Pension funds with a minimum guarantee: a stochastic control approach

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Abstract

In this paper we propose and study a continuous time stochastic model of optimal allocation for a defined contribution pension fund with a minimum guarantee. Usually, portfolio selection models for pension funds maximize the expected utility from final wealth over a finite horizon (the retirement time), whereas our target is to maximize the expected utility from current wealth over an infinite horizon since we adopt the point of view of the fund manager.

In our model the dynamics of wealth takes directly into account the flows of contributions and benefits and the level of wealth is constrained to stay above a "solvency level". The fund manager can invest in a riskless asset and in a risky asset but borrowing and short selling are prohibited.

We concentrate the analysis on the effect of the solvency constraint, analyzing in particular what happens when the fund wealth reaches the allowed minimum value represented by the solvency level.

The model is naturally formulated as an optimal stochastic control problem and is treated by the dynamic programming approach. We show that the value function of the problem is a regular solution of the associated Hamilton-Jacobi-Bellman equation. Then we apply verification techniques to get the optimal allocation strategy in feedback form and to study its properties. We finally give a special example with explicit solution.

Contents

Keywords: Defined contribution pension fund, minimum guarantee, stochastic optimal control, dynamic programming, Hamilton-Jacobi-Bellman equation, viscosity solutions.

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1 Introduction

This paper deals with the management of a defined contribution[1] pension fund with a minimum guarantee. In particular we want to find the optimal portfolio strategy assuming that the manager can invest in two assets (a risky and a riskless one, in a standard Black and Scholes market) and maximizes an intertemporal utility function depending on the current wealth over an infinite horizon.

Our problem is similar to optimal portfolio selection problems but it has some special features due to the nature and the social target of the pension funds: in particular the presence of contributions and benefits, the presence of constraints on the investment strategies, the presence of solvency constraints (see Section 2 for further explanations).

Differently from previous approaches, to our knowledge, our asset allocation model takes the point of view of the pension fund manager instead of the one of the representative worker[2]. Moreover, we require that the wealth of the running pension fund remains above a prescribed level (that we call "solvency level") at any time. We analyze the effect of such constraint on the admissible and on the optimal strategies: in particular we show that, for sufficiently high

[1] Traditionally, most of pension plans created in the past were based on defined benefits, but the situation has changed in recent years and actually we are going towards a rapid development of defined contributions pension plans. For this reason only recently the literature is oriented to models concerning defined contribution pension schemes.

[2] In the literature one can find models that adopt the point of view of the fund manager and also take account of all main features of the problem, but, due to their complexity, they are not studied from the theoretical point of view and stochastic optimal control techniques have not been applied to them. See e.g. [Sbaraglia et al, 2003] and the references therein.

solvency level, the optimal portfolio strategies do not become trivial (i.e. the fund manager can still reinvest in the risky asset), even after that the solvency level has been reached (see later in this introduction and also Section 5 for explanations).

The model is formulated as a stochastic optimal control problem where the control variable is the proportion of wealth to invest in the risky asset. The main goal is to study the properties of optimal allocation strategies: this is achieved applying the dynamic programming approach (for a general reference on stochastic control see [Fleming & Soner, 1993] and [Yong & Zhou, 1999]).

The first analysis on optimal selection portfolio in a continuous time model and using the dynamic programming approach was faced by R. C. Merton. Maximizing expected utility from consumption and from final wealth, he proved in [Merton, 1969] that explicit solutions exist if the individual utility function belongs to the CRRA (*constant relative risk aversion*) family, and in [Merton, 1971] if it belongs to the HARA (*hyperbolic absolute risk aversion*) family.

Stochastic optimization approaches to defined contribution plans with the constraint that the wealth must not be inferior to a minimum guarantee at a terminal date (so-called European guarantee) have been introduced in [Boulier, Huang & Taillard, 2001] and [Deelstra, Grasselli & Koehl, 2003]. In these models they assume that the terminal date corresponds to the retirement of a representative worker and they apply the traditional Merton approach maximizing the total expected discounted utility from final wealth exceeding the promised guarantee. More recently in [El Karoui, Jeanblanc & Lacoste, 2005] the authors solve an optimal allocation problem for an investor which maximizes utility from final wealth but is constrained to stay above the guarantee at every intermediate date (so-called American guarantee).

In addition, those models concerning a single representative participant to the pension fund (see, e.g., [Cairns, Blake & Dowd, 2000], [Boulier, Huang & Taillard, 2001], [Vigna & Haberman, 2001], [Haberman & Vigna, 2002], [Deelstra, Grasselli & Koehl, 2003], and [Battocchio & Menoncin, 2004]) do not take into account the dynamical evolution of contributions and benefits which are related with new workers that adhere to the pension fund and those that have accrued the right of pension. In our model this evolution is naturally incorporated in the model.

Since this is a first step in this direction, we decided to introduce some simplifying hypotheses on other features that, for now, we are not interested to focus on. These features are the study of the demographic risk and the evolution of the interest rate, as we are going to explain below.

Concerning the first point, we leave the inclusion of demographic risk to future research. One reason for this restriction is that the demographic risk can be diversified independently from the financial risk, e.g., by a reinsurance, and therefore can be neglected in determining the optimal portfolio policy.

About the second point, in our framework the interest rate is assumed to be constant. This is a restriction with respect to other works on the same subject as [Boulier, Huang & Taillard, 2001] and [Deelstra, Grasselli & Koehl, 2003] in the case of a pension fund with a minimum guarantee, and [Cairns, Blake & Dowd, 2000], [Vigna & Haberman, 2001, Haberman & Vigna, 2002] and [Battocchio & Menoncin, 2004] in absence of a minimum guarantee. We aim to incorporate a stochastic interest rate model in a further work.

From the mathematical point of view our problem is a stochastic optimal control problem with constraints on the control and on the state (deriving for the presence of investment and solvency constraints: see Section 3 for the precise statement). Differently, to our knowledge, from other papers on optimal portfolio problems (see, e.g., [Karatzas, Lehoczky, Sethi & Shreve, 1986], [Sethi & Taksar, 1992], [Sethi, Taksar & Presman 1992], [Zariphopoulou, 1994], [Cadenillas & Sethi, 1997], [Choulli, Taksar & Zhou, 2003] and [El Karoui, Jeanblanc & Lacoste, 2005]), within our model the boundary of the state space is not always an absorbing barrier: the optimal strategies can touch the boundary and come back in the interior keeping the same state dynamics.[3] This important modeling issue involves some nontrivial technical problems in the study of optimal strategies (see Section 4, subsections 4.3, 4.4).

To avoid technical complications we concentrate the analysis on a running pension fund which has entered in a stationary regime (see Section 3, Hypothesis 3.3) so also the solvency level becomes a constant l.

The core of the dynamic programming approach is the study of the associated Hamilton-Jacobi-Bellman equation and its relationship with the control problem which is performed in Section 4. We follow the path used in other papers on stochastic control problems of similar kind:

- We first prove some basic properties of the value function (finiteness, concavity, monotonicity, continuity: see Subsection 4.1);
- We then prove that the value function V is a viscosity solution of Hamilton-Jacobi-Bellman equation (Theorem 4.13);
- Furthermore we prove that $V \in C^2$ in the interior of the state space, so it is a classical solution of the Hamilton-Jacobi-Bellman equation there (Theorem 4.16);
- Finally we apply such results to prove a verification theorem that gives the existence and uniqueness of the optimal feedback map (Theorem 4.24).

Due to the specific feature of our problem (and this mainly comes from the nature of state constraints) we cannot apply to it other results given in the literature. So we carefully prove all the results we need. The results on finiteness, concavity, monotonicity, continuity are proved adapting well known arguments from previous papers, sometimes with nontrivial arrangements. The proof of Theorem 4.13 uses a straightforward adaptation of standard techniques. The proof of Theorem 4.16 uses some ideas of [Choulli, Taksar & Zhou, 2003] with a substantial change due to the fact that in such paper a uniformly elliptic equation is studied while our equation is not uniformly elliptic.

The verification theorem deserves a deeper explanation. To avoid that the set of admissible strategies is empty, we prove in Section 3.1 that the inequality $rl \geq A$ must be satisfied, where r is the interest rate (so rl is the return from the constrained solvency level) and A is the balance between contribution and benefit rate. We assume this along the paper. However the case of $rl = A$ is qualitatively different from the case of $rl > A$, especially concerning the behavior of optimal strategies near l. This is also strictly related to the structure of the instantaneous

[3] To be precise also in [Sethi & Taksar, 1992] the state process can come back in the interior after touching the boundary: however this is obtained taking different state dynamics when the boundary is reached, so using a completely different setting.

utility function U near l. We develop the analysis of the optimal strategies in two cases which seem to us more relevant from the economic point of view[4]:

- $rl = A$ and $U'(l^+) = +\infty$. The situation is more similar to what happens in standard portfolio problems: starting the wealth at l, the only admissible and so optimal policy is to keep it forever in the riskless asset (trivial strategy) so the wealth remains at the constant level l. Starting at a wealth grater than l, the optimal wealth path always remains strictly above l. Since this case is more standard we have chosen to study it only in a special case where the value function (and so the optimal feedback map) can be found explicitly (see Subsection 4.4)

- $rl > A$, and both $U(l)$ and $U'(l^+)$ are finite. Starting the wealth at l, there are many admissible strategies and along the optimal one the fund manager can reinvest in the risky asset. Starting at a wealth grater than l, the boundary l is reached with positive probability, and after touching l the manager can still reinvest in the risky asset. In this case the value function cannot be given explicitly even in the case of power utility (see Remark 4.25) and the verification theorem is non trivial. In Section 4.3 the verification theorem (Theorem 4.24) is proved using an *ad hoc* approximation procedure and sharp results on the behavior of the value function at l (Corollary 4.21).

