Duality and valuation of derivatives in a semimartingale setting

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The Theme

Call option in FX market: Euro/Dollar

Gives you the right to buy Euros paying in Dollars.
At the same time a right to sell Dollars getting Euros.

Payoff $(S_T - K)^+$ (S_t) exchange rate, K strike

$$
\begin{aligned}
(S_T - K)^+ &= KS_T\Big(\frac{1}{K} - \frac{1}{S_T}\Big)^+ \\
&= KS_T\big(K' - S'_T\big)^+
\end{aligned}
$$

$\uparrow$

Dollar/Euro rate

Call price = $K\cdot$ put price (in the dual market)

$\longrightarrow$ duality principle

Brief literature survey

- Carr (1994) put-call duality in BS-setting and for diffusions
- Chesney and Gibson (1995) two-factor diffusion model
- Bates (1997) diffusions and jump-diffusions
- Schroder (1999) various payoffs in diffusions and jump-diffusions
- Carr, Ellis, and Gupta (1998) static hedging strategies for exotic derivatives
- Carr and Chesney (1996) put-call for American options
- Detemple (2001) American options with general payoffs
- Henderson and Wojakowski (2002) Asian options
- Eberlein and Papapantoleon (2005a,b) Exotic options for Lévy and time-inhomogeneous Lévy models
- Vanmaele, Deelstra, Liinev, Dhaene, Goovaerts (2006) Forward start Asian options
- Fajardo and Mordecki (2006a,b) Lévy models
- Vecer (2002), Vecer and Xu (2004) Asian options (PIDE)
- Eberlein, Kluge, and Papapantoleon (2006) Interest rate options
- Eberlein, Papapantoleon, Shiryaev (2006) Semimartingales

Exponential semimartingale models

Let $\mathcal{B} = (\Omega, \mathcal{F}, \mathbf{F}, P)$ be a stochastic basis, where $\mathcal{F} = \mathcal{F}_T$ and $\mathbf{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$. We model the price process of a financial asset as an exponential semimartingale

$$S_t = \mathrm{e}^{H_t}, \quad 0 \leq t \leq T.$$

$H = (H_t)_{0 \leq t \leq T}$ is a *semimartingale* with canonical representation

$$H = H_0 + B + H^c + h(x) * (\mu^H - \nu) + (x - h(x)) * \mu^H$$

or, in detail

$$H_t = H_0 + B_t + H_t^c + \int_0^t \int_{\mathbb{R}} h(x) \mathrm{d}(\mu^H - \nu) + \int_0^t \int_{\mathbb{R}} (x - h(x)) \mathrm{d}\mu^H,$$

where

- $h = h(x)$ is a truncation function; canonical choice $h(x) = x1_{\{|x|\leq 1\}}$;
- $B = (B_t)_{0\leq t\leq T}$ is a predictable process of bounded variation;
- $H^c = (H^c_t)_{0\leq t\leq T}$ is the continuous martingale part of H;
- $\nu = \nu(\omega; \mathrm{d}t, \mathrm{d}x)$ is the predictable compensator of the random measure of jumps $\mu^H = \mu^H(\omega; \mathrm{d}t, \mathrm{d}x)$ of H.

For the processes B, $C = \langle H^c \rangle$, and the measure ν we use the notation

$$\mathbb{T}(H|P) = (B, C, \nu)$$

which will be called the *triplet of predictable characteristics* of the semimartingale H with respect to the measure P.

Assumption: The truncation function satisfies the *antisymmetry* property

$$h(-x) = -h(x).$$

Alternative model description

$\mathcal{E}(X) = (\mathcal{E}(X)_t)_{0 \le t \le T}$ stochastic exponential

$$\begin{aligned} S_t &= \mathcal{E}(\widetilde{H})_t, \quad 0 \le t \le T \\ dS_t &= S_{t-} d\widetilde{H}_t \end{aligned}$$

where

$$\widetilde{H}_t = H_t + \frac{1}{2}\langle H^c \rangle_t + \int_0^t \int_{\mathbb{R}} (e^x - 1 - x)\mu^H(\mathrm{d}s, \mathrm{d}x)$$

Note

$$\mathcal{E}(\widetilde{H})_t = \exp(\widetilde{H}_t - \frac{1}{2}\langle \widetilde{H}^c \rangle_t) \prod_{0 < s \le t} (1 + \Delta \widetilde{H}_s) \exp(-\Delta \widetilde{H}_s)$$

Asset price positive only if $\Delta \widetilde{H} > -1$.

Martingale and dual martingale measures

Assumption ($\mathbb{ES}$)

The process $1_{\{x>1\}} e^x * \nu$ has bounded variation.

Then, H is exponentially special and

$$S = e^H \in \mathcal{M}_{\text{loc}}(P) \Leftrightarrow B + \frac{C}{2} + (e^x - 1 - h(x)) * \nu^H = 0.$$

Moreover, we assume that $S \in \mathcal{M}(P)$, therefore $ES_T = 1$. Define on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T})$ a *new* probability measure P' with

$$\frac{dP'}{dP} = S_T.$$

Since $S > 0$ (P-a.s.), we have $P \ll P'$ and

$$\frac{dP}{dP'} = \frac{1}{S_T}.$$

Introduce the process

$$S' = \frac{1}{S}.$$

Then, denoting by H' the *dual* of the semimartingale H, i.e. $H' = -H$, we have

$$S' = \mathrm{e}^{H'}.$$

Proposition

Suppose $S = \mathrm{e}^{H} \in \mathcal{M}(P)$ i.e. S is a P-martingale. Then the process $S' \in \mathcal{M}(P')$ i.e. S' is a P'-martingale.

Lemma

Let f be a predictable, bounded process. The triplet of predictable characteristics of the stochastic integral process $J = \int_0^{\cdot} f \mathrm{d}H$, denoted by $\mathbb{T}(J|P) = (B_J, C_J, \nu_J)$, is

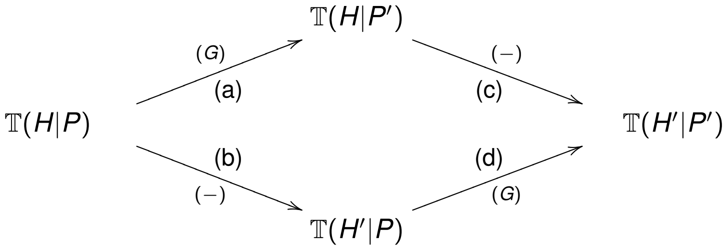
$$\begin{aligned} B_J &= f \cdot B + [h(fx) - fh(x)] * \nu \\ C_J &= f^2 \cdot C \\ 1_A(x) * \nu_J &= 1_A(fx) * \nu, \quad A \in \mathcal{B}(\mathbb{R}). \end{aligned}$$

Theorem

The triplet $\mathbb{T}(H'|P') = (B', C', \nu')$ *can be expressed via the triplet* $\mathbb{T}(H|P) = (B, C, \nu)$ *by the following formulae:*

$$\begin{aligned} B' &= -B - C - h(x)(e^x - 1) * \nu \\ C' &= C \\ 1_A(x) * \nu' &= 1_A(-x)e^x * \nu, \quad A \in \mathcal{B}(\mathbb{R}). \end{aligned}$$

Structure of the proof:



$\xrightarrow{(G)}$: Girsanov's theorem, $\xrightarrow{(-)}$: dual of a semimartingale.

