Optimal Static Hedging of Equity & Credit Risk using Derivatives

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The Problem

Trader wants to hedge a primary derivatives position *statically* for a set of future hedging dates, using a set of given derivatives as hedging instruments.

- Static hedging: easy to implement, less transactions
- Ideally: Robustness of the hedge to model risk
- Aim: Optimal (approximative) static hedge
- Examples for hedging instruments: Vanilla options, ZCB, CDS, forward staring options...
- Examples for primary instruments: more exotic options, convertible bonds, capped/floored cliquets...

Goals

What are we trying to achieve?

- A ‘generic’ static hedging method with few requirements on
  - the structure of the underlying model, and
  - the primary derivative and the hedge instruments;
- That uses a given set of hedge instruments.
- Tractable computation of hedges.
- Investigate whether hedges are robust.
  (in- and outside the hedging model).

Optimal Static Hedge

- Optimality criterion: Minimise variance of hedging P/L under *pricing measure*,
- for future horizons $T_1 < \cdots T_k \cdots < T_M$.
- All hedges initiated at time zero, no rebalancing - but may be unwound before expiry.
- For each horizon T_k, minimise variance of P/L from primary and hedge instruments for present and future dates:

$$\forall k, \quad \min_{\theta_k} \mathrm{Var}\left[H_k + \sum_{i>k} \theta_i \cdot Y_k^i + \theta_k \cdot Y_k^k \right]$$

- Solution: co-initial linear regression

Minimum Variance Hedging

	Complete	Incomplete (multiple period)
static hedge	Breeden/Litzenberger	*
dynamic hedge	Black/Scholes	Föllmer/Sonderm. Lamberton/Bouleau

- Comparison: Dynamic Minimum Variance hedging $\longrightarrow$ sequential (conditional) regression. While (*) $\longrightarrow$ co-initial regression.
- Method (*) needs few assumptions on hedging instruments and the instruments to be hedged.

A Hedging Example

Consider a convertible bond (CB1):

- Notional 100
- Semi annual coupon, 5%
- Conversion 0.9 of shares on coupon dates
- Bond is callable at 120 on coupon dates

In addition:

- Spot stock price 100.
- Bond recovery in default 40%.

Hedge the bond value at its coupon dates.

Hedging Example Setup

To investigate the quality of the hedge, we consider two different models

1. A *hedger's model*:
   To compute hedges, after the model has been calibrated to market prices of hedging instruments.
2. A (hypothetical) *"real market" model*:
   To simulate market prices, and investigate hedge performance and robustness.

The Hedger's Model (DD+)

- One factor model, with piecewise constant parameters for vol, skew & spread.
- For period $t \in (t_{k-1}, t_k]$, the (pre-default) stock price follows displaced-diffusion-type dynamics

$$dS_t = S_t(r_k + \lambda_k)\,dt + (S_t\sigma_k q_k + (1 - q_k)F_k)\,dW_t$$

- with additional jumps to $S_t = 0$ at default, intensity being $\lambda_k = a_k(S_{t_{k-1}}/S_0)^{-p}$,
- with $p > 0$ set exogenously ($\rightarrow$ Andersen/Buffum '02).

The Hedger's Model (DD+)

Model properties:

- sufficient parameters to match market option prices and CDS spreads.
- fast forward calibration (appropriate choice of a_k, q_k, σ_k) and pricing of primary and hedge instruments for the hedge computation.

The ‘Real Market’ Model (EH)

- Extended Heston SV-model to include default,
- (pre-default) stock price S_t given by

$$\frac{dS_t}{S_t} = (r(t) + \lambda(\Sigma_t))dt + \sqrt{\Sigma(t)}dW_t^S,$$
$$d\Sigma_t = \kappa(\theta - \Sigma_t)dt + \sigma^\Sigma \sqrt{\Sigma_t}\, dW_t^\Sigma,$$
$$\lambda(\Sigma) = \xi\Sigma\,,$$
$$d\langle W^S, W^\Sigma\rangle_t \;=\; \rho dt\,,$$

- where $\lambda(\Sigma_t)$ is the intensity of default.
- Affine two factor model. Semi-analytic formulas for CDS and European option prices.

Note

Our goal is not to demonstrate the properties of the models.

But to investigate the quality of the convertible bond hedge numerically.