The work is organized in 6 sections as follows.

In Section 2 we describe the model separating the presentation in five subsections to improve its readability.

In Section 3 we set up the stochastic control problem and (in Subsection 3.1) we discuss the structure of the set of admissible strategies depending on the solvency level l.

In Section 4 we develop the dynamic programming approach proving various results about the properties of the value function (Subsection 4.1), the fact that it is a regular solution of the Hamilton-Jacobi-Bellman equation (Subsection 4.2), the verification theorem, and the optimal feedback policies for $rl > A$ (Subsection 4.3). In Subsection 4.4 we consider the case $rl = A$ and solve analytically the Hamilton-Jacobi-Bellman equation for a specific utility function finding the value function and the optimal policies.

In Section 5 we give an analysis of the optimal policies underlying the role of the solvency level.

Section 6 concludes the paper.

2 The model

We consider a continuous-time model where the financial market is competitive,[5] frictionless, viable, default free and continuously open. The fund manager maximizes the intertemporal

[4] When $rl = A$ the solvency level is the minimum possible to guarantee that admissible strategies exist so it makes sense that utility from wealth rapidly increase at $x = l$ since the solvency constraint coincides with a kind of "no bankruptcy" constraint. On the other hand if $rl > A$ the solvency level is not the minimum possible and is strictly higher than the "no bankruptcy" constraint, so it is less stringent to have rapidly increasing utility at $x = l$ and also it makes sense to have finite utility at l.

[5] This hypothesis imply that the investor is price taker. This is usual in literature regarding financial management models of pension funds and it is realistic if the agent does not invest a big amount of money. As a matter of fact, the volume of assets exchanged by pension funds is such that they could affect the price of assets (i.e. investor may be price maker) but we do not deal with this fact here.

expected utility from the fund wealth over an infinite horizon and faces the following trading constraints: borrowing and short positions are not allowed and the pension fund wealth must be greater of a suitable positive value at each time. We call this value (possibly time dependent) *solvency level*.[6] This constraint is imposed so that the pension fund always affords to pay at least a given fraction of the due pension, and in particular to avoid the bankruptcy.

As we said in the introduction we concentrate the analysis on the financial issues, so we assume that the population of fund members is a stationary open collectivity: there can be new entries, nevertheless there will be no changes during the time in the quantity as well as in the distribution per age class.

The time horizon of our model is infinite and independent from the work life of the fund members since we adopt the point of view of the manager of a pension fund that always operates, leaving out the date of retirement of the single participant.[7]

2.1 Dynamics of wealth

To set up the mathematical model we consider a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $\{\mathcal{F}_t^B\}_{t\geq 0}$, where $t \geq 0$ is the time variable. The space Ω is the set of all possible states of nature, the set $\mathcal{F}$ is a σ-algebra and $\mathbb{P}$ is a probability measure (the observed one), defined on the measurable space $(\Omega,\ \mathcal{F})$. The filtration $\{\mathcal{F}_t^B\}_{t\geq 0}$, describing the information structure, is generated by the trajectories of a one-dimensional standard Brownian motion $B(t)$, $t \geq 0$, defined on the same probability space and completed with the addition of the null measure sets of $\mathcal{F}$. Moreover we assume that $\mathcal{F}_{+\infty}^B = \mathcal{F}$. Sometimes we will use a starting point $s \geq 0$. In this case $\{\mathcal{F}_t^B\}_{t\geq s}$ will be the complete filtration generated by $B^s(t) = B(t) - B(s)$.

The financial market is composed of two kinds of assets: a riskless asset and a risky asset.

Hypothesis 2.1 *The price of the riskless asset $S^0(t)$, $t \geq 0$, evolves according to the equation*

$$dS^0(t) = rS^0(t)dt,$$

where $r > 0$ is the instantaneous spot rate of return.

Hypothesis 2.2 *The price of risky asset $S^1(t)$, $t \geq 0$, follows an Itô process and satisfies the equation*

$$dS^1(t) = \mu S^1(t)dt + \sigma S^1(t)dB(t),$$

where $\mu > 0$ is the instantaneous rate of expected return and $\sigma > 0$ is the instantaneous rate of volatility.

The drift μ verifies the relation $\mu = r + \sigma\lambda$, where λ is the instantaneous risk premium of the market, i.e. the price that the market assigns to the randomness expressed by the standard Brownian motion B. We assume that $\lambda > 0$, so $\mu > r$.

[6] In order to avoid misunderstandings, it seems relevant to precise that with this concept we do not refer to the so called *solvency margin*, i.e. the shareholders equity of an insurance company whose method of calculation is normally imposed by the supervisory authority of a given country, but to a meaning which is fully explained in Subsection 2.4.

[7] This fact, together with the demographic stationarity hypothesis, drives us to neglect the so called *lifestyle strategy*, treated, e.g., in [Vigna & Haberman, 2001, Haberman & Vigna, 2002], that suggests to invest the whole fund wealth in risky assets when the worker is young and to gradually switch into riskless assets when the date of retirement approaches. Also in [Battocchio & Menoncin, 2004] is found an optimal decision policy coherent with the lifestyle strategy.

The state variable, represented by $X(t)$, $t \geq 0$, is the $\{\mathcal{F}_t^B\}_{t\geq 0}$-adapted process that gives the amount of the pension fund wealth at any time. We suppose that the pension fund starts its activity at the date $t = 0$ and that at this time it owns a starting amount of wealth $x_0 \geq 0$.

The control variable, denoted by $\theta(t)$, $t \geq 0$, is the $\{\mathcal{F}_t^B\}_{t\geq 0}$-adapted process that represents the proportion of fund wealth to invest in the risky asset (so $\theta(t) \in [0,1]$ for every t due to the borrowing and short selling constraints). Therefore the dynamics of wealth is expressed, formally, by the following state equation

$$\begin{cases} dX(t) = \dfrac{\theta(t)X(t)}{S^1(t)} dS^1(t) + \dfrac{[1-\theta(t)]\,X(t)}{S^0(t)}\, dS^0(t) + c(t)dt - b(t)dt, \quad t \geq 0, \\ X(0) = x_0 \geq 0, \end{cases} \tag{1}$$

where $\frac{\theta(t)X(t)}{S^1(t)}$ and $\frac{[1-\theta(t)]X(t)}{S^0(t)}$ are the quantities in portfolio of risky and riskless asset, respectively; while the non-negative integrable function $c(t)$, $t \geq 0$, indicates the flow of contributions and the non-negative function $b(t)$, $t \geq 0$, represents the flow of benefits.

As we said in the introduction, we assume that a *solvency constraint* must be respected (see Subsection 2.4 for further explanations).

Hypothesis 2.3 *The process X describing the fund wealth is subject to the following constraint*

$$X(t) \geq l(t) \quad \mathbb{P}\text{-}a.s., \qquad \forall t \geq 0, \tag{2}$$

where the positive function $l(t)$, $t \geq 0$ represents the solvency level.

By standard arguments the state equation (1) can be rewritten in the following way

$$\begin{cases} dX(t) = \left\{ [\theta(t)\sigma\lambda + r]\, X(t) + c(t) - b(t) \right\} dt + \theta(t)\sigma X(t) dB(t), \quad t \geq 0, \\ X(0) = x_0 \geq l_0 \geq 0, \end{cases} \tag{3}$$

with the constraint that $X(t) \geq l(t)$ $\mathbb{P}$-a.s., for any $t \geq 0$, and $l_0 = l(0)$.

2.2 Contributions

In the population stationarity hypothesis, the flow of contributions $c(\cdot)$ can be considered exogenous. We assume that the workers who enter in the pension fund are a homogeneous class[8] and their flow is constant in time. Moreover we suppose that each participant adheres for a length of time represented by an exogenous constant $T > 0$. We observe that T is not necessarily the time of retirement.[9] Similarly there is a fixed number N of fund members after time T.[10] The flow of new members per unit of time is $\frac{N}{T}$.

Hypothesis 2.4 *The payment of aggregate contributions occurs at any time according to the following relation*

$$c(t) = \begin{cases} \dfrac{t}{T}\alpha N w(t) & \textit{if} \quad 0 \leq t \leq T, \\ \alpha N w(t) & \textit{if} \quad t > T, \end{cases}$$

[8] We mean a class of people that have the same characteristics (same age at the entry date, same professional qualification, same level of skill, and so on).

