\Lambda_\tau}f(X_\tau)\mathbf{1}_{\{\tau<t\}}\right],$$

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- This is true if f is continuous.
- For an arbitrary f,

$$\begin{aligned} \mathbb{E}_x \left| f(X_s) - f(X_t) \right| &\le \mathbb{E}_x \left| f(X_s) - \varphi(X_s) \right| + \mathbb{E}_x \left| \varphi(X_s) - \varphi(X_t) \right| \\ &\quad + \mathbb{E}_x \left| \varphi(X_t) - f(X_t) \right|. \end{aligned}$$

- Therefore, it suffices to prove that $\lim_{s\to t, s<t} \mathbb{E}_x |f(X_s) - f(X_t)| = 0$.
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- Therefore, we need to prove that, given $\varepsilon > 0$ one can find a bounded continuous function φ such that

$$\sup_{t/2 \leq s \leq t} \mathbb{E}_x |f(X_s) - \varphi(X_s)| \leq \varepsilon.$$

- This can be deduced from the following estimate, where $P_t h(x) = \mathbb{E}_x h(X_t)$.

$$\int_I \left(\frac{d}{dx}(P_t h)(x) \right)^2 \frac{dx}{p'(x)} \leq \frac{1}{t} ||h||_{L^2(m)}.$$

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- The previous estimate is deduced from a similar estimate for the resolvent $(U_\rho)_\rho$ of the semi-group, where $U_\rho h(x) = \mathbb{E}_x \left[\int_0^\infty e^{-\rho t} h(X_t) dt \right]$

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- Note that $U_\rho h$ is the unique bounded solution of the ordinary differential equation

$$\frac{\sigma^2(x)}{2} u''(x) + b(x)u'(x) - \rho u(x) + h(x) = 0.$$