Real World Parameters

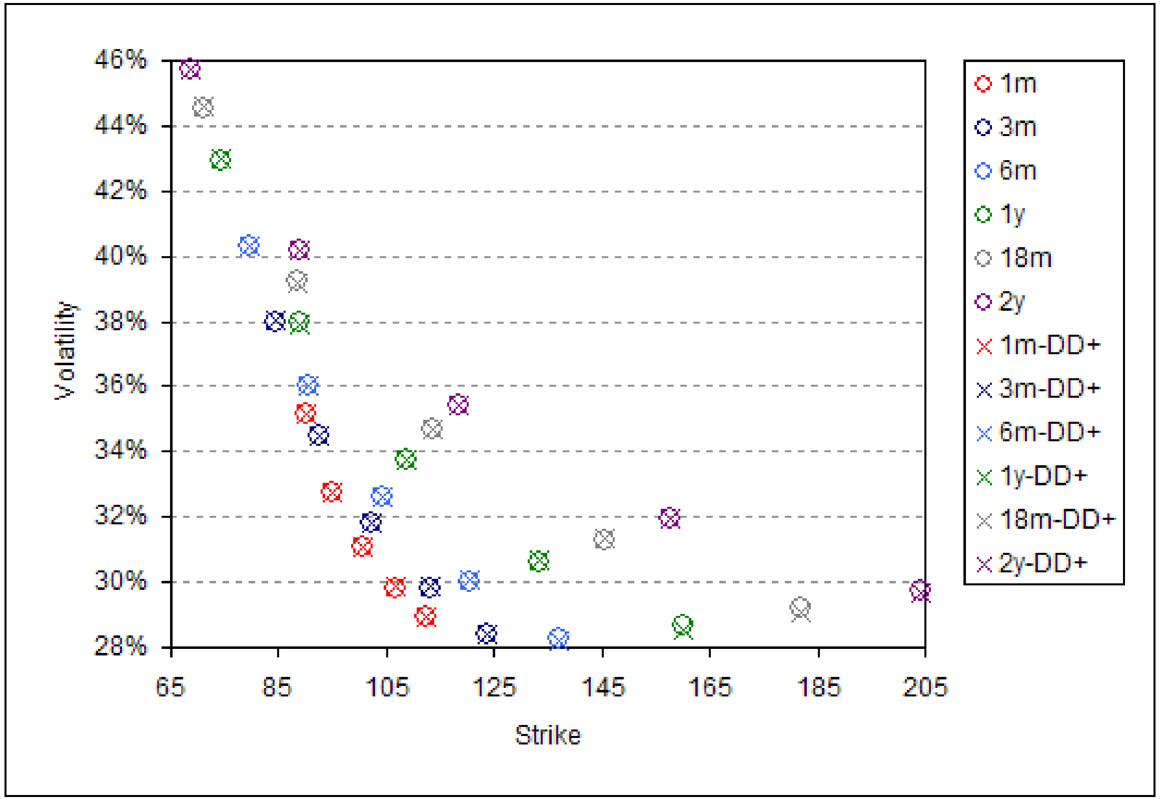
Assume the ‘real market’ prices equity and credit derivatives according to the EH model with parameters:

- interest rate $r = 4\%$
- variance mean reversion rate and level $\kappa = 1.0$, $\theta = 0.35^2$
- volatility of variance $\sigma^{\Sigma} = 0.3$
- variance/stock price correlation $\rho = -0.6$
- factor of variance in intensity $\xi = 0.35$.

Also assume:

- Initial stock price is 100 and volatility is 30%
- On default recovery is 40% of face value

DD+ Model Calibration



- First 6 data series are the target vols given by the market, second 6 series are calibrated by DD+ model.
- Calibration of vol, skew and spread parameters is

DD+ Model Calibration ctd

- CDS term structure matched very closely by calibrating the model to risky zero coupon bonds given by the EH model during option calibration.
- CDS spreads, with 40% recovery

Maturity	Intensity	CDS Spread
1m	3.1960%	1.9610%
3m	3.2802%	2.0017%
6m	3.3893%	2.0877%
1y	3.5595%	2.1791%
18m	3.6829%	2.2573%
2y	3.7742%	2.3047%

DD+ Model Hedge Portfolio

Hedge CB1 at coupon dates, time 2.0, 1.5, 1.0, 0.5, using hedge instruments with maturity equal to hedge time

- calls, puts strikes matching those in calibration options
- par CDS
- zero coupon bonds to set expected PL to zero.