[9] We may think to T as the average time that the members spend in the fund, taking account also of the fraction of those that decide to transfer their positions to another pension fund before their retirement.

[10] We may say that each unit of time (e.g., year) $\frac{N}{T}$ new members enter in the fund and that (after time T) exactly $\frac{N}{T}$ members exit of it.

where $\alpha \in (0,1)$ represents the average contribution rate and $w(t) \geq 0$, $t \geq 0$, the average per capita wage rate of the fund members. We will take $w(\cdot)$ equal to a constant $w > 0$ for simplicity. The flow of contributions of new members per unit of time is constant and we call it $\bar{c}(\cdot) \equiv \frac{1}{T}\alpha N w$.

The above hypothesis is a bit restrictive because the stochastic wage is an important and additional source of uncertainty for the fund manager. We observe that the introduction of an extra source of risk renders the market incomplete, as discussed and studied in [Cairns, Blake & Dowd, 2000] in absence of guarantee and in a continuous and finite time horizon. We leave the investigation of the full problem to future research. Here we consider the constant w as a real wage, i.e. the nominal wage discounted from a constant inflation rate; therefore we assume a point of view essentially in line with [Boulier, Huang & Taillard, 2001] where the (nominal) wage is a deterministic function of the time and continuously increasing at a constant inflation rate.

Notice also that, with some technical complications, the part of our model concerning the contribution could be extended to the frameworks of [Deelstra, Grasselli & Koehl, 2003] and, in absence of minimum guarantee, of [Battocchio & Menoncin, 2004], where the flow of contributions is stochastic but the corresponding sources of risk are hedgeable.

2.3 Benefits

In our financial setting is natural to consider the minimum guarantee rate of return regulated by the market (and so fixed exogenously); for an optimal design see [Deelstra, Grasselli & Koehl, 2004]. Consequently the aggregate benefit function $b(\cdot)$ is exogenous if it depends on the guarantee, while it is endogenous if it also depends on the wealth level. In this work, as a first step, we assume that the payment of the surplus exceeding the guarantee is not provided. Moreover in demographic stationarity the benefits are paid only after time $T > 0$. Then we will work with the following assumption.

Hypothesis 2.5 *The aggregate benefits are paid at any time according to the following relation*

$$b(t) = \begin{cases} 0 & \text{if} \quad 0 \leq t < T, \\ g(t) & \text{if} \quad t \geq T, \end{cases}$$

where $g(\cdot)$ is the aggregate flow of money at any time that the pension fund pays to its fund members after retirement as a minimum guarantee.

Due to the above hypothesis the guarantee assumes a particular importance for the fund risk management. The demographic stationarity hypothesis drives us to write it as follows.

Hypothesis 2.6 *The flow of minimum guarantee is given by*

$$g(t) = \int_{t-T}^{t} \bar{c}(u)\, e^{\delta(t-u)} du = \int_{t-T}^{t} \frac{1}{T}\alpha N w e^{\delta(t-u)} du, \qquad t \geq T,$$

where $\delta > 0$ is the guaranteed rate of return.

Having assumed that the riskless interest rate r is constant, we are clearly forced to assume the following.

Hypothesis 2.7 *We have*

$$r \geq \delta.$$

The previous inequality could be justified thinking to the fact that often the participants to the pension fund do not have time nor access to the financial market as the fund manager. Moreover, sometimes workers are forced by law to adhere to a pension fund, so that they are constrained to accept a rate of return less than the rate paid by the riskless asset. Finally, we recall that in the actual market, but it is not the case of our framework that has neither transaction nor information costs, the fund manager can usually get higher interest rate than the fund members.

Hypotheses 2.4 and 2.6 imply that the benefits paid from the pension fund at any time are

$$b(t) = \begin{cases} 0 & \text{if} \quad 0 \leq t < T, \\ \alpha N w \dfrac{e^{\delta T} - 1}{\delta T} > \alpha N w & \text{if} \quad t \geq T. \end{cases}$$

This means that, for $t > T$, we always have $b(t) > c(t)$ as expected. In fact, if the guaranteed rate of return is positive and the stationary demographic hypothesis holds, then the aggregate flow of money paid as guarantee must be greater than the aggregate flow of contributions. Therefore in the presence of minimum guarantee and stationary demographic hypothesis, current contributions do not allow to pay current benefits. Nevertheless we will see (Remark 3.10) that thanks to the fact that $r \geq \delta$ the fund manager can always pay the benefits.

Remark 2.8 *A more realistic form of the benefits (that is part of our current work) is*

$$b\left(t, X(\cdot)|_{[t-T,t]}\right) = \begin{cases} 0 & \textit{if} \quad 0 \leq t < T, \\ g(t) + S\left(t, X(\cdot)|_{[t-T,t]}\right) & \textit{if} \quad t \geq T, \end{cases} \tag{4}$$

where $S(\cdot,\cdot)$ that we define surplus*, is a function depending on the time t and on the fund level within the interval of time $[t-T, t]$.*

With this form of the benefits the equation for the wealth process X becomes a stochastic delay differential equation that can be treated with the tool of stochastic optimal control in infinite dimension (see, e.g., [Goldys & Gozzi, 2006, Gozzi & Marinelli, 2006, Fuhrman & Tessitore, 2004]).

In this new framework, if we assume in (4) *that the minimum guarantee g is identically zero, the model is fit to describe a simple defined contribution pension fund.* ■

2.4 Solvency level

A solvency level may be imposed by law or by a supervisory authority to avoid improper behavior of the fund manager and to guarantee that she/he is able to pay at least part of the due benefits at each time $t \geq 0$. Without imposing this constraint the fund manager is allowed to use strategies that may bring her/him to mismatches with the social target of the pension fund.

In our case we assume that:

- at the beginning the pension fund should hold a given minimum startup level $l_0 \geq 0$;
- in the first period, i.e. for all $t \in [0, T)$, when benefits are still not paid (due to our demographic hypothesis) the solvency level is l_0 plus a fraction of the contribution paid up to time t and compounded with rate δ;

- at any time $t \geq T$ the solvency level is l_0 plus a fraction of the contributions paid in the interval $[t-T,t]$ and compounded with rate δ;

So we have the following.

Hypothesis 2.9 *The solvency level imposed in* (2) *is represented by the function*

$$l(t) = l_0 + \zeta \int_{(t-T)\wedge 0}^{t} \bar{c}\,(u)\, e^{\delta(t-u)} du = l_0 + \zeta \int_{(t-T)\wedge 0}^{t} \frac{1}{T} \alpha N w e^{\delta(t-u)} du, \quad t \geq 0, \tag{5}$$

where $\zeta \in [0,1]$.[11]

Straightly from (5), we get

$$l(t) = \begin{cases} l_0 + \zeta \alpha N w \dfrac{e^{\delta t} - 1}{\delta T} & \text{if} \quad 0 \leq t < T, \\ l_0 + \zeta \alpha N w \dfrac{e^{\delta T} - 1}{\delta T} & \text{if} \quad t \geq T. \end{cases}$$

Remark 2.10 *The choice of the parameters* l_0 *and* ζ *is very important: if they are too low the fund manager could not be able to pay benefits, if they are too high the constraint on the choice of the asset allocation becomes too binding. See on this also Remark 3.11.* ■

2.5 Expected utility

We assume that the intertemporal preferences of the fund manager are represented by a time-additively separable utility function J. This function is the expected discounted integral, with a subjective discount rate $\rho > 0$, of a given instantaneous utility function U depending on the current wealth X of the fund

$$J = \mathbb{E} \int_0^{+\infty} e^{-\rho t} U\left(X\left(t\right)\right) dt.$$

Clearly more general types of utility functionals J may be used (more general form of U, risk sensitive functionals, and so on) but this would go beyond the scope of this paper. Here we stress the dependence of the utility of the manager from the fund wealth which is the key variable for this problem.

3 The stochastic control problem

Now we formulate and study our problem as a stochastic optimal control problem. First of all we observe that the initial time $t = 0$ has been chosen as the first time of operation of the fund. However it also makes sense to look to a pension fund that is already running after a given amount of time $s > 0$ so to establish an optimal decision policy from s on. For this reason we set an initial time $s \geq 0$, a given amount of wealth x at time s, and consider the following

[11] If we set $\zeta = 1$ then the pension fund can always pay the current minimum guarantee (and so, by Hypothesis 2.5, the current benefits) to the members that have just reached the right to receive their pension. The case of $\zeta > 1$ corresponds to a solvency level higher than the current benefits that the fund manager must guarantee. We consider this setting as financially meaningless and therefore we exclude it from our discussion.

equation for the dynamics of the wealth (in accordance with (3) and with the hypotheses just stated)

$$\begin{cases} dX(t) = \{[\theta(t)\sigma\lambda + r]\, X(t) + c\,(t) - b\,(t)\}\, dt + \theta(t)\sigma X(t) dB^s(t), \quad t \geq s, \\ X\,(s) = x \geq l\,(s)\,, \end{cases}$$

subject to the state constraint $X(t) \geq l(t)$ $\mathbb{P}$-a.s., for any $t \geq s$. The function $c\,(\cdot)$ is given by Hypothesis 2.4, $b\,(\cdot)$ is given by Hypotheses 2.5, and 2.6 and $l\,(\cdot)$ is given by Hypothesis 2.9.

