Advances in Mathematics of Finance,
II General AMAMEF Conference

Bedlewo (Poland), April 30 - May 5, 2007

PENSION FUNDS WITH A MINIMUM GUARANTEE: A STOCHASTIC CONTROL MODEL

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PLAN OF THE TALK

1. The problem
2. The literature
3. The model
4. Dynamic programming approach and HJB equation
5. Optimal feedback strategies
6. Explicit solution in a special case
7. Future work

1 - THE PROBLEM

- The object: a pension fund with **defined contributions and a minimum guarantee**.

- Adopt the viewpoint of the management of the fund: determine an **optimal dynamic portfolio allocation** θ for the fund.

- The model is formulated as an **optimal stochastic control problem** where the agent maximizes the intertemporal utility from the fund's wealth X over an **infinite horizon**.

- The problem **is not a standard optimal portfolio problem** as the pension funds have various particular features (presence of **contributions and benefits, solvency constraints, constraints on the portfolio strategy**).

2 - THE LITERATURE

Some papers on defined contribution pension fund without minimum guarantee:

(2000) Gerber H.U., and Shiu E.S.W., Investing for retirement: optimal capital growth and dynamic asset allocation, *North American Actuarial Journal*, **4**, 1, 42–62

(2000) Cairns A.J G. , Blake D., and Dowd K., Optimal dynamic asset allocation for defined-contribution pension plans, *Proceedings of the 10th AFIR International Colloquium*, 131–154

(2001) Vigna E., and Haberman S., Optimal investment strategy for defined contribution pension schemes, *Insurance: Mathematics and Economics*, **28**, 233–262

(2002) Haberman S., and Vigna E., Optimal investment strategies and risk measures in defined contribution pension schemes, *Insurance: Mathematics and Economics*, **31**, 35–69

(2004) Battocchio P., and Menoncin F., Optimal pension management in a stochastic framework, *Insurance: Mathematics and Economics*, **34**, 79–95

Defined contribution pension fund with minimum guarantee:

(2001) Boulier J.F., Huang S.J., and Taillard G., Optimal management under stochastic interest rates: the case of a protected pension fund, *Insurance: Mathematics and Economics*, **28**, 173-179.

(2003) Deelstra G., Grasselli M., and Koehl P.F., Optimal investment strategies in the presence of a minimum guarantee, *Insurance: Mathematics and Economics*, **33**, 189–207.

(2003) Sbaraglia, S.; Papi, M.; Briani, M.; Bernaschi, M.; Gozzi, F., A model for the optimal asset-liability management for insurance companies, *Int. J. Theor. Appl. Finance* , **6**, No. 3, 277-299.

In particular [Boulier et al., 2001] and [Deelstra et al., 2003] study the optimal management over the accumulation phase:

- in a complete financial market
- in a continuous and finite time horizon
- assuming as terminal date the time of retirement of a representative agent (i.e. single cohort)
- by considering the guarantee as a contingent claim
- by applying a martingale and duality approach
- by using the CRRA utility function

They find explicit solutions by maximizing the expected utility function of the terminal wealth under the constraint that the terminal wealth must exceed the minimum guarantee

Moreover [Boulier et al., 2001] consider

- the contribution flow is a deterministic process
- the guarantee has a very specific form
- the Vasiček model for the term structure of interest rates

On the contrary [Deelstra et al., 2003] assume that

- the contribution flow is a stochastic process but generated by the market (since the market is complete)
- the guarantee is a general process
- the interest rates follow the affine dynamics in the one-dimensional version, which include as a special case the CIR model and the Vasiček model

[Sbaraglia et al., 2003]: discrete time model that describes the ALM of a real fund of an italian insurance company (INA).

Optimal portfolio problem in presence of

- contributions and benefits,
- constraints on the portfolio strategy,
- solvency level.

This has been the departure point of the present work.

3 - THE MODEL

SECURITY MARKET

- The security market is assumed to be complete, frictionless, arbitrage free, continuously open and default free

- The security market is composed of two kinds of assets: a riskless asset and a risky asset

- Randomness is described by a one-dimensional standard Brownian motion $B(t)$, $t \geq 0$, defined on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t^B\}_{t \geq 0}, \mathbb{P})$

- The interest rate is constant: this restriction is done for simplicity to focus on the other characteristics of the model.