Symmetry of Markets

If the original market (S, P) and the dual market (S', P') satisfy

$$\mathrm{Law}(S|P) = \mathrm{Law}(S'|P')$$

then we say these markets are *symmetric*.

In cases where the triplets $\mathbb{T}(H|P)$ and $\mathbb{T}(H'|P')$ determine these laws completely (e.g. for Lévy processes H and H')

$$\text{symmetry holds iff } \nu' = \nu$$

The equation in the Theorem is then

$$1_A(x) * \nu = 1_A(-x)e^x * \nu, \qquad A \in B(\mathbb{R})$$

Example 1: Diffusion models

$$dS_t = S_t\sigma(t, S_t)\,\mathrm{d}W_t, \qquad S_0 = 1$$

local volatility models (Dupire (1994), Skiadopoulos (2001))

$$H_t = \int_0^t \sigma(u, e^{H_u})\,\mathrm{d}W_u - \frac{1}{2}\int_0^t \sigma^2(u, e^{H_u})\,\mathrm{d}u$$

$$\Rightarrow \quad B = -\frac{1}{2}\int_0^{\cdot} \sigma^2(u, e^{H_u})\,\mathrm{d}u, \quad C = \int_0^{\cdot} \sigma^2(u, e^{H_u})\,\mathrm{d}u, \quad \nu \equiv 0$$

Theorem $\Rightarrow B' = -B - C = -\frac{1}{2}\int_0^{\cdot} \sigma^2(u, e^{H_u}\,\mathrm{d}u)$, $C' = C$, $\nu' \equiv 0$

Example 2: Purely discontinuous Lévy models

$$S_t = e^{H_t}, \qquad \mathbb{T}(H,P) = (B, 0, \nu)$$

local characteristics: $B_t(\omega) = bt,\ \nu(\omega; \mathrm{d}t, \mathrm{d}x) = \mathrm{d}t F(\mathrm{d}x)$,
F Lévy measure

$$S \in \mathcal{M}_{\mathrm{loc}}(P) \Leftrightarrow b = -\int_{\mathbb{R}} (e^X - 1 - h(x)) F(\mathrm{d}x)$$

Actually: $S \in \mathcal{M}(P)$

Parametric models: $F(\mathrm{d}x) = e^{\vartheta x} f(x)\,\mathrm{d}x \qquad f$ even
Generalized hyperbolic (includes hyperbolic, NIG, VG, ...)
CGMY, Meixner

Dual process H' : $\displaystyle\int 1_A(x) F'(\mathrm{d}x) = \int 1_A(-x) e^{(1+\vartheta)x} f(x)\,\mathrm{d}x$

$$b' = -\int_{\mathbb{R}} (e^x - 1 - h(x)) F'(\mathrm{d}x)$$

European options (1)

Theorem

The prices of standard call and put options satisfy the following duality relations:

$$\mathbb{C}_T(S;K) = K\,\mathbb{P}'_T(K';S')$$

and

$$\mathbb{P}_T(K;S) = K\,\mathbb{C}'_T(S';K').$$

Proof: Using the dual measure

$$\begin{aligned}\mathbb{C}_T(S;K) &= E\left[S_T\frac{(S_T-K)^+}{S_T}\right] = E'\left[\frac{(S_T-K)^+}{S_T}\right] = E'[(1-KS'_T)^+]\\ &= KE'\left[\left(\frac{1}{K}-S'_T\right)^+\right] = KE'[(K'-S'_T)^+],\end{aligned}$$

where $K' = \frac{1}{K}$. $\square$

European options (2)

Corollary

Call and put prices in a dual pair of markets (S, P) and (S', P') satisfy a *call-call parity*

$$\mathbb{C}_T(S; K) = K\,\mathbb{C}'_T(S'; K') + 1 - K$$

and a *put-put parity*

$$\mathbb{P}_T(K; S) = K\,\mathbb{P}'_T(K'; S') + K - 1$$

Proof: Combine with classical call-put parity

$$\mathbb{C}_T(S; K) = \mathbb{P}_T(K; S) + 1 - K$$ □

American options

Consider payoff processes $(e^{-\lambda t} f_t)_{0 \le t \le T}$, $\lambda \ge 0$

Price of the American option

$$\widehat{\mathbb{V}}_T(S) = \sup_{\tau \in \mathcal{M}_T} E[e^{-\lambda\tau} f_\tau]$$

where $\mathcal{M}_T$ is the class of stopping times τ with $0 \le \tau \le T$

Standard call $\quad f_\tau = (S_\tau - K)^+$

Standard put $\quad f_\tau = (K - S_\tau)^+$

Duality relations

$$\widehat{\mathbb{C}}_T(S;K) = K\,\widehat{\mathbb{P}}'_T(K';S')$$

$$\widehat{\mathbb{P}}_T(K;S) = K\,\widehat{\mathbb{C}}'_T(S';K')$$

Floating strike lookback options (1)

Payoff of a call: $\left(S_T - \alpha \inf_{0 \le t \le T} S_t\right)^+$ for an $\alpha \ge 1$

Assume $H' = (H'_t)_{0 \le t \le T}$ satisfies the *reflection principle*

$$\mathrm{Law}\left(\sup_{t \le T} H'_t - H'_T | P'\right) = \mathrm{Law}(-\inf_{t \le T} H'_t | P')$$

(holds for Lévy processes), then

$$\mathbb{C}_T(S; \alpha \inf S) = \alpha \mathbb{P}'_T\left(\frac{1}{\alpha}; \inf S'\right)$$

Value of a *floating strike* lookback call

$\rightarrow$ value of a *fixed strike* lookback put

Floating strike lookback options (2)