The control strategy $\theta(\cdot)$ is an $\{\mathcal{F}_t^B\}_{t\geq s}$-adapted process with values in $[0, 1]$. Control strategies could also be considered in weak form as explained in [Yong & Zhou, 1999, Chapter 2, p. 64] but we will not do it here.

Remark 3.1 *The state equation, for any $\{\mathcal{F}_t^B\}_{t\geq s}$-adapted process θ, has a unique strong solution on the filtered probability space $(\Omega,\ \mathcal{F},\ \{\mathcal{F}_t^B\}_{t\geq s},\ \mathbb{P})$ (see, e.g., Problem 6.15 in [Karatzas & Shreve, 1991], pp. 360–361). We denote its value at time t by $X\,(t; s, x, \theta)$. We have*

$$X\,(t; s, x, \theta) = Z\,(t) \left[x + \int_s^t \frac{c\,(u) - b\,(u)}{Z\,(u)} du \right],$$

where

$$Z\,(t) = \exp \left(\int_s^t [\theta(u)\sigma\lambda + r]\, du + \int_s^t \theta(u)\sigma dB\,(u) - \frac{1}{2} \int_s^t \theta^2(u)\sigma^2 du \right).$$

■

The set of admissible control strategies for initial time $s \geq 0$ and initial point x, is then

$$\Theta_{ad}(s, x) = \left\{ \theta : [s, +\infty) \times \Omega \longrightarrow [0, 1] \text{ adapted to } \{\mathcal{F}_t^B\}_{t\geq s} \mid X\,(t; s, x, \theta) \in [l\,(t)\,, +\infty),\ t \geq s \right\}.$$

Remark 3.2 *The set $\Theta_{ad}(s, x)$ could be empty due to the presence of the state constraint. Later (Lemma 3.9) we will show conditions that guarantee the nonemptiness of $\Theta_{ad}(s, x)$.* ■

Note that assuming $s \geq T$ means that the fund is entered in a stationary demographic regime which gives an autonomous state equation. The case of $s < T$ can be studied within our model but a deep analysis of it would introduce some more technical difficulties. Then we concentrate our study on a running pension fund assuming the following.

Hypothesis 3.3 *$s \geq T$.*

In this case the state equation becomes

$$\begin{cases} dX(t) = \{[\theta(t)\sigma\lambda + r]\, X(t) - A\}\, dt + \theta(t)\sigma X(t) dB(t), \quad t \geq s, \\ X\,(s) = x \geq l, \end{cases} \tag{6}$$

where we set

$$A = \alpha N w \left(\frac{e^{\delta T} - 1}{\delta T} - 1 \right) > 0 \quad \text{and} \quad l = l\,(T) = l_0 + \zeta \alpha N w \frac{e^{\delta T} - 1}{\delta T} = l_0 + \zeta\,(A + \alpha N w)\,.$$

The total expected discounted utility coming from wealth is given as follows

$$J\,(s, x; \theta\,(\cdot)) = \mathbb{E} \int_s^{+\infty} e^{-\rho t} U\,(X\,(t; s, x, \theta))\, dt,$$

where fund manager's utility function U satisfies the following.

Hypothesis 3.4 *(i)* $U:(l,+\infty)\to\mathbb{R}$ *belong to class* $C^2((l,+\infty);\mathbb{R})$ *and* $U'>0$, $U''<0$.

(ii) For given $C>0$ *and* $\beta\in[0,1)$ *we have* $U(x)\le C(1+x)^\beta$, *where*

$$\rho>\beta r+\frac{\lambda^2}{2}\cdot\frac{\beta}{1-\beta}. \tag{7}$$

Remark 3.5 *Let us give some comment on the above Hypothesis 3.4.*

- *First of all recall that* $l(s)=l$ *for* $s\ge T$. *So the utility function is defined where the wealth process* $X(\cdot)$ *must live (apart from the extremum* l *where* U *may take value* $-\infty$*). If we take* $s<T$ *then* U *need to be defined also for some* $x<l$ *(maybe depending on time).*
- *We do not assume that Inada like conditions (i.e.* $\lim_{x\to l^+}U'(x)=+\infty$ *and* $\lim_{x\to+\infty}U'(x)=0$*) hold for the utility function* U. *We will do it only in the special case of Subsection 4.4.*
- *All utility functions of the form* $U(x)=\frac{(x-x_0)^\gamma}{\gamma}$, *for* $x_0\le l$ *and* $\gamma\in(-\infty,0)\cup(0,1)$ *always satisfy Hypothesis 3.4-(i). In the case when* $x_0=l$ *they also satisfy the Inada like conditions mentioned above. This fact will be used later in Subsection 4.4 to give examples.*
- *Hypothesis 3.4-(ii) guarantees the finiteness of the value function, as is proved in Proposition 4.4. A more general condition could be given on the line of what is done, in a different case, in [Karatzas, Lehoczky, Sethi & Shreve, 1986, Section 2]. As we will see in Subsection 4.4 in the case of* $U(x)=\frac{(x-l)^\gamma}{\gamma}$, *condition* (7) *may be sharp and even cases with* $\rho\le0$ *may be treated (when* $\gamma<0$*).* ■

To solve our stochastic control problem we have to find the optimal pair (X^*,θ^*), i.e. the solution of the following problem.

$$\textit{maximize } J(s,x;\theta(\cdot)) \qquad \text{over } \theta(\cdot)\in\Theta_{ad}(s,x). \tag{8}$$

Let us give the following.

Definition 3.6 *An admissible control strategy* $\theta^*(\cdot)\in\Theta_{ad}(s,x)$ *is called optimal for* x *if*

$$+\infty>J(s,x;\theta^*(\cdot))>-\infty$$

and

$$J(s,x;\theta^*(\cdot))\ge J(s,x;\theta(\cdot)),\qquad\forall\theta(\cdot)\in\Theta_{ad}(s,x).$$

The corresponding state trajectory $X^*(\cdot)=X(\cdot;s,x,\theta^*)$ *is the optimal trajectory and the pair* (X^*,θ^*) *is known as optimal pair.*

Definition 3.7 *The control* $\theta_\epsilon(\cdot)$ *is called* ϵ*-optimal for the initial condition* (s,x) *if it is admissible and*

$$J(s,x;\theta_\epsilon(\cdot))>\sup_{\theta(\cdot)\in\Theta(s,x)}J(s,x;\theta_\epsilon(\cdot))-\epsilon.$$

Remark 3.8 *Our problem could also be formulated with a finite time horizon* T_1 *as follows.*

$$\textit{Maximize } \mathbb{E}\left[\int_s^{T_1} e^{-\rho t} U\left(X\left(t;s,x,\theta\right)\right)dt + e^{-\rho T_1} U_{T_1}\left(X\left(T_1;s,x,\theta\right)\right)\right] \quad \textit{over } \theta\left(\cdot\right)\in\Theta_{ad}\left(s,x\right), \tag{9}$$

where U_{T_1} *is another fund manager's utility function satisfying Hypothesis 3.4-(ii). If we take* $s=0$ *and* $T_1=T$ *then* (9) *reminds the problem faced by [Boulier, Huang & Taillard, 2001] and by [Deelstra, Grasselli & Koehl, 2003] but with some different features: constant interest rate, constraints over the strategies, and solvency constraint.* ■

3.1 The set of admissible strategies

The set of admissible controls for initial time $s\geq T$ and initial point $x\geq l$, is

$$\Theta_{ad}\left(s,x\right)=\left\{\theta:[s,+\infty)\times\Omega\longrightarrow[0,1] \text{ adapted to } \{\mathcal{F}_t^B\}_{t\geq s} \mid X\left(t;s,x,\theta\right)\in[l,+\infty),\ t\geq s\right\}.$$

This set is independent of s in the following sense. Let $s>T$. Given any adapted control strategy $\theta:[T,+\infty)\times\Omega\rightarrow[0,1]$, there exists a process $\psi:[T,+\infty)\times C\left([T,+\infty);\mathbb{R}\right)\rightarrow\mathbb{R}$ which is progressively measurable and such that

$$\theta\left(t\right)=\psi\left(t,B^T\left(\cdot\wedge t\right)\right), \qquad \forall t\geq T$$

(see [Yong & Zhou, 1999] Theorem 2.10, p. 18). Then, defining the control strategy $\bar{\theta}:[s,+\infty)\times\Omega\rightarrow[0,1]$ as

$$\bar{\theta}\left(t\right)=\psi\left(t-s+T,B^T\left(\cdot\wedge\left(t-s+T\right)\right)\right), \qquad \forall t\geq s,$$

we have that $\bar{\theta}\in\Theta_{ad}\left(s,x\right)$ if and only if $\theta\in\Theta_{ad}\left(T,x\right)$. Indeed the random processes $X\left(\cdot;T,x,\theta\right)$ and $X\left(\cdot+s-T;s,x,\bar{\theta}\right)$ have the same law as they are weak solutions of the same equation, thanks to the time homogeneity of the state equation (6). This establishes a one-to-one correspondence between $\Theta_{ad}\left(T,x\right)$ and $\Theta_{ad}\left(s,x\right)$.