We obtained a hedge portfolio...

DD+ Model Hedge Portfolio

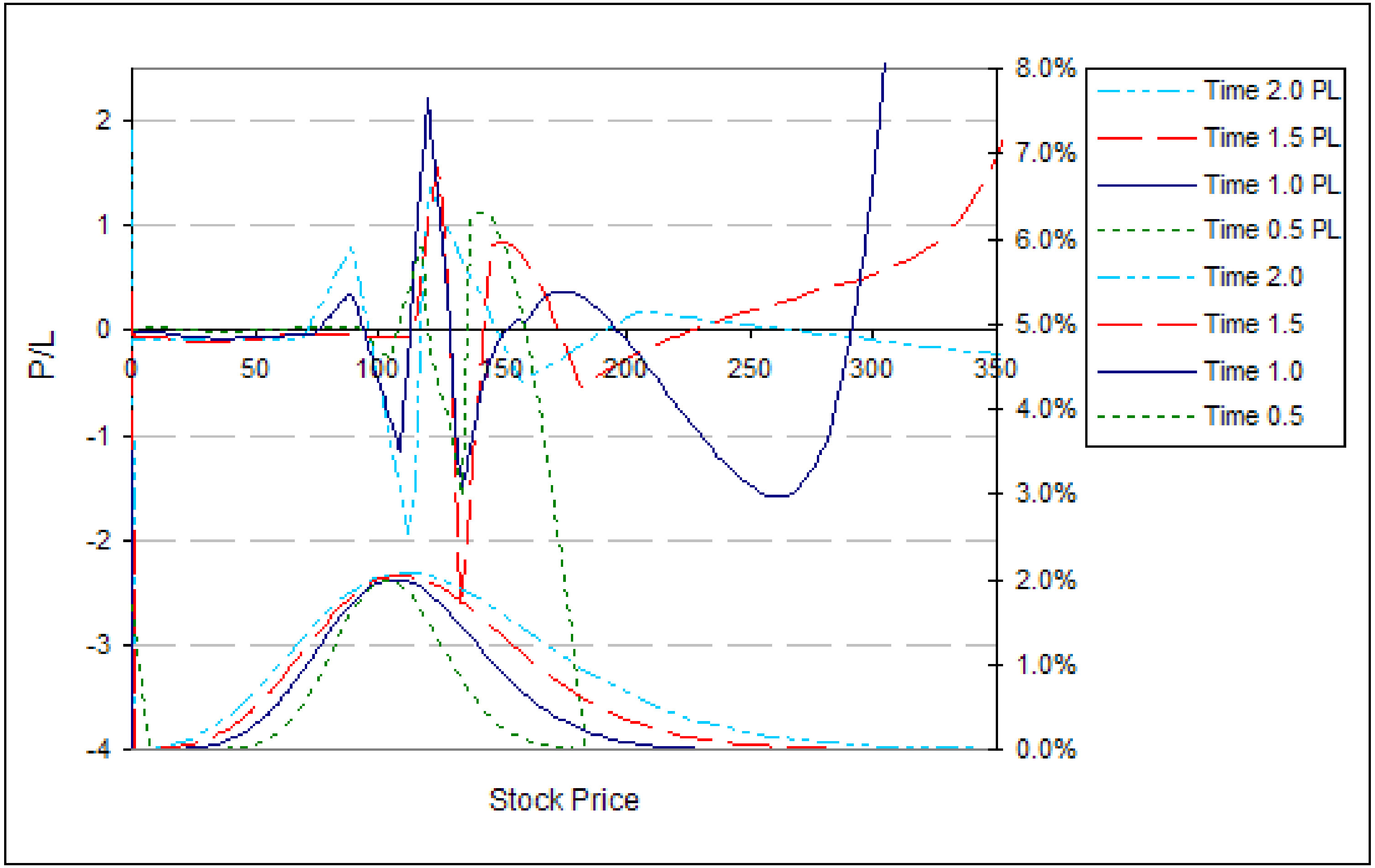
#	Trade	Expiry	Notional	Details
1	Par CDS	2.00	102.36196	Spread 0.023038
2	OPTION	2.00	0.044629	CALL strike 68.82
3	OPTION	2.00	-0.164464	CALL strike 89.03
4	OPTION	2.00	-0.829655	CALL strike 118.53
5	OPTION	2.00	0.064043	CALL strike 157.80
6	OPTION	2.00	-0.017375	CALL strike 204.16
7	ZCB	2.00	-101.4171	
8	Par CDS	1.50	-1.043921	Spread 0.022445
9	OPTION	1.50	0.159955	CALL strike 113.60
10	OPTION	1.50	-0.23741	CALL strike 145.55
11	OPTION	1.50	0.085212	CALL strike 181.92
12	ZCB	1.50	-1.381842	