Proof:

$$\begin{aligned}\mathbb{C}_T(S;\alpha \inf S) &= E\left[\left(S_T - \alpha \inf_{t\le T} S_t\right)^+\right] = E\left[S_T\left(1-\frac{\alpha \inf_{t\le T} S_t}{S_T}\right)^+\right]\\ &= E'\left[\left(1-\alpha e^{\inf_{t\le T} H_t - H_T}\right)^+\right]\\ &= E'\left[\left(1-\alpha e^{H'_T - \sup_{t\le T} H'_t}\right)^+\right]\end{aligned}$$

The process $H' = (H'_t)_{0\le t\le T}$ satisfies the *reflection principle*:

$$\mathrm{Law}\left(\sup_{t\le T} H'_t - H'_T \mid P'\right) = \mathrm{Law}\left(-\inf_{t\le T} H'_t \mid P'\right)$$

$$\begin{aligned}\mathbb{C}_T(S;\alpha \inf S) &= \alpha E'\left[\left(\frac{1}{\alpha} - e^{\inf_{t\le T} H'_t}\right)^+\right]\\ &= \alpha E'\left[\left(\frac{1}{\alpha} - \inf_{t\le T} S'_t\right)^+\right]\\ &= \alpha \mathbb{P}'_T\left(\frac{1}{\alpha}; \inf S'\right)\end{aligned}$$

□

Floating strike lookback options (3)

Payoff of a put: $\left(\beta \sup_{0 \le t \le T} S_t - S_T\right)^+$ for a $0 < \beta \le 1$

Assume $H' = (H'_t)_{0 \le t \le T}$ satisfies

$$\mathrm{Law}\left(H'_T - \inf_{t \le T} H'_t | P'\right) = \mathrm{Law}\left(\sup_{t \le T} H'_t | P'\right)$$

(holds for Lévy processes), then

$$\mathbb{P}_T(\beta \sup S; S) = \beta \mathbb{C}'_T\left(\sup S'; \frac{1}{\beta}\right)$$

Value of a *floating strike* lookback put
$\longrightarrow$ value of a *fixed strike* lookback call

Floating strike Asian options

Payoff of a call: $\left(S_T - \frac{1}{T}\int_0^T S_t \, \mathrm{d}t\right)^+$

Assume $H' = (H'_t)_{0 \le t \le T}$ satisfies

$$\mathrm{Law}(H'_T - H'_{(T-t)-}; 0 \le t < T | P') = \mathrm{Law}(H'_t; 0 \le t < T | P')$$

(holds for Lévy processes), then

$$\mathbb{C}_T(S; \frac{1}{T}\int S) = \mathbb{P}'_T\left(1; \frac{1}{T}\int S'\right)$$

Value of a *floating strike* Asian call
→ value of a *fixed strike* Asian put

Similarly $\mathbb{P}_T\left(\frac{1}{T}\int S; S\right) = \mathbb{C}'_T\left(\frac{1}{T}\int S', 1\right)$

Forward-start options

Payoff of a call: $(S_T - S_t)^+$

Payoff of a put: $(S_t - S_T)^+$

Assume $H' = (H'_t)_{0 \leq t \leq T}$ satisfies

$$\mathrm{Law}(H'_T - H'_{(T-t)-}; 0 \leq t < T | P') = \mathrm{Law}(H'_t; 0 \leq t < T | P')$$

then

$$\mathbb{C}_{t,T}(S; S) = \mathbb{P}'_{T-t}(1; S')$$

and

$$\mathbb{P}_{t,T}(S; S) = \mathbb{C}'_{T-t}(S'; 1)$$

Value of a *forward-start* call

$\rightarrow$ value of a *plain vanilla* put

Semimartingale semantics for a multidimensional PIIAC

$L = (L_t)_{0 \le t \le T}$ is a *process* with *independent increments* and *absolutely continuous* characteristics (PIIAC). The *canonical representation* is

$$L_t = \int_0^t b_s \mathrm{d}s + \int_0^t c_s^{1/2} \mathrm{d}W_s + \int_0^t \int_{\mathbb{R}^d} h(x) \mathrm{d}(\mu^L - \nu) + \int_0^t \int_{\mathbb{R}^d} (x - h(x)) \mathrm{d}\mu^L,$$

where $W = (W_t)_{0 \le t \le T}$ is a P-standard Brownian motion on $\mathbb{R}^d$. The *triplet of predictable characteristics* of L with respect to P, $\mathbb{T}(L|P) = (B, C, \nu)$, is

$$B_t = \int_0^t b_s \mathrm{d}s, \quad C_t = \int_0^t c_s \mathrm{d}s, \quad \nu([0,t] \times A) = \int_0^t \int_A \lambda_s(\mathrm{d}x)\mathrm{d}s,$$

where $A \in \mathcal{B}(\mathbb{R}^d)$; it also determines the *law* of L. The triplet (b, c, λ) is called the *triplet of differentiable characteristics* of L.

Integrability assumptions

Assumption ($\mathbb{AC}$)

The triplets (b_t, c_t, λ_t) *satisfy*

$$\int_0^T \left(|b_t| + \|c_t\| + \int_{\mathbb{R}^d} (1 \wedge |x|^2) \lambda_t(\mathrm{d}x) \right) \mathrm{d}t < \infty.$$

Assumption ($\mathbb{EM}$)

There exists a constant $M > 1$, such that the Lévy measures λ_t satisfy

$$\int_0^T \int_{\|x\|>1} \exp\langle u, x\rangle \lambda_t(\mathrm{d}x)\mathrm{d}t < \infty, \qquad \forall u \in [-M, M]^d.$$

Moreover, without loss of generality, we assume $\int_{\|x\|>1} \exp\langle u, x\rangle \lambda_t(\mathrm{d}x) < \infty$ for all $t \in [0, T]$ and $u \in [-M, M]^d$.

Multiasset price model

Each component S^i of the vector of asset price processes $S = (S^1, \ldots, S^d)^\top$ is an exponential time-inhomogeneous Lévy process

$$S^i_t = S^i_0 \exp L^i_t, \qquad 0 \le t \le T.$$

The driving process $L = (L_t)_{0 \le t \le T}$ is an $\mathbb{R}^d$-valued time-inhomogeneous Lévy process that satisfies Assumption ($\mathbb{EM}$), with canonical decomposition

$$L_t = \int_0^t b_s \, ds + \int_0^t c_s^{1/2} \mathrm{d}W_s + \int_0^t \int_{\mathbb{R}^d} x(\mu^L - \nu)(\mathrm{d}s, \mathrm{d}x).$$

Using Assumption ($\mathbb{EM}$) we get

$$S^i \in \mathcal{M}_{\text{loc}}(P) \Leftrightarrow \langle \mathbf{e}_i, B \rangle + \frac{1}{2} \langle \mathbf{e}_i, C\mathbf{e}_i \rangle + \left(\mathrm{e}^{\langle \mathbf{e}_i, x \rangle} - 1 - \langle \mathbf{e}_i, x \rangle \right) * \nu = 0.$$