We now give a lemma on the nonemptiness of the set of admissible strategies.

Lemma 3.9 *Given any* $x\geq l$ *the set of admissible strategies* $\Theta\left(T,x\right)$ *is nonempty if and only if the control* $\theta(\cdot)\equiv0$ *is admissible. This happens if and only if*

$$x\geq\frac{A}{r}. \tag{10}$$

Moreover the set of admissible strategies $\Theta\left(T,x\right)$ *is nonempty for every* $x\geq l$ *if and only if*

$$l\geq\frac{A}{r} \qquad \Longleftrightarrow \qquad l_0+\zeta\left(A+\alpha Nw\right)\geq\frac{A}{r}. \tag{11}$$

Proof. Let $x\geq l$. It is clear that if $\theta(\cdot)\equiv0$ is admissible at (T,x) then $\Theta\left(T,x\right)$ is nonempty. We prove the opposite. Assume that $\Theta\left(T,x\right)$ is nonempty and let θ be an admissible strategy. By the Girsanov Theorem under the probability $\tilde{\mathbb{P}}=\exp\left(-\lambda B^T\left(t\right)-\frac{1}{2}\lambda^2\left(t-T\right)\right)\cdot\mathbb{P}$ (which is equivalent to $\mathbb{P}$) the process $\tilde{B}^T\left(t\right)=\lambda\left(t-T\right)+B^T\left(t\right)$ is a Brownian motion starting at T and we have

$$X\left(t\right)=x+\int_T^t rX\left(\tau\right)d\tau-\int_T^t Ad\tau+\int_T^t\theta\left(t\right)\sigma X\left(\tau\right)d\tilde{B}^T\left(\tau\right). \tag{12}$$

Now $X(t) \geq l$ $\mathbb{P}$-a.s., and so also $\tilde{\mathbb{P}}$-a.s.; taking the expectation $\tilde{\mathbb{E}}$ under $\tilde{\mathbb{P}}$ we get $\tilde{\mathbb{E}}[X(t)] \geq l$. But from (12) we have

$$\tilde{\mathbb{E}}[X(t)] = x + \int_T^t r\tilde{\mathbb{E}}[X(\tau)]\,d\tau - \int_T^t A d\tau. \tag{13}$$

This implies that the deterministic function $g(t) = \tilde{\mathbb{E}}[X(t)]$ satisfies the same ordinary differential equation as $X(t)$ when the control strategy is $\theta \equiv 0$, so the claim is proved.

Now for $\theta(\cdot) \equiv 0$ the state equation (6) becomes the following deterministic equation

$$\begin{cases} dX(t) = (rX(t) - A)\,dt, & t \geq T, \\ X(T) = x \geq l. \end{cases} \tag{14}$$

It is then easy to see that $X(t) \geq l$ $\mathbb{P}$-a.s., for any $t \geq T$, if and only if

$$rx - A \geq 0 \Longleftrightarrow x \geq \frac{A}{r}.$$

This gives the first statement. For the second one it is enough to observe that, if $l \geq \frac{A}{r}$ the set of admissible strategies is never empty when the starting point is $x \geq l$. But this would mean

$$l_0 + \zeta(A + \alpha N w) \geq \frac{A}{r}$$

which is the "if" part of the claim. Viceversa assume that $l < \frac{A}{r}$. Then there is no admissible strategy starting at $x = l$. Indeed the null strategy $\theta(\cdot) \equiv 0$ is not admissible by the argument above and so $\Theta(T, x) = \emptyset$. ∎

Remark 3.10 *If a pension fund starts at $s = 0$ with wealth x_0 and uses the control strategy $\theta(\cdot) \equiv 0$ then its wealth at time T is*

$$X(T) = e^{rT}x_0 + \frac{\alpha N w}{r}\left(\frac{e^{rT} - 1}{rT} - 1\right).$$

This implies that the null strategy is admissible from time 0 if and only if

$$e^{rT}x_0 + \frac{\alpha N w}{r}\left(\frac{e^{rT} - 1}{rT} - 1\right) \geq \frac{A}{r},$$

i.e.

$$e^{rT}x_0 \geq \frac{\alpha N w}{r}\left(\frac{e^{\delta T} - 1}{\delta T} - \frac{e^{rT} - 1}{rT}\right).$$

Since $r \geq \delta$ thanks to Hypothesis 2.7, it is easy to check that $\frac{e^{\delta T}-1}{\delta T} \leq \frac{e^{rT}-1}{rT}$. This means in particular that, for $l_0 = 0$ and $x_0 = 0$, the null strategy is admissible, as expected. ∎

Remark 3.11 *Lemma 3.9 substantially states that the good solvency level must be such that the return rl from it is greater than A, i.e. the balance between contribution and benefit rate.*[12] *In the case of stochastic interest rates and demographic risk this would be a stochastic constraint.*

[12] In other words the solvency level l must be above the present value $\frac{A}{r}$ of the perpetual annuity, which is obtained discounting at the instantaneous risk free rate r the balance between benefit and contribution rate A, i.e. the present value of the total outcomes, over the whole time horizon. This may remind what happens, in a different setting, in [Sethi, Taksar & Presman 1992] and [Cadenillas & Sethi, 1997] where models with subsistence consumption are considered.

We observe that in (11) *all quantities are given by the market except for* l_0 *and* ζ *which may be chosen by a supervisory authority:*[13] *this choice should always satisfy* (11) *and may vary depending on the goals of the authority itself* [14] *(see also Remark 2.10).* ■

Remark 3.12 *When* $rl = A$ *then* $\Theta(T, l)$ *is made only by the null strategy since by* (13) *any admissible strategy must have mean value* l*. If* $rl > A$ *then* $\Theta(T, l)$ *contains also other strategies (e.g.,* $\theta(t) = G(X(t))$*, where* G *is given in (44)).* ■

From now on we will always assume that $\Theta(T, x)$ is nonempty over all $[l, +\infty)$, i.e. that (11) holds. We will often divide the two cases $rl = A$ and $rl > A$ since they have different features.

4 Dynamic Programming

The value function associated to the control problem (8) is given, for $s \geq 0$, by

$$V(s, x) = \sup_{\theta(\cdot) \in \Theta_{ad}(s,x)} \mathbb{E} \int_s^{+\infty} e^{-\rho t} U(X(t; s, x, \theta))\, dt, \qquad x \in [l(s), +\infty). \tag{15}$$

and Bellman's optimality principle is as follows (see, e.g., [Soner, 2004], Section 1.3, for a proof).

Theorem 4.1 *The value function* V *satisfies the dynamic programming equation, i.e. for every* $s \geq 0$*,* $x \in [l(s), +\infty)$ *and* $\tau \geq s$ *stopping time, the following functional equation holds*

$$V(s, x) = \sup_{\theta(\cdot) \in \Theta_{ad}(s,x)} \mathbb{E} \left[\int_s^{\tau} e^{-\rho t} U(X(t; s, x, \theta))\, dt + V(\tau, X(\tau; s, x, \theta)) \right]. \tag{16}$$

In general the dynamic programming equation is hard to treat. Then one usually studies its differential form, i.e. the Hamilton-Jacobi-Bellman (hereafter HJB) equation. To this regard, we introduce the following Hamiltonian function

$$H(s, x, p, Q) = \sup_{\theta \in [0,1]} H_{cv}(s, x, p, Q; \theta), \quad s \geq 0, \quad x \in [l(s), +\infty), \quad p, Q \in \mathbb{R},$$

where

$$\begin{aligned} H_{cv}(s, x, p, Q; \theta) &= e^{-\rho s} U(x) + p([\theta\sigma\lambda + r]x + c(s) - b(s)) + \frac{1}{2}\theta^2\sigma^2 x^2 Q \\ &= \begin{cases} e^{-\rho s} U(x) + p([\theta\sigma\lambda + r]x + \alpha w) + \frac{1}{2}\theta^2\sigma^2 x^2 Q & \text{if } s \in [0, T), \\ e^{-\rho s} U(x) + p([\theta\sigma\lambda + r]x - A) + \frac{1}{2}\theta^2\sigma^2 x^2 Q & \text{if } s \geq T. \end{cases} \end{aligned}$$

The HJB equation associated to our problem is then the following

$$-v_s(s, x) - H(s, x, v_x(s, x), v_{xx}(s, x)) = 0, \quad s \geq 0, \quad x \in [l(s), +\infty),$$

Since we take $s \geq T$ then $l(s) \equiv l$ and, using the properties of the set of admissible strategies (see the beginning of Subsection 3.1), we can prove the following.