DD+ Model Hedge Portfolio

#	Trade	Expiry	Notional	Details
13	Par CDS	1.00	-1.084435	Spread 0.021725
14	OPTION	1.00	0.025512	CALL strike 74.12
15	OPTION	1.00	-0.10441	CALL strike 88.93
16	OPTION	1.00	0.404295	CALL strike 108.87
17	OPTION	1.00	-0.418216	CALL strike 133.29
18	OPTION	1.00	0.053135	CALL strike 159.91
19	ZCB	1.00	-1.345131	
20	Par CDS	0.50	-1.111014	Spread 0.020525
21	OPTION	0.50	-0.02296	CALL strike 90.43
22	OPTION	0.50	0.108459	CALL strike 104.34
23	OPTION	0.50	0.383026	CALL strike 120.39
24	OPTION	0.50	-0.677842	CALL strike 136.94
25	ZCB	0.50	-1.3163	

DD+ Model Hedge Performance

Hedged portfolio P/L



DD+ Model Hedge Performance

Time	P/L Mean	Variance	Skew	Kurtosis
2.0	0.000000	**0.443657**	-0.7267	4.14161
1.5	0.000000	**0.432481**	-1.2046	8.60745
1.0	0.000000	**0.533097**	0.84642	4.42283
0.5	0.000000	**0.242496**	-0.9768	12.6853

Zero bonds are chosen for zero P/L mean. P/L variance should be used to judge the *quality* of the hedge.

Hedge Robustness: No Hedging

Time	P/L Mean	Variance	Skew	Kurtosis
2.00	0.000000	**1346.182**	0.92058	6.40640
1.50	0.000000	**825.0955**	0.34701	6.54585
1.00	0.000000	**435.4891**	-0.5850	8.27132
0.50	0.000000	**165.0104**	-2.6315	18.2412

No hedges used - zero coupon bonds used to set the PL to zero.

Hedge Robustness: Parameter Risk

Lets assume that the market moves after the hedge has been initiated i.e., the EH model parameters change.

- Vol. mean rev. level θ goes from 0.35^2 to 0.36^2,
- spread factor ξ goes from 0.35 to 0.5
- initial spot volatility goes from 0.3 to 0.35

These model parameters result in 4.5% to 8% increase in option vols and almost doubling of CDS spreads.

DD+ model is recalibrated and the hedge performance is evaluated.

Hedge Robustness: Parameter Risk

P/L statistic after a change, in DD+ model.

Time	P/L Mean	Variance	Skew	Kurtosis
2.0	-0.00147	**0.402973**	-0.7818	5.23895
1.5	-0.18047	**0.572386**	-2.0936	12.7339
1.0	-0.45498	**0.696917**	-0.6579	3.50667
0.5	-0.68086	**0.747691**	-2.6296	16.5599

Hedge Robustness: Model Risk

We consider how well the hedge performs when we compute the hedged portfolio in the EH model.

Time	P/L Mean	Variance	Skew	Kurtosis
2.0	0.039114	**0.507971**	-0.523523	3.905897
1.5	-0.213219	**0.714340**	-1.628541	7.805858
1.0	-0.322333	**1.642980**	-0.26542	4.644496
0.5	-0.338016	**1.338578**	-2.071667	10.23964

To Summarise so far...

- Using a 1 factor model - qualitatively different to the *real market model* - we have computed a static hedge that is robust up to reasonable parameter changes,
- with commensurate performance in the *real market*.
- Interpretation: Static hedge instruments were appropriate for the convertible bond.
- Consider another example...