Using that every exponential time-inhomogeneous Lévy process that is a local martingale is indeed a martingale, we conclude that, for all $i \in \{1, \dots, d\}$

$$S^i = \mathrm{e}^{L^i} \in \mathcal{M}(P) \Leftrightarrow B^i + \frac{1}{2} C^{ii} + \left(\mathrm{e}^{x^i} - 1 - x^i \right) * \nu = 0$$

Multiasset price model (2)

Theorem

Let $L = (L_t)_{0 \le t \le T}$ be an $\mathbb{R}^d$-valued PIIAC that satisfies Assumption $(\mathbb{EM})$, with characteristics $\mathbb{T}(L|P) = (B, C, \nu)$. Let u, v be vectors in $\mathbb{R}^d$ such that $v \in (-M, M)^d$ and $u + v \in [-M, M]^d$. Define the measure P'

$$\frac{\mathrm{d}P'}{\mathrm{d}P} = \frac{\mathrm{e}^{\langle v, L_T \rangle}}{E[\mathrm{e}^{\langle v, L_T \rangle}]}.$$

Then, the process $L^u = (L^u_t)_{0 \le t \le T}$, where $L^u_t := \langle u, L_t \rangle$, is a 1-dimensional PIIAC with characteristics $\mathbb{T}(L^u|P') = (B^u, C^u, \nu^u)$ with

$$\begin{aligned} b^u_s &= \langle u, b_s \rangle + \langle u, c_s v \rangle + \int_{\mathbb{R}^d} \langle u, x \rangle \big(\mathrm{e}^{\langle v, x \rangle} - 1\big) \lambda_s(\mathrm{d}x) \\ c^u_s &= \langle u, c_s u \rangle \\ \lambda^u_s(E) &= \lambda'_s(\{x \in \mathbb{R}^d : \langle u, x \rangle \in E\}), \qquad E \in \mathcal{B}(\mathbb{R}), \end{aligned}$$

where λ'_s is a measure defined by

$$\lambda'_s(A) = \int_A \mathrm{e}^{\langle v, x \rangle} \lambda_s(\mathrm{d}x), \qquad A \in \mathcal{B}(\mathbb{R}^d).$$

Application: Multiasset options

Example: Swap option (Margrabe)

Theorem

We can relate the value of a swap, with payoff $(S^1_T - S^2_T)^+$, *and a plain vanilla option via the following duality:*

$$\mathbb{M}\big(S^1_0, S^2_0; C, \nu\big) = S^1_0\, \mathbb{P}\big(1, S^2_0/S^1_0; C', \nu'\big)$$

where the characteristics (C', ν') *are given in the previous Theorem for* $v = (1, 0)^\top$ *and* $u = (-1, 1)^\top$.

Proof: Using asset S^1 to form the Radon–Nikodym derivative

$$\mathbb{M} = E\left[\left(S^1_T - S^2_T\right)^+\right] = S^1_0\, E\left[\frac{S^1_T}{S^1_0}\left(1 - \frac{S^2_T}{S^1_T}\right)^+\right]$$

$$= S^1_0\, E\left[\mathrm{e}^{L^1_T}\left(1 - \frac{S^2_T}{S^1_T}\right)^+\right] = S^1_0\, E'\left[\left(1 - \frac{S^2_T}{S^1_T}\right)^+\right],$$

where $v = (1, 0)^\top$. Now, note that

$$\frac{S_t^2}{S_t^1} = \frac{S_0^2}{S_0^1} \frac{\mathrm{e}^{L_t^2}}{\mathrm{e}^{L_t^1}} = \frac{S_0^2}{S_0^1} \mathrm{e}^{\langle u, L_t \rangle}, \qquad 0 \le t \le T$$

where $u = (-1, 1)^\top$ and

$$\mathrm{e}^{\langle u, L \rangle} \in \mathcal{M}(P') \quad \text{since} \quad \mathrm{e}^{\langle u, L \rangle} \mathrm{e}^{\langle v, L \rangle} = \mathrm{e}^{L^2} \in \mathcal{M}(P).$$

Then, we have that

$$\mathbb{M} = S_0^1 E' \left[\left(1 - S_T'\right)^+ \right]$$

where S' is an exponential PIIAC with characteristics C' and ν'. □

Model with interest rates

Asset price processes

$$S_t^i = S_0^i \exp[(r - \delta^i)t + L_t^i]$$

where $L = (L^1, \ldots, L^d)$ is a PIIAC with triplet (B, C, ν)

payoff of a Margrabe option: $(S_T^1 - S_T^2)^+$

value $\mathbb{M}(S_0^1, S_0^2; r, \delta, C, \nu) = \mathrm{e}^{-rT} E[(S_T^1 - S_T^2)^+]$

then

$$\mathbb{M}(S_0^1, S_0^2; r, \delta, C, \nu) = E[S_T^1] e^{C_T} \mathbb{P}(K, S_0^2/S_0^1, \delta^1, r, C', \nu')$$

where $K = e^{-C_T}$ and C_T is a constant.

Basic interest rates

$B(t,T)$: price at time $t \in [0, T]$ of a default-free zero coupon bond with maturity $T \in [0, T^*]$ $\qquad (B(T,T) = 1)$

$f(t,T)$: instantaneous forward rate

$$B(t,T) = \exp\left(-\int_t^T f(t,u)\,\mathrm{d}u\right)$$

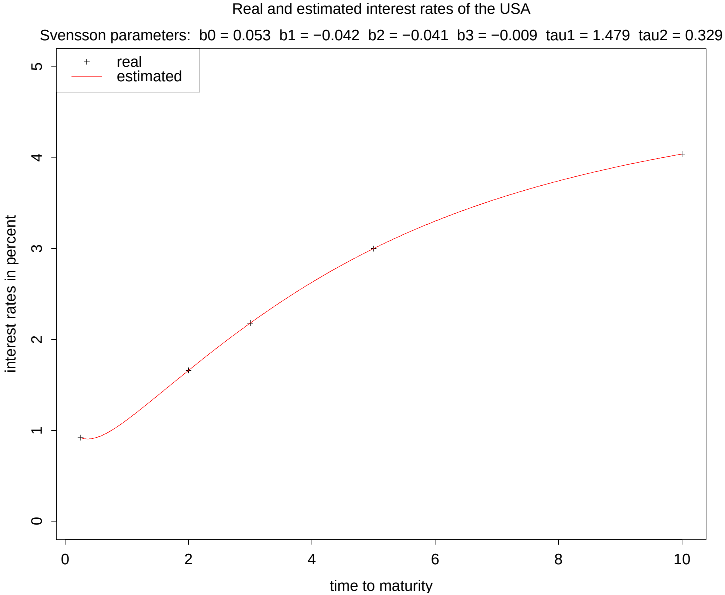
$L(t,T)$: default-free forward Libor rate for the interval T to $T+\delta$ as of time $t \le T$ $\quad$ (δ-forward Libor rate)