[13] In other words the authority fix the liquidity l_0 needed to start a pension fund and the percentage ζ of the accrued contribution that must be stored to regulate wisely the pension fund.

[14] For example high $rl - A$ will force the fund manager to keep more prudential behaviours in order to avoid default but would restrict her/his investment strategies and the value would be smaller (see Remark 4.25). Also high l_0 would decrease the number of new entries in the market, and so on.

Proposition 4.2 *For every $s \geq T$ and $x \geq l$ we have*

$$V(s,x) = e^{-\rho(s-T)} V(T,x). \tag{17}$$

Setting

$$V_T(x) = e^{\rho T} V(T,x), \tag{18}$$

we also obtain

$$V_T(x) = \sup_{\theta(\cdot)\in\Theta_{ad}(T,x)} J_T(x;\theta(\cdot)) \tag{19}$$

with

$$J_T(x;\theta(\cdot)) = \mathbb{E}\int_T^{+\infty} e^{-\rho(t-T)} U(X(t;T,x,\theta))\,dt.$$

Moreover, for every stopping time $\hat{t} \geq T$ we have

$$V_T(x) = \sup_{\theta(\cdot)\in\Theta_{ad}(T,x)} \mathbb{E}\left[\int_T^{\hat{t}} e^{-\rho(t-T)} U(X(t;T,x,\theta))\,dt + e^{-\rho(\hat{t}-T)} V_T\left(X\left(\hat{t};T,x,\theta\right)\right)\right]. \tag{20}$$

Proof. We give only a sketch, as the arguments are quite standard. Performing the change of variable $t' = t-(s-T)$ on the right side of (15), we get

$$V(s,x) = \sup_{\theta(\cdot)\in\Theta_{ad}(s,x)} \mathbb{E}\int_T^{+\infty} e^{-\rho(t'+s-T)} U\left(X\left(t'+s-T;s,x,\theta\right)\right) dt'.$$

Since both the state equation and the set of admissible controls are autonomous for $s \geq T$ (see the beginning of Subsection 3.1) then there is a one to one correspondence between $\Theta_{ad}(s,x)$ and $\Theta_{ad}(T,x)$ such that, for any $\theta \in \Theta_{ad}(s,x)$ and calling $\bar{\theta}$ the corresponding strategy in $\Theta_{ad}(T,x)$, the trajectories $X(\cdot+s-T;s,x,\theta)$ and $X\left(\cdot\,;T,x,\bar{\theta}\right)$ have the same law. Hence

$$V(s,x) = \sup_{\theta(\cdot)\in\Theta_{ad}(T,x)} \mathbb{E}\int_T^{+\infty} e^{-\rho(t'+s-T)} U\left(X\left(t';T,x,\theta\right)\right) dt' = e^{-\rho(s-T)} V(T,x).$$

Statement (19) simply follows by the definitions of V_T and V. Statement (20) follows setting $s = T$ and $\tau = \hat{t}$ in the dynamic programming equation (16), and using (17) and (18). ■

Remark 4.3 *By arguing as in the proof above it is easy to see that*

$$V_T(x) = \sup_{\theta(\cdot)\in\tilde{\Theta}_{ad}(0,x)} \mathbb{E}\int_0^{+\infty} e^{-\rho t} U\left(\tilde{X}(t;0,x,\theta)\right) dt,$$

where $\tilde{X}(t;0,x,\theta)$ is the solution of

$$\begin{cases} dX(t) = \{[\theta(t)\sigma\lambda + r]X(t) - A\}\,dt + \theta(t)\sigma X(t)dB(t), & t \geq 0, \\ X(0) = x \geq l, \end{cases}$$

and

$$\tilde{\Theta}_{ad}(0,x) = \left\{\theta : [0,+\infty)\times\Omega \longrightarrow [0,1] \text{ adapted to } \{\mathcal{F}_t^B\}_{t\geq 0} \mid \tilde{X}(t;0,x,\theta) \in [l,+\infty),\ t \geq 0\right\}.$$

So one could use this form of V_T to study its properties. We decide to keep the form starting at T to avoid too many changes of variables. ■

Therefore taking $s \geq T$ we reduce ourselves to the study of the above infinite horizon problem starting at T. The associated HJB equation is, formally (the meaning of the equation at l will be specified later in Subsection 4.2),

$$\rho v(x) - H_0\left(x, v'(x), v''(x)\right) = 0, \qquad x \in [l, +\infty), \tag{21}$$

where

$$H_0\left(x, v'(x), v''(x)\right) = \sup_{\theta \in [0,1]} H_{0,cv}\left(x, v'(x), v''(x); \theta\right), \quad x \in [l, +\infty),$$

with

$$H_{0,cv}\left(x, p, Q; \theta\right) = U\left(x\right) + p\left(\left[\theta\sigma\lambda + r\right]x - A\right) + \frac{1}{2}\theta^2\sigma^2x^2Q.$$

We now concentrate our analysis on this last case. Note that calling $\mathcal{L}^\theta$ the parabolic operator defined, for $f \in C^2\left([l, +\infty); \mathbb{R}\right)$, by

$$\mathcal{L}^\theta f\left(x\right) = \frac{1}{2}\theta^2\sigma^2x^2 f''(x)\left(x\right) + \left(\left[\theta\sigma\lambda + r\right]x - A\right)f'(x)\left(x\right), \qquad x \geq l, \tag{22}$$

we can write

$$H_{0,cv}\left(x, p, Q; \theta\right) = \mathcal{L}^\theta v\left(x\right) + U\left(x\right), \qquad x \geq l.$$

To calculate the Hamiltonians we observe that the function

$$H_{1,cv}\left(x, p, Q; \theta\right) = p\theta\sigma\lambda x + \frac{1}{2}\theta^2\sigma^2x^2Q$$

when $p \geq 0$, $Q \leq 0$, $p^2 + Q^2 > 0$ has a unique maximum point over $\theta \in [0, 1]$ given by

$$\theta^* = -\frac{\lambda p}{\sigma x Q} \wedge 1$$

(where we mean that for $Q = 0$ $\theta^* = 1$) and

$$H_1\left(x, p, Q\right) = \sup_{\theta \in [0,1]} H_{1,cv}\left(x, p, Q; \theta\right) = \begin{cases} -\frac{\lambda^2 p^2}{2Q} & \text{if} \quad \theta^* < 1, \\ p\sigma\lambda x + \frac{1}{2}\sigma^2x^2Q & \text{if} \quad \theta^* = 1. \end{cases}$$

When $p = Q = 0$ each $\theta \in [0, 1]$ is a maximum point and $H_1\left(x, p, Q\right) = 0$. Then the HJB equation (21) can be written as

$$\rho v(x) - U\left(x\right) - v'\left(x\right)\left(rx - A\right) - H_1\left(x, v'\left(x\right), v''\left(x\right)\right) = 0, \qquad x \in [l, +\infty). \tag{23}$$

4.1 Properties of the value function

In this section we discuss and prove some basic properties (finiteness, concavity, monotonicity, continuity) of the value function $V_T\left(x\right) = e^{\rho T} V\left(T, x\right)$, where $V(\cdot, \cdot)$ is defined in (15). For simplicity in what follows we will write $V_T\left(x\right) = V\left(x\right)$ omitting the subscript T; similarly we will write $B(t)$ for $B^T(t)$. Throughout this section we will always assume that all previously stated Hypotheses 2.1, 2.2, 2.3, 2.4, 2.5, 2.6, 2.7, 2.9, 3.3, 3.4 hold.

Proposition 4.4 *Let $rl \geq A$. Then $V\left(\cdot\right) > -\infty$ on $(l, +\infty)$. Moreover*

(i) when $rl = A$: $V\left(l\right) > -\infty$ if and only if $U\left(l\right) > -\infty$;

(ii) when $rl > A$: $V(l) > -\infty$ if and only if U is integrable in a right neighborhood of l.

Finally $\exists\, C > 0$ such that $V(x) \le C(1+x)^{\beta}$ $(x \ge l)$, where β is given by Hypothesis 3.4-(ii).

Proof. *Estimates from below for $x > l$.*

First of all we show that $V > -\infty$ on $(l, +\infty)$. Indeed, since the null strategy is always admissible we have, for every $x \ge l$,

$$V(x) \ge J_T(x;0) = \int_T^{+\infty} e^{-\rho(t-T)} U(X(t;T,x,0))\, dt.$$

But, recalling that $X(t;T,x,0)$ satisfies (14), we have

$$X(t;T,x,0) = e^{r(t-T)}x - A\frac{e^{r(t-T)}-1}{r} = e^{r(t-T)}\left[x - \frac{A}{r}\right] + \frac{A}{r}.$$

Since (10) holds then $x - \frac{A}{r} \ge 0$, so $X(t;T,x,0) \ge x$ for every $t \ge T$ and

$$V(x) \ge J_T(x;0) \ge \int_T^{+\infty} e^{-\rho(t-T)} U(x)\, dt = \frac{U(x)}{\rho}, \tag{24}$$

which gives the claim.