Forward Starting Option

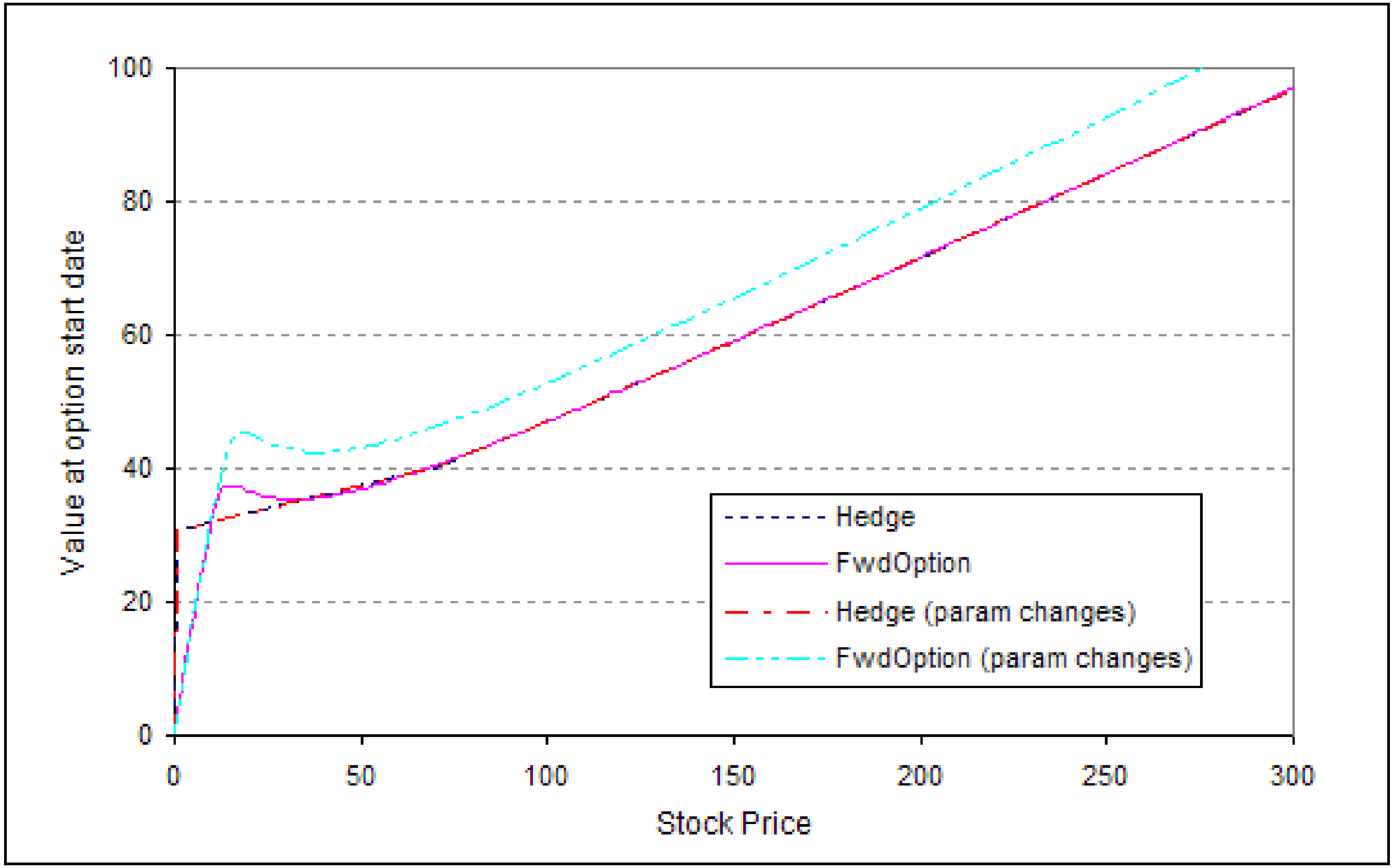
- Consider a 2-year expiry forward starting call option,
  starting in 18 months from today with strike set ATM then:
- Forward call option payoff at maturity

$$\text{FWD Option PayOff} = (S_{2y} - S_{18m})^+ \qquad (1)$$

- Statically hedge the option value at 18 months with same 18 month maturity hedge instruments as before: European calls, puts and CDS.