$$L(t,T) := \frac{1}{\delta}\left(\frac{B(t,T)}{B(t,T+\delta)} - 1\right)$$

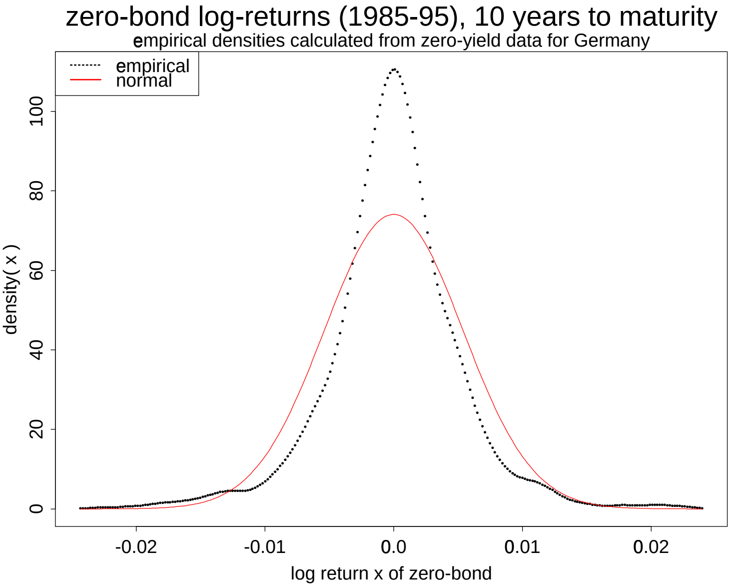
$F_B(t,T,U)$: forward process for the two maturities $T < U$

$$F_B(t,T,U) := \frac{B(t,T)}{B(t,U)}$$

$$\Longrightarrow \quad 1 + \delta L(t,T) = \frac{B(t,T)}{B(t,T+\delta)} = F_B(t,T,T+\delta)$$



Termstructure, February 17, 2004



zero-bond log-returns (1985-95), 10 years to maturity
empirical densities calculated from zero-yield data for Germany
empirical
normal
density(x)
0
20
40
60
80
100
-0.02
-0.01
0.0
0.01
0.02
log return x of zero-bond

The driving process

$L = (L^1, \ldots, L^d)$ is a d-dimensional time-inhomogeneous Lévy process, i.e. L has independent increments and the law of L_t is given by the characteristic function

$$\mathbb{E}[\exp(\mathrm{i}\langle u, L_t\rangle)] = \exp \int_0^t \theta_s(\mathrm{i}u)\,\mathrm{d}s \qquad \text{with}$$

$$\theta_s(z) = \langle z, b_s\rangle + \frac{1}{2}\langle z, c_s z\rangle + \int_{\mathbb{R}^d} \left(e^{\langle z,x\rangle} - 1 - \langle z, x\rangle\right) F_s(\mathrm{d}x),$$

where $b_t \in \mathbb{R}^d$, c_t is a symmetric nonnegative-definite $d \times d$-matrix, and F_t is a Lévy measure.

Integrability: $$\int_0^{T^*} \left(|b_s + ||c_s|| + \int_{\{|x|\le 1\}} |x|^2 F_s(\mathrm{d}x)\right) \mathrm{d}s < \infty$$

$$\int_0^{T^*} \int_{\{|x|>1\}} \exp(ux) F_s(\mathrm{d}x)\,\mathrm{d}s < \infty \quad \text{for } u \le M$$

Description in terms of modern stochastic analysis

$L = (L_t)$ is a special semimartingale with canonical representation

$$L_t = \int_0^t b_s \, \mathrm{d}s + \int_0^t c_s^{1/2} \, \mathrm{d}W_s + \int_0^t \int_{\mathbb{R}^d} x(\mu^L - \nu)(\mathrm{d}s, \mathrm{d}x)$$

and characteristics

$$A_t = \int_0^t b_s \, \mathrm{d}s, \qquad C_t = \int_0^t c_s \, \mathrm{d}s, \qquad \nu(\mathrm{d}s, \mathrm{d}x) = F_s(\mathrm{d}x) \, \mathrm{d}s$$

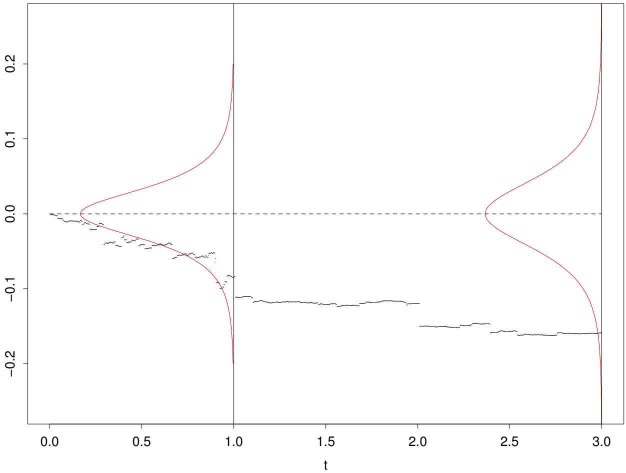
$W = (W_t)$ is a standard d-dimensional Brownian motion ,

μ^L the random measure of jumps of L and ν is the compensator of μ^L

L is also called a process with independent increments and absolutely continuous characteristics (PIIAC).