Estimates from below for $x = l$, case (i).

The above arguments also says that $V(l) > -\infty$ when $U(l) > -\infty$. Moreover, when $rl = A$ the only admissible strategy at $x = l$ is the null one (Remark 3.12) that keeps the state in l at every time; so when $U(l) = -\infty$ also $V(l) = -\infty$.

Estimates from below for $x = l$, case (ii).

Assume now that $rl > A$, so

$$V(l) \ge J_T(l;0) = \int_T^{+\infty} e^{-\rho(t-T)} U\left(e^{r(t-T)}\left[l - \frac{A}{r}\right] + \frac{A}{r}\right) dt. \tag{25}$$

By setting $z = e^{r(t-T)}\left[l - \frac{A}{r}\right] + \frac{A}{r}$ the above integral becomes equal to a given constant multiplied by

$$\int_l^{+\infty} \left(z - \frac{A}{r}\right)^{-\frac{\rho}{r}-1} U(z)\, dz$$

which is $> -\infty$. Indeed, take any $z_0 > l$, the integrability of U in a neighborhood of l says that $\int_l^{z_0} \left(z - \frac{A}{r}\right)^{-\frac{\rho}{r}-1} U(z)\, dz$ is finite, while for the term $\int_{z_0}^{+\infty} \left(z - \frac{A}{r}\right)^{-\frac{\rho}{r}-1} U(z)\, dz$ we have two cases. Either U remains negative over all $(l, +\infty)$ and in this case the integral is $> -\infty$ thanks to the term $\left(z - \frac{A}{r}\right)^{-\frac{\rho}{r}-1}$, or becomes positive after a certain point and in this case the integral is immediately $> -\infty$.

Take now U which is not integrable in a right neighborhood of l. To prove the claim it is enough to show that, for every $\theta(\cdot) \in \Theta(T, l)$, setting $X(t) = X(t;T,l,\theta)$ we have

$$\mathbb{E}\int_T^{T+1} e^{-\rho(t-T)} U(X(t))dt = -\infty.$$

By the Girsanov Theorem under the probability $\tilde{\mathbb{P}} = \exp\left(-\lambda B(t) - \frac{1}{2}\lambda^2 (t-T)\right) \cdot \mathbb{P}$, the process $\tilde{B}(t) = \lambda(t-T) + B(t)$ is a Brownian motion starting at T and we have

$$X(t) = l + \int_T^t rX(\tau)\, d\tau - \int_T^t A d\tau + \int_T^t \theta(t)\,\sigma X(\tau)\, d\tilde{B}(\tau),$$

so by a comparison result (see, e.g., in [Karatzas & Shreve, 1991] Proposition 2.18, p. 293) we get that $l \leq X(t) \leq Y(t)$, $t \geq T$ a.e., $\mathbb{P}$-a.s., where Y satisfies

$$Y(t) = l + \int_T^t rY(\tau)\, d\tau + \int_T^t \theta(t)\,\sigma Y(\tau)\, d\tilde{B}(\tau).$$

Using the Jensen inequality, the Girsanov Theorem and then the Schwarz inequality we obtain

$$\begin{aligned} &\mathbb{E}\int_T^{T+1} e^{-\rho(t-T)} U(X(t))dt \leq \mathbb{E}\int_T^{T+1} e^{-\rho(t-T)} U(Y(t))dt \leq \int_T^{T+1} e^{-\rho(t-T)} U(\mathbb{E}[Y(t)])dt \\ &\leq \int_T^{T+1} e^{-\rho(t-T)} U\left(\tilde{\mathbb{E}}\left[e^{\lambda\tilde{B}(t)-\frac{1}{2}\lambda^2(t-T)}Y(t)\right]\right) dt \leq \int_T^{T+1} e^{-\rho(t-T)} U\left(e^{\frac{1}{2}\lambda^2(t-T)}\left(\tilde{\mathbb{E}}\left[Y(t)^2\right]\right)^{\frac{1}{2}}\right) dt. \end{aligned}$$

Applying the Itô formula to the process Y we easily find the estimate

$$\tilde{\mathbb{E}}\left[Y(t)^2\right] \leq l^2 e^{(2r+\sigma^2)(t-T)},$$

so we finally get

$$\mathbb{E}\int_T^{T+1} e^{-\rho(t-T)} U(X(t))dt \leq \int_T^{T+1} e^{-\rho(t-T)} U\left(l e^{\left(r+\frac{1}{2}\sigma^2+\frac{1}{2}\lambda^2\right)(t-T)}\right) dt.$$

Applying a change of variable like the one done in formula (25) we get the claim.

Estimates from above.
First, if $\lim_{z\to+\infty} U(z) =: \bar{U} < +\infty$ then, for every $x \geq l$,

$$V(x) \leq \int_T^{+\infty} e^{-\rho(t-T)} \bar{U} dt = \frac{\bar{U}}{\rho},$$

so in this case V is finite and bounded. Assume now that $\bar{U} = +\infty$. Let $\theta \in \Theta(T,x)$ and set $X(t) = X(t;T,x,\theta)$; by Hypothesis 3.4-(ii) we have

$$J_T(x;\theta(\cdot)) = \mathbb{E}\int_T^{+\infty} e^{-\rho(t-T)} U(X(t))\, dt \leq \int_T^{+\infty} e^{-\rho(t-T)} C\left(1 + \mathbb{E}\left[X(t)^\beta\right]\right) dt. \tag{26}$$

Applying the Girsanov Theorem under the probability $\tilde{\mathbb{P}}$ as above, we have

$$X(t) = x + \int_T^t rX(\tau)\, d\tau - \int_T^t A d\tau + \int_T^t \theta(t)\,\sigma X(\tau)\, d\tilde{B}(\tau),$$

so by the Gronwall lemma applied to the function $g(t) = \tilde{\mathbb{E}}X(t)$ we get $\tilde{\mathbb{E}}X(t) \leq e^{r(t-T)}x$ for $t \geq T$. Now

$$\mathbb{E}\left[X(t)^\beta\right] = \tilde{\mathbb{E}}\left[X(t)^\beta \exp\left(\lambda\tilde{B}(t) - \frac{1}{2}\lambda^2(t-T)\right)\right],$$

and by the Schwarz inequality

$$\tilde{\mathbb{E}}\left[X(t)^{\beta}\exp\left(\lambda\tilde{B}(t)-\frac{1}{2}\lambda^{2}(t-T)\right)\right]$$
$$\leq\left(\tilde{\mathbb{E}}\left[X(t)\right]\right)^{\beta}\cdot\left(\tilde{\mathbb{E}}\left[\exp\left(\lambda\tilde{B}(t)-\frac{1}{2}\lambda^{2}(t-T)\right)\frac{1}{1-\beta}\right]\right)^{1-\beta}=e^{\left(\beta r+\frac{\beta}{1-\beta}\cdot\frac{\lambda^{2}}{2}\right)(t-T)}.$$

This implies by (26), for some $C'>0$,

$$J_{T}(x;\theta(\cdot))\leq\int_{T}^{+\infty}e^{-\rho(t-T)}C\left(1+e^{\left(\beta r+\frac{\beta}{1-\beta}\cdot\frac{\lambda^{2}}{2}\right)(t-T)}\right)dt$$
$$\leq C'\left(1+\int_{T}^{+\infty}e^{\left(-\rho+\beta r+\frac{\beta}{1-\beta}\cdot\frac{\lambda^{2}}{2}\right)(t-T)}dt\right)$$

which gives the claim. ■

We now have the following result.

Proposition 4.5 *The value function V is concave.*

Proof. Take two initial values x_1 and x_2 such that $x_1,\ x_2\geq l$. Suppose, without loss of generality, that $x_1<x_2$ and set $x_\eta=\eta x_1+(1-\eta)x_2$, $\eta\in[0,1]$. We have to prove that

$$V(x_\eta)\geq\eta V(x_1)+(1-\eta)V(x_2)\qquad\mathbb{P}\text{-a.s.}. \tag{27}$$

Let us suppose $\theta_1(\cdot)\in\Theta_{ad}(T,x_1)$ and $\theta_2(\cdot)\in\Theta_{ad}(T,x_2)$ be ϵ-optimal for x_1 and for x_2, respectively. Then $X_1(\cdot)=X(\cdot\,;T,x_1,\theta_1)$ and $X_2(\cdot)=X(\cdot\,;T,x_2,\theta_2)$ satisfy the state equation (6). We have

$$\begin{aligned}\eta V(x_1)+(1-\eta)V(x_2)&<\eta\left[J_T(x_1;\theta_1(\cdot))+\epsilon\right]+(1-\eta)\left[J_T(x_2;\theta_2(\cdot))+\epsilon\right]\\&=\epsilon+\eta J_T(x_1;\theta_1(\cdot))+(1-\eta)J_T(x_2;\theta_2(\cdot))\\&=\epsilon+\eta\mathbb{E}\int_T^{+\infty}e^{-\rho(t-T)}U\left(X_1(t)\right)dt+(1-\eta)\mathbb{E}\int_T^{+\infty}e^{-\rho(t-T)}U\left(X_2(t)\right)dt\\&=\epsilon+\mathbb{E}\int_T^{+\infty}e^{-\rho(t-T)}\left[\eta U\left(X_1(t)\right)+(1-\eta)U\left(X_2(t)\right)\right]dt.\end{aligned}$$