Hypothesis 1 *The price of the riskless asset $S^0(t)$, evolves according to the equation*

$$\frac{dS^0(t)}{S^0(t)} = rdt, \qquad S^0(0) = 1,$$

where $r \geq 0$ is the instantaneous spot rate of return

Hypothesis 2 *The price of risky asset $S^1(t)$ follows an Itô process and satisfies the equation*

$$\frac{dS^1(t)}{S^1(t)} = (r + \sigma\lambda)dt + \sigma dB(t),$$

where

- B *is a standard Wiener process*
- $\sigma > 0$ *is the instantaneous rate of volatility*
- λ *is the instantaneous risk premium. We assume $\lambda > 0$.*

DYNAMICS OF WEALTH

Let $X(t)$, $t \geq 0$, be the process that describes the amount of pension fund wealth (**State variable**).

Let $\theta(t)$, $t \geq 0$, be the $\{\mathcal{F}_t^B\}_{t\geq 0}$-adapted process which represents the proportion of wealth invested in the risky asset (**Control variable**).

The fund starts at $t = 0$ but we may look at it when it is already working. So we are given initial data $t_0 \geq 0$, $x_0 \geq 0$ and we assume that the wealth process satisfies the equation:

$$\begin{cases} dX(t) = \Big\{ \left[\theta(t)\sigma\lambda + r\right] X(t) + c(t) - b(t) \Big\} dt + \theta(t)\sigma X(t) dB(t), \quad t \geq t_0 \\ X(t_0) = x_0 \end{cases}$$

This the standard wealth equation with the extra terms given by the flow of **contributions** $c(\cdot)$ and **benefits** $b(\cdot)$.

CONTRIBUTIONS AND BENEFITS

As a first step we assume **demographic stationarity**

Hypothesis 3 *The flow of aggregate contributions is given by:*

$$c(t) := \frac{t \wedge T}{T} \alpha \, N \, w, \quad 0 < \alpha < 1, \quad \forall t \geq 0,$$

where

- α *is the average contribution rate;*
- T *is the average time spent in the fund by members;*
- $N \in \mathbb{N}$ *is the average number of fund members after* T*.*
- $w > 0$ *is the average per capita wage bill earned by the fund members (see* [Boulier et al., 2001]*)*

Hypothesis 4 *The flow of aggregate benefits is given by:*

$$b(t) := \begin{cases} 0 & \textit{if} \quad 0 \leq t < T \\ g(t) + s\left(t, X\left(\cdot\right)\big|_{[t-T,t]}\right) & \textit{if} \quad t \geq T \end{cases}$$

where

- $g(\cdot)$ *is the flow of minimum guarantee;*
- $s(\cdot, \cdot)$ *is the 'surplus' function. At time* $t \geq T$ *it depends on the fund wealth level in the time period* $[t - T, t]$.

The minimum guarantee

Hypothesis 5 *The minimum guarantee is, for $t \geq T$:*

$$g(t) := \int_{t-T}^{t} \overline{c}(t) e^{\delta(t-u)} du, \quad \eta \geq 0,$$

where $\delta \geq 0$ is the instantaneous guaranteed rate of return and $\overline{c}(t)$ is flow of contributions of new members per unit of time. By demographic stationarity $\overline{c}(t) = \frac{1}{T}\alpha N w$.

It follows:

$$g(t) = \alpha N w \, \frac{e^{\delta T} - 1}{\delta T} > \alpha N w, \quad \forall t \geq T$$

The surplus

Taking a nonzero surplus function the state equation becomes a **delay differential equation** and the related problem becomes very complex as it requires techniques of **stochastic control in infinite dimension** (*work in progress with S. Federico: see the poster session*).

To begin the study of the problem we consider first the following less realistic case

Hypothesis 6 *No surplus is provided. So* $b(t) = g(t)$.

Remark 1 For $t \geq T$ the pension fund is in a demographic stationary regime: in this case the equation for the wealth is autonomous.