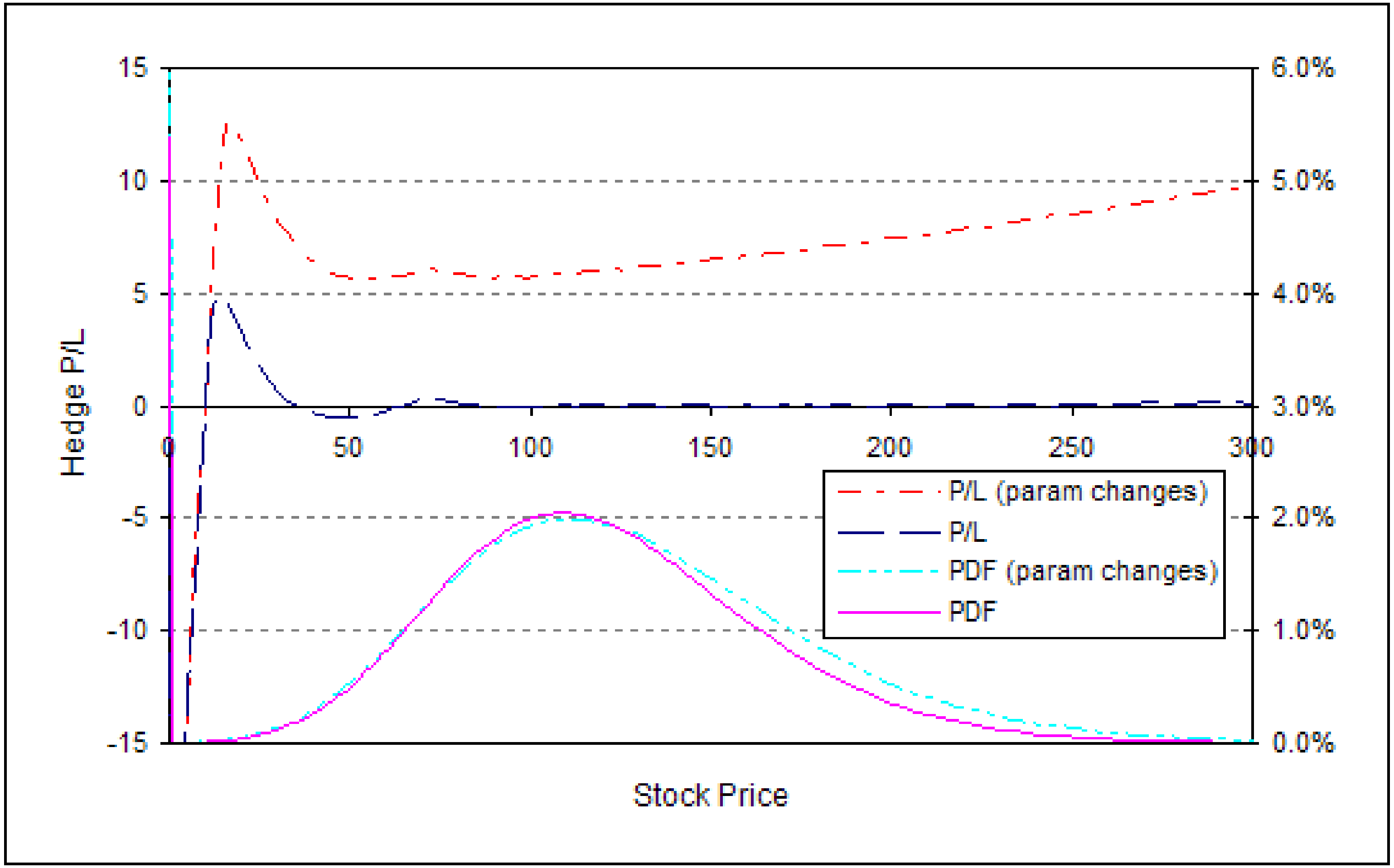
Forward Starting Option

Fwd starting option and hedge at the option start date in the calibrated DD+ and after parameter changes.



Forward Starting Option

P/L for the hedged forward starting option in the calibrated DD+ model



The End

- Conclusions
  - Static minimum variance hedge of derivatives by a wider range of other derivertives is fairly tractable to compute by co-initial regression.
  - Robustness of the hedge
    $\leftrightarrow$ appropriate hedge instruments.
- Outlook / Questions
  - Hedging other derivative products.
    Interesting suggestions ?
  - How new is this? Further references?
    (Preprint is online)
- **Thank you!**

Supporting slides

Additional information:

- the real world model
- the hedgers model