Simulation of a Lévy process

NIG(10,0,0.100,0) on [0,1]
NIG(10,0,0.025,0) on [1,3]



Dynamics of the forward rates

(Eb–Raible (1999), Eb–Özkan (2003), Eb–Jacod–Raible (2005), Eb–Kluge (2006)

$$\mathrm{d}f(t,T) = \alpha(t,T)\,\mathrm{d}t - \sigma(t,T)\,\mathrm{d}L_t \qquad (0 \le t \le T \le T^*)$$

$\alpha(t,T)$ and $\sigma(t,T)$ satisfy measurability and boundedness conditions and $\alpha(s,T) = \sigma(s,T) = 0$ for $s > T$

Define $\quad A(s,T) = \int_{s\wedge T}^{T} \alpha(s,u)\,\mathrm{d}u \quad$ and $\quad \Sigma(s,T) = \int_{s\wedge T}^{T} \sigma(s,u)\,\mathrm{d}u$

Assume $0 \le \Sigma^i(s,T) \le M \qquad (1 \le i \le d)$

For most purposes we can consider deterministic α and σ

Implications

Savings account and default-free zero coupon bond prices are given by

$$B_t = \frac{1}{B(0,t)} \exp\left(\int_0^t A(s,T)\,\mathrm{d}s - \int_0^t \Sigma(s,t)\,\mathrm{d}L_s \right) \quad \text{and}$$

$$B(t,T) = B(0,T) B_t \exp\left(-\int_0^t A(s,T)\,\mathrm{d}s + \int_0^t \Sigma(s,T)\,\mathrm{d}L_s \right).$$

If we choose $A(s,T) = \theta_s(\Sigma(s,T))$, then bond prices, discounted by the savings account, are martingales.

In case $d = 1$, the martingale measure is unique (see Eberlein, Jacod, and Raible (2005)).

Pricing of European options

$$B(t,T) = B(0,T)\exp\left[\int_0^t (r(s)+\theta(\Sigma(s,T)))\,ds + \int_0^t \Sigma(s,T)dL_s\right]$$

where $r(t) = f(t,t)$ short rate

$V(0,t,T,w)$ time-0-price of a European option with maturity t and payoff $w(B(t,T),K)$

$$V(0,t,T,w) = \mathbb{E}_{\mathbf{P}^*}[B_t^{-1}w(B(t,T),K)]$$

Forward measure associated with date $T \leq T^*$

Density $\dfrac{d\mathbb{P}_T}{d\mathbb{P}^*} = \dfrac{1}{B_T B(0,T)}$ or $\mathbb{E}_{\mathbb{P}^*}\left[\dfrac{d\mathbb{P}_T}{d\mathbb{P}^*} \mid \mathcal{F}_t\right] = \dfrac{B(t,T)}{B_t B(0,T)}$

For the case of the Lévy term structure model this equals

$$\exp\left(\int_0^t \Sigma(s,T)\,\mathrm{d}L_s - \int_0^t A(s,T)\,\mathrm{d}s\right)$$

Compensator of μ^L under $\mathbb{P}_T$: $\nu^T(\mathrm{d}t,\mathrm{d}x) = e^{\langle \Sigma(t,T),x\rangle}\nu(dt,dx)$

Standard Brownian motion under $\mathbb{P}_T$: $W_t^T = W_t - \int_0^t c_s^{1/2}\Sigma(s,T)ds$

Pricing formula for caps

$$w(B(t,T),K) = (B(t,T)-K)^+$$

Call with strike K and maturity t on a bond that matures at T

$$\begin{aligned} C(0,t,T,K) &= \mathbb{E}_{\mathbb{P}^*}[B_t^{-1}(B(t,T)-K)^+] \\ &= B(0,t)\mathbb{E}_{\mathbb{P}_t}[(B(t,T)-K)^+] \end{aligned}$$

Assume $X = \int_0^t (\Sigma(s,T)-\Sigma(s,t))dL_s$ has a Lebesgue density and choose $R < -1$ s.t. $M_t^X(-R) < \infty$, then

$$\begin{aligned} C(0,t,T,K) &= \frac{1}{2\pi} KB(0,t)\exp(R\xi) \\ &\quad \times \int_{-\infty}^{\infty} e^{iu\xi}(R+iu)^{-1}(R+1+iu)^{-1}M_t^X(-R-iu)du \end{aligned}$$

where ξ is a constant
Analogous for the corresponding put and for swaptions

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where ξ is a constant
Analogous for the corresponding put and for swaptions

Duality in the Lévy forward rate model

Denote the value of a call option on a zero coupon bond by

$$V_c\left(B(0,T); \frac{B(0,U)}{B(0,T)}, K; C, \nu\right) = \mathbb{E}\left[\frac{1}{B_T}(B(T,U)-K)^+\right],$$

and similarly for a put option

$$V_p\left(B(0,T); \frac{B(0,U)}{B(0,T)}, K; C, \nu\right) = \mathbb{E}\left[\frac{1}{B_T}(K-B(T,U))^+\right].$$

Theorem

Assume that bond prices are modeled according to the Lévy forward rate model. Then, the value of a call and a put option on a bond are related via:

$$V_c\left(B(0,T); \frac{B(0,U)}{B(0,T)}, K; C, \nu\right) = V_p\left(B(0,T); K, \frac{B(0,U)}{B(0,T)}; C, -f\nu\right)$$

where $f(s,x) = \exp\left((\Sigma(s,U)+\Sigma(s,T))x\right)$*.*

Idea of proof: Define the constant $D := \mathbb{E}\left[\frac{B(T,U)}{(B_T)^2}\right]$ and the measure $\widetilde{\mathbb{P}}$ via

$$\frac{\mathrm{d}\widetilde{\mathbb{P}}}{\mathrm{d}\mathbb{P}} := \frac{B(T,U)}{D\,(B_T)^2} = \eta_T.$$

$\mathbb{P} \sim \widetilde{\mathbb{P}}$ and the density process $(\eta_t)_{t\in[0,T]}$ is $\eta_t = \mathbb{E}\left[\frac{B(T,U)}{D(B_T)^2}\middle|\mathcal{F}_t\right]$. Using Girsanov's theorem for semimartingales we deduce the $\widetilde{\mathbb{P}}$-characteristics of the driving process L. Now,

$$\begin{aligned} V_c &= \mathbb{E}\left[\frac{1}{B_T}(B(T,U)-K)^+\right] \\ &= \mathbb{E}\left[\frac{B(T,U)}{D\,(B_T)^2} KDB_T(K^{-1} - B(T,U)^{-1})^+\right] \end{aligned}$$

and changing measure from $\mathbb{P}$ to $\widetilde{\mathbb{P}}$, we get that

$$V_c = \widetilde{\mathbb{E}}\left[KDB_T(K^{-1} - B(T,U)^{-1})^+\right].$$

This can be re-written as

$$V_c = \widetilde{\mathbb{E}}\left[\frac{1}{\widehat{B}_T}\left(\widehat{K} - \widehat{B}(T,U)\right)^+\right],$$

for $(\widehat{B}_T)^{-1} := \frac{B(0,T)}{B(0,U)} DB_T$, $\widehat{K} := \frac{B(0,U)}{B(0,T)}$ and $\widehat{B}(T,U) := K\frac{B(0,U)}{B(0,T)} B(T,U)^{-1}$.