We study this case so we assume that we look at the fund starting from the time $t_0 \geq T$. The equation for the wealth becomes

$$\begin{cases} dX(t) = \Big\{ \left[\theta(t)\sigma\lambda + r\right] X(t) - A \Big\} dt + \theta(t)\sigma X(t) dB(t), \quad t \geq t_0 \\ X(t_0) = x_0 \end{cases}$$

where

$$A = \alpha N w \left[\frac{e^{\delta T} - 1}{\delta T} - 1 \right] > 0$$

is the balance between benefits and contributions flow. ■

SOLVENCY CONSTRAINTS

Let $X(t)$, $t \geq 0$, be the process that describes the amount of pension fund wealth.

Hypothesis 7 *The process X is subject to the following constraint:*

$$X(t) \geq l(t) \quad \mathbb{P} - a.s., \qquad \forall t \geq 0,$$

where the non negative function l represents the solvency constraint

Remark 2 The previous hypothesis avoids "improper" behavior of the fund manager. ∎

We assume that the solvency level is given by a startup level l_0 plus a "share" of the due minimum guarantee in a unit of time.

$$l(t) = l_0 + \zeta \int_{(t-T)\wedge 0}^{t} \bar{c}(u) e^{\delta(t-u)} du \qquad\qquad t \geq 0,$$

where $l_0 \geq 0$ and $0 \leq \zeta \leq T$.

Note that for $t \geq T$ $l(t)$ is constant and

$$l(t) = l(T) = \zeta \alpha N w \; \frac{e^{\delta T} - 1}{\delta T}$$

Another possible choice (treatable in our setting with some more work) is that $l(t) = l_0$ plus a share of the contributions of active workers, evaluated at the rate of return of minimum guarantee δ.

In this case we would put $c(u)$ instead of $\bar{c}(u)$ in the above integral.

MAXIMIZING THE OBJECTIVE

We want to maximize the objective

$$J(t_0, x_0; \theta(\cdot)) = \mathbb{E}\left[\int_{t_0}^{+\infty} e^{-\rho t} U(X(t; t_0, x_0, \theta))\, dt\right] \text{ s.t. } X(t_0) = x_0 \geq l(T) =: l_T$$

where

- the discount rate is $\rho > 0$;
- $U : [l_T, +\infty) \longrightarrow \mathbb{R} \cup \{-\infty\}$ is strictly increasing, strictly concave, belongs to $C^2((0, +\infty))$ and satisfies Inada's conditions: $\lim_{y \to 0^+} DU(y) = +\infty$ and $\lim_{y \to +\infty} DU(y) = 0$.

The set of admissible strategies is

$$\Theta_{ad}(t_0, x_0) := \Big\{\theta : [l_T, +\infty) \times \Omega \longrightarrow [0,1] \text{ adapted to } \{\mathcal{F}_t^B\}_{t\geq t_0} \text{ s.t. } X(t; t_0, x_0, \theta) \in [l_T, +\infty),\ t \geq t_0\Big\}$$

This set is nonempty for every $x_0 \geq l_T$ **if and only if**

$$r l_T \geq A.$$

We will assume this from now on.

4 - DP APPROACH AND HJB EQUATION

We define the value function:

$$V(t,x) := \sup_{\theta(\cdot)\in\Theta_{ad}(t,x)} J\left(t,x;\theta\left(\cdot\right)\right)$$

Since the problem is autonomous for $t \geq T$ then

$$V(t,x) = e^{-\rho(t-T)}V(T,x) \qquad \forall t \geq T,\ x \geq l_T.$$

Then it is enough to study the function of one variable $V_T\left(x\right) := V\left(T,x\right)$ defined for $x \geq l_T$.