The strictly concavity of U implies that

$$\eta U\left(X_1(t)\right)+(1-\eta)U\left(X_2(t)\right)<U\left(\eta X_1(t)+(1-\eta)X_2(t)\right),\qquad\forall t\geq T.$$

Consequently if we set $X_\eta(\cdot)=\eta X_1(\cdot)+(1-\eta)X_2(\cdot)$ then we get

$$\eta V(x_1)+(1-\eta)V(x_2)<\epsilon+\mathbb{E}\int_T^{+\infty}e^{-\rho(t-T)}U\left(X_\eta(t)\right)dt.$$

If there exists $\theta_\eta(\cdot)\in\Theta(T,x_\eta)$ such that $X_\eta(\cdot)=X(\cdot;T,x_\eta,\theta_\eta)$ then we would have

$$\epsilon+\mathbb{E}\int_T^{+\infty}e^{-\rho(t-T)}U\left(X_\eta(t)\right)dt=\epsilon+J_T(x_\eta;\theta_\eta(\cdot))\leq\epsilon+V(x_\eta),$$

i.e.

$$\eta V(x_1) + (1-\eta)V(x_2) < \epsilon + V(x_\eta)$$

and therefore, by the arbitrariness of ϵ, the claim (27) would be proved.

To find such a $\theta_\eta(\cdot)$, let us write the equation satisfied by X_η. Recalling (6) it follows

$$\begin{aligned} dX_\eta(t) &= \eta dX_1(t) + (1-\eta)dX_2(t) \\ &= \eta\left[\Big(\left(\theta_1(t)\sigma\lambda + r\right)X_1(t) - A\Big)dt + \theta_1(t)\sigma X_1(t)dB(t)\right] \\ &\quad + (1-\eta)\left[\Big(\left(\theta_2(t)\sigma\lambda + r\right)X_2(t) - A\Big)dt + \theta(t)\sigma X_2(t)dB(t)\right] \\ &= \left\{\left[\eta\frac{X_1(t)}{X_\eta(t)}\theta_1(t) + (1-\eta)\frac{X_2(t)}{X_\eta(t)}\theta_2(t)\right]\sigma\lambda X_\eta(t) + rX_\eta(t) - A\right\}dt \\ &\quad + \left[\eta\frac{X_1(t)}{X_\eta(t)}\theta_1(t) + (1-\eta)\frac{X_2(t)}{X_\eta(t)}\theta_2(t)\right]dB(t). \end{aligned}$$

Then defining the control $\theta_\eta(t) = a(t)\,\theta_1(t) + d(t)\,\theta_2(t)$, where $a(\cdot) = \eta\frac{X_1(\cdot)}{X_\eta(\cdot)}$ and $d(\cdot) = (1-\eta)\frac{X_2(\cdot)}{X_\eta(\cdot)}$, we have

$$\begin{cases} dX_\eta(t) = \left\{\left[\theta_\eta(t)\sigma\lambda + r\right]X_\eta(t) - A\right\}dt + \theta_\eta(t)\sigma X_\eta(t)dB(t), & t \geq T, \\ X_\eta(T) = \eta x_1 + (1-\eta)x_2 = x_\eta, \quad x_\eta \geq l, \end{cases}$$

so we get $X_\eta(\cdot) = X(\cdot; T, x_\eta, \theta_\eta)$. The admissibility of θ_η is clear since:

- (i) for every $t \geq T$, we have $\theta_1(t), \theta_2(t) \in [0,1]$ and $a(t) + d(t) = 1$, so by convexity of $[0,1]$ we get $\theta_\eta(t) \in [0,1]$;
- (ii) by construction $X_\eta(t) \geq l$ $\mathbb{P}$-a.s., for any $t \geq T$.

The claim follows. ■

We now prove the monotonicity of V.

Proposition 4.6 *The value function V is strictly increasing.*

Proof. First we verify that the value function V is increasing showing that

$$l \leq x_1 < x_2 \implies V(x_1) \leq V(x_2) \quad \mathbb{P}\text{-a.s.},$$

Take any $\theta(\cdot) \in \Theta_{ad}(T, x_1)$. By Proposition 2.18 in [Karatzas & Shreve, 1991], p. 293, we have

$$X(\cdot\,; T, x_1, \theta) \leq X(\cdot\,; T, x_2, \theta) \quad \mathbb{P}\text{-a.s.},$$

so also $\theta(\cdot) \in \Theta_{ad}(T, x_2)$. Moreover, by the strict monotonicity of the utility function U we have

$$x_1 < x_2 \implies U\left(X\left(\cdot\,; T, x_1, \theta\right)\right) \leq U\left(X\left(\cdot\,; T, x_2, \theta\right)\right) \ \mathbb{P}\text{-a.s.} \implies J_T\left(x_1; \theta(\cdot)\right) \leq J_T\left(x_2; \theta(\cdot)\right).$$

Since $\Theta_{ad}(T, x_1) \subseteq \Theta_{ad}(T, x_2)$ the above implies $V(x_1) \leq V(x_2)$.

The strict monotonicity of the value function V is a direct consequence of monotonicity and concavity (see, e.g., the proof in [Zariphopoulou, 1994], p. 63). Indeed, if V is not strictly monotone then it must be constant on a half line $[\bar{x}, +\infty)$. We show that this cannot be true.

By (24) we have, for every $y \geq l$,

$$V(y) \geq \frac{U(y)}{\rho}.$$

So, if $\lim_{z\to+\infty} U(z) = +\infty$ then $\lim_{z\to+\infty} V(z) = +\infty$ and the claim follows.

Take now $\lim_{z\to+\infty} U(z) =: \bar{U} < +\infty$. In this case we must have

$$V(\bar{x}) = \lim_{y\to+\infty} V(y) \geq \frac{\bar{U}}{\rho}.$$

On the other hand, for every $\theta \in \Theta_{ad}(T, \bar{x})$, we get

$$J_T(\bar{x}; \theta(\cdot)) = \mathbb{E} \int_T^{+\infty} e^{-\rho(t-T)} U(X(t; T, \bar{x}, \theta))\, dt \leq \int_T^{+\infty} e^{-\rho(t-T)} U\left(e^{(\lambda\sigma+r)(t-T)}\bar{x}\right) dt.$$

Fix $T_1 > T$. We have, calling $U_{T_1} = U\left(e^{(\lambda\sigma+r)(T_1-T)}\bar{x}\right) < \bar{U}$,

$$\begin{aligned} J_T(\bar{x}; \theta(\cdot)) &\leq \int_T^{T_1} e^{-\rho(t-T)} U_{T_1} dt + \int_{T_1}^{+\infty} e^{-\rho(t-T)} \bar{U} dt \\ &= \frac{U_{T_1}}{\rho}\left[1 - e^{-\rho(T_1-T)}\right] + \frac{\bar{U}}{\rho} e^{-\rho(T_1-T)} < \frac{\bar{U}}{\rho} \leq V(\bar{x}). \end{aligned}$$

This is a contradiction and so the claim follows. ■

Finally we prove the continuity.

Proposition 4.7 *The value function V is continuous in $(l, +\infty)$ and Lipschitz continuous in $[a, +\infty)$, for any $a > l$. Moreover if $rl > A$ and $V(l) > -\infty$ then V is uniformly continuous in $[l, +\infty)$.*

Proof. The Lipschitz continuity of V on the interval $[a, +\infty)$, for any $a > l$, is a straightforward consequence of concavity and strict monotonicity.

It remains to prove the continuity of V in l when $rl > A$ and $V(l) > -\infty$, i.e. when U is integrable in a right neighborhood of l (Proposition 4.4 - (ii)). We have to show that

$$\lim_{x\to l^+} [V(x) - V(l)] = 0.$$

Since $rl > A$, the control strategy $\theta(\cdot) \equiv 0$ at the starting point l gives rise to a trajectory which is strictly increasing.

Let $x > l$. Applying the control $\theta(\cdot) \equiv 0$ to the state equation (6) with initial point $X(T) = l$, the corresponding trajectory is deterministic and it reaches the point x at time $\hat{t}$ such that $X\left(\hat{t}; T, l, 0\right) = x$, i.e.

$$e^{r\left(\hat{t}-T\right)}\left[l - \frac{A}{r}\right] + \frac{A}{r} = x \qquad \Longleftrightarrow \qquad \hat{t} = T + \frac{1}{r}\ln\frac{rx - A}{rl - A}.$$

Now, by the dynamic programming principle we have

$$V(l) = \sup_{\theta(\cdot)\in\Theta_{ad}(T