Showing that $\widehat{B}_T$ and $\widehat{B}(T,U)$ have dynamics analogous to that of B_T and $B(T,U)$ concludes the proof. □

Equivalent formulation

time-T_i payoff of a caplet: $N\delta(L(T_i,T_i)-K)^+$

Recall $1+\delta L(T_i,T_i)=\frac{B(T_i,T_i)}{B(T_i,T_{i+1})}$

$$\begin{aligned}\delta(L(T_i,T_i)-K)^+ &= (1+\delta L(T_i,T_i)-(1+\delta K)) \\ &= \left(\frac{1}{B(T_i,T_{i+1})}-\mathcal{K}\right)\end{aligned}$$

time-T_i value of this payoff

$$B(T_i,T_{i+1})\left(\frac{1}{B(T_i,T_{i+1})}-\mathcal{K}\right)^+=\mathcal{K}\left(\frac{1}{\mathcal{K}}-B(T_i,T_{i+1})\right)^+$$

$\rightarrow$ payoff of a put option on a bond with strike $\dfrac{1}{1+\delta K}$

Analogously for a floorlet.

Equivalent formulation

Value of a *floorlet* with strike K maturing at time T_i that settles in arrears at time T_{i+1}

$$\mathbb{FL}(L(0,T_i),K;C,\nu) = \mathbb{E}\left[\frac{1}{B_{T_{i+1}}}\delta(K - L(T_i,T_i))^+\right]$$
$$= (1+\delta K)\mathbb{E}\left[\frac{1}{B_{T_i}}(B(T_i,T_{i+1}) - \mathcal{K})^+\right]$$

where $L(0,T_i) = \frac{1}{\delta}\left(\frac{B(0,T_i)}{B(0,T_{i+1})} - 1\right)$ initial forward Libor rate

Therefore

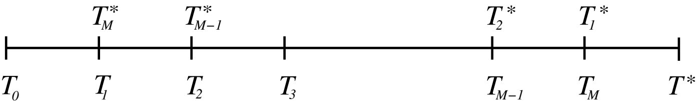
$$\mathbb{FL}(L(0,T_i),K;C,\nu) = C\,\mathbb{CL}(K,L(0,T_i);C,-f\nu)$$

where $C = \dfrac{1+\delta K}{1+\delta L(0,T_i)}$

The Lévy Libor model

(Eb–Özkan (2005))

Tenor structure $T_0 < T_1 < \cdots < T_M < T_{M+1} = T^*$
with $T_{i+1} - T_i = \delta$, set $T_i^* = T^* - i\delta$ for $i = 1, \ldots, M$



Assumptions

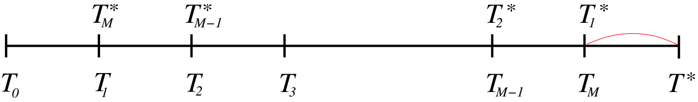
(LR.1): For any maturity T_i there is a bounded deterministic function $\lambda(\cdot, T_i)$, which represents the volatility of the forward Libor rate process $L(\cdot, T_i)$.

(LR.2): We assume a strictly decreasing and strictly positive initial term structure $B(0, T)$ $(T \in]0, T^*])$. Consequently the initial term structure of forward Libor rates is given by

$$L(0, T) = \frac{1}{\delta} \left(\frac{B(0, T)}{B(0, T + \delta)} - 1 \right)$$

Backward Induction (1)

Given a stochastic basis $(\Omega, \mathcal{F}_{T^*}, \mathbb{P}_{T^*}, (\mathcal{F}_t)_{0 \le t \le T^*})$



We postulate that under $\mathbb{P}_{T^*}$

$$L(t, T_1^*) = L(0, T_1^*) \exp\left(\int_0^t \lambda(s, T_1^*) dL_s^{T^*} \right)$$

where $$L_t^{T^*} = \int_0^t b_s ds + \int_0^t c_s^{1/2} dW_s^{T^*} + \int_0^t \int_{\mathbb{R}} x(\mu^L - \nu^{T^*,L})(ds, dx)$$

is a non-homogeneous Lévy process with random measure of jumps μ^L and $\mathbb{P}_{T^*}$-compensator $\nu^{T^*,L}(ds, dx) = F_s(dx)ds$

Backward Induction (2)

In order to make $L(t, T_1^*)$ a $\mathbb{P}_{T^*}$-martingale, choose the drift characteristic (b_s) s.t.

$$\int_0^t \lambda(s, T_1^*) b_s ds = -\frac{1}{2}\int_0^t c_s \lambda^2(s, T_1^*) ds - \int_0^t \int_{\mathbb{R}} \left(e^{\lambda(s, T_1^*)x} - 1 - \lambda(s, T_1^*)x \right) \nu^{T^*,L}(ds, dx)$$

Transform $L(t, T_1^*)$ in a stochastic exponential

$$L(t, T_1^*) = L(0, T_1^*)\mathcal{E}(H(t, T_1^*))$$

where

$$H(t, T_1^*) = \int_0^t \lambda(s, T_1^*) c_s^{1/2} dW_s^{T^*} + \int_0^t \int_{\mathbb{R}} \left(e^{\lambda(s, T_1^*)x} - 1 \right) (\mu^L - \nu^{T^*,L})(ds, dx)$$

Backward Induction (3)

Equivalently

$$dL(t, T_1^*) = L(t-, T_1^*)\Big(\lambda(t, T_1^*)c_t^{1/2}dW_t^{T^*} + \int_{\mathbb{R}} \left(e^{\lambda(t,T_1^*)x} - 1\right)(\mu^L - \nu^{T^*,L})(dt, dx)\Big)$$

with initial condition

$$L(0, T_1^*) = \frac{1}{\delta}\left(\frac{B(0, T_1^*)}{B(0, T^*)} - 1\right)$$

Backward Induction (4)

Recall $F_B(t, T_1^*, T^*) = 1 + \delta L(t, T_1^*)$, therefore,

$$\begin{aligned} dF_B(t, T_1^*, T^*) &= \delta dL(t, T_1^*) \\ &= F_B(t-, T_1^*, T^*)\Bigg(\underbrace{\frac{\delta L(t-, T_1^*)}{1 + \delta L(t-, T_1^*)}\lambda(t, T_1^*)}_{=\alpha(t, T_1^*, T^*)} c_t^{1/2} dW_t^{T^*} \\ &\quad + \int_{\mathbb{R}} \underbrace{\frac{\delta L(t-, T_1^*)}{1 + \delta L(t-, T_1^*)}\left(e^{\lambda(t, T_1^*)x} - 1\right)}_{=\beta(t, x, T_1^*, T^*) - 1} (\mu^L - \nu^{T^*, L})(dt, dx)\Bigg) \end{aligned}$$