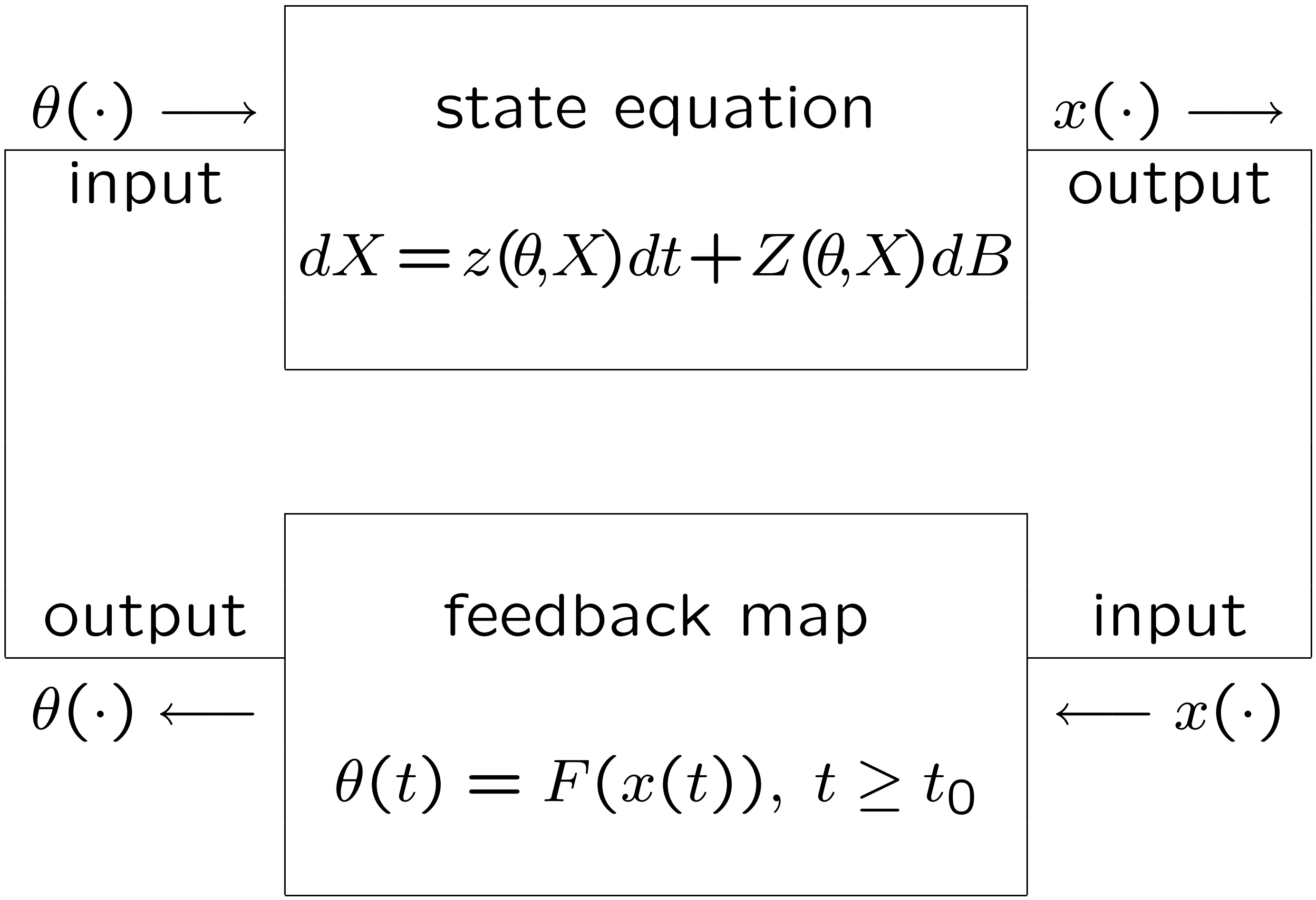
Dynamic Programming

A - The value function V satisfies the DP Equation

$$V_T(x) = \sup_{\theta(\cdot)\in\Theta_{ad}(T,x)} \mathbb{E}\Bigg[\int_T^\tau e^{-\rho t} U\left(X(t;T,x,\theta)\right) dt + \\ + e^{-\rho(\tau - T)} V_T\left(X(\tau;T,x,\theta)\right)\Bigg], \quad x \in [l_T, +\infty), \quad \forall \tau \in (T, +\infty)$$

B - Derive from it the Hamilton-Jacobi-Bellman (HJB) equation and show that the value function is a classical solution of this equation.

C - Apply a verification theorem to get the optimal strategies in feedback form.


state equation
$dX = z(\theta, X)\,dt + Z(\theta, X)\,dB$
$\theta(\cdot) \longrightarrow$
input
$x(\cdot) \longrightarrow$
output
feedback map
$\theta(t) = F(x(t)),\ t \geq t_0$
output
$\theta(\cdot) \longleftarrow$
input
$\longleftarrow x(\cdot)$

The HJB equation associated with our stochastic control problem is given by:

$$\rho v(x) - H\Big(x, Dv(x), D^2v(x)\Big) = 0, \; \forall x \in [l_T, +\infty),$$

where

$$H\Big(x, Dv(x), D^2v(x)\Big) := \sup_{\theta\in[0,1]} H_{cv}\Big(x, Dv(x), D^2v(x); \theta\Big)$$

$$= \sup_{\theta\in[0,1]} \Big\{U(x) + \Big[(\theta\sigma\lambda + r)x - A\Big]Dv(x) + \frac{1}{2}\theta^2\sigma^2x^2D^2v(x)\Big\}$$

where $A = \alpha N w \left[\frac{e^{\delta T}-1}{\delta T} - 1\right] > 0$.

THE PROPERTIES OF THE VALUE FUNCTION

We provide first conditions for the finiteness of V and show that it is

- concave,
- strictly increasing,
- continuous on the interval $(l_T, +\infty)$ (also in l_T if it is finite in l_T).

Then, studying the HJB equation we prove that

- V is the unique concave viscosity solution of the HJB equation,
- V belongs to $\mathsf{C}\Big([l_T, +\infty); \mathbb{R}\Big) \cap \mathsf{C}^2\Big((l_T, +\infty); \mathbb{R}\Big)$.

We can then determine the optimal control policies in feedback form

5 - OPTIMAL FEEDBACK STRATEGIES

We take now the case $rl > A$ while in the case $rl = A$ we give an example with explicit solution.

In this case the boundary point $x = l_T$ is not an absorbing point. So one can reach it and then going back into the interior.

THEOREM When $rl > A$ there exists a **unique optimal strategy** given by the feedback map

$$G(x) := G_0\left(x, DV(x), DV^2(x)\right), \quad x \geq l_T,$$

where

$$\begin{aligned} G_0(x, DV(x), DV^2(x)) &= \arg\max_{\theta\in[0,1]} H_{cv}\left(x, DV(x), D^2V(x); \theta\right) \\ &= \min\left\{1, \left|\frac{\lambda}{\sigma} \frac{DV(x)}{xD^2V(x)}\right|\right\} \end{aligned}$$

with the agreement that $G(l_T) = 0$.

As in Merton's model G_0 is explicitly linked to the payoff of every unit of risk $\frac{\lambda}{\sigma}$, and to elasticity of DV w.r.t. the wealth i.e. $-\frac{DV}{xD^2V}$

6 - EXPLICIT SOLUTION IN AN EXAMPLE

Let us consider the following CRRA utility function

$$U(y) = \frac{y^\gamma}{\gamma}, \qquad \gamma \in (-\infty, 0) \cup (0, 1)$$

where $y := x - \frac{A}{r}$ and $\frac{A}{r}$ is the present value of the whole balance between future benefit and contribution flows

Given suitable constraints on the solvency function l, our HJB equation is solved by

$$V(x) = \frac{\left(x - \frac{A}{r}\right)^\gamma}{\gamma\left[\rho - \gamma\left(r + \frac{\lambda^2}{2(1-\gamma)}\right)\right]}, \qquad \rho - \gamma\left(r + \frac{\gamma\lambda^2}{2(1-\gamma)}\right) > 0$$

The optimal feedback map becomes

$$G_0(x) = \min\left\{1, \frac{\lambda}{\sigma\,(1-\gamma)\,x}\left(x - \frac{A}{r}\right)\right\}$$

7 - FUTURE TARGETS

Research project with

Salvatore Federico (Scuola Normale, Pisa, Italy),

Ben Goldys (UNSW, Sydney, Australia).

- To take into account a surplus function s not zero (see poster of S.F.)

- To take stochastic interest rates
- To release the hypotheses of demographic stationarity
- To introduce a stochastic wage