Define the forward martingale measure associated with T_1^*

$$\frac{\mathrm{d}\mathbb{P}_{T_1^*}}{\mathrm{d}\mathbb{P}_{T^*}} = \mathcal{E}_{T_1^*}(M^1) \qquad \text{where}$$

$$\begin{aligned} M_t^1 &= \int_0^t \alpha(s, T_1^*, T^*) c_s^{1/2} dW_s^{T^*} \\ &\quad + \int_0^t \int_{\mathbb{R}} (\beta(s, x, T_1^*, T^*) - 1)\, (\mu^L - \nu^{T^*, L})(ds, dx) \end{aligned}$$

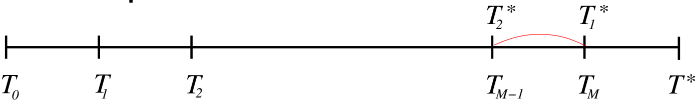
Backward Induction (5)

Then $W_t^{T_1^*} = W_t^{T^*} - \int_0^t \alpha(s, T_1^*, T^*) c_s^{1/2} ds$

is the forward Brownian motion for date T_1^* and

$\nu^{T_1^*, L}(dt, dx) = \beta(t, x, T_1^*, T^*)\nu^{T^*, L}(dt, dx)$ is the $\mathbb{P}_{T_1^*}$-compensator for μ^L.

Second step



We postulate that under $\mathbb{P}_{T_1^*}$

$$L(t, T_2^*) = L(0, T_2^*) \exp\left(\int_0^t \lambda(s, T_2^*) dL_s^{T_1^*}\right) \quad \text{where}$$

$$L_t^{T_1^*} = \int_0^t b_s^{T_1^*} ds + \int_0^t c_s^{1/2} dW_s^{T_1^*} + \int_0^t \int_{\mathbb{R}} x(\mu^L - \nu^{T_1^*, L})(ds, dx)$$

Backward Induction (6)

Second measure change

$$\frac{d\mathbb{P}_{T_2^*}}{d\mathbb{P}_{T_1^*}} = \mathcal{E}_{T_2^*}(M^2)$$

where

$$M_t^2 = \int_0^t \alpha(s, T_2^*, T_1^*) c_s^{1/2} dW_s^{T_1^*} + \int_0^t \int_{\mathbb{R}} \left(\beta(s, x, T_2^*, T_1^*) - 1\right)(\mu^L - \nu^{T_1^*, L})(ds, dx)$$

This way we get for each time point T_j^* in the tenor structure a Libor rate process which is under the forward martingale measure $\mathbb{P}_{T_{j-1}^*}$ of the form

$$L(t, T_j^*) = L(0, T_j^*) \exp\left(\int_0^t \lambda(s, T_j^*) dL_s^{T_{j-1}^*}\right)$$

Forward process model (1)

Postulate

$$1 + \delta L(t, T_1^*) = (1 + \delta L(0, T_1^*)) \exp\left(\int_0^t \lambda(s, T_1^*) dL_s^{T^*} \right)$$

equivalently

$$F_B(t, T_1^*, T^*) = F_B(0, T_1^*, T^*) \exp\left(\int_0^t \lambda(s, T_1^*) dL_s^{T^*} \right)$$

In differential form

$$\begin{aligned} \mathrm{d}F_B(t, T_1^*, T^*) \;=\; & F_B(t-, T_1^*, T^*)\Big(\lambda(t, T_1^*) c_t^{1/2} dW_t^{T^*} \\ & + \int_{\mathbb{R}} \Big(\mathrm{e}^{\lambda(t, T_1^*)x} - 1\Big)(\mu^L - \nu^{T^*,L})(dt, dx)\Big) \end{aligned}$$

Forward process model (2)

Define the forward martingale measure associated with T_1^*

$$\frac{d\mathbb{P}_{T_1^*}}{d\mathbb{P}_{T^*}} = \mathcal{E}_{T_1^*}(\widetilde{M}^1)$$

where

$$\widetilde{M}_t^1 = \int_0^t \lambda(s, T_1^*) c_s^{1/2} dW_s^{T^*} + \int_0^t \int_{\mathbb{R}} \left(\mathrm{e}^{\lambda(s,T_1^*)x} - 1\right)(\mu^L - \nu^{T^*,L})(ds, dx).$$

Forward process model (3)

Then $W_t^{T_1^*} = W_t^{T^*} - \int_0^t \lambda(s, T_1^*) c_s^{1/2} ds$ is the forward Brownian motion for date T_1^* and

$\nu^{T_1^*,L}(dt, dx) = \exp(\lambda(t, T_1^*)x)\, \nu^{T^*,L}(dt, dx)$ is the $\mathbb{P}_{T_1^*}$-compensator of μ^L.

Continuing this way we get for each time point T_j^* in the tenor structure a Libor rate process under $\mathbb{P}_{T_{j-1}^*}$ in the form

$$1 + \delta L(t, T_j^*) = (1 + \delta L(0, T_j^*)) \exp\left(\int_0^t \lambda(s, T_j^*) dL_s^{T_{j-1}^*} \right).$$

with successive compensators

$$\nu^{T_j^*,L}(dt, dx) = \exp\left(\sum_{i=1}^{j} \lambda(t, T_i^*) x \right) F_t(dx) dt.$$

Consequence of this alternative approach: negative Libor rates can occur

Duality in the Lévy Libor model

Value of a *caplet* with strike K maturing at time T_i that settles in arrears at time T_{i+1}

$$\mathbb{CL}(L(0,T_i),K;C,\nu^{T_{i+1}}) = B(0,T_{i+1})\mathbb{E}_{\mathbb{P}_{T_{i+1}}}[\delta(L(T_i,T_i)-K)^+]$$

Duality result

$$\mathbb{CL}(L(0,T_i),K;C,\nu^{T_{i+1}}) = \mathbb{FL}(K,L(0,T_i);C,-f\nu^{T_{i+1}})$$

where $f(s,x) = \exp(\lambda(s,T_i)x)$

Duality in the Lévy forward process model

Value of a call option on the forward process with strike K which is settled in arrears at time T_{i+1}

$$\mathbb{C}(F(0,T_i,T_{i+1}),K;C,\nu^{T_{i+1}}) = B(0,T_{i+1})\mathbb{E}_{\mathbb{P}_{T_{i+1}}}[(F(T_i,T_i,T_{i+1})-K)^+]$$

Duality for call and put options on the forward process

$$\mathbb{C}(F(0,T_i,T_{i+1}),K;C,\nu^{T_{i+1}}) = \mathbb{P}(K,F(0,T_i,T_{i+1});C,-f\nu^{T_{i+1}})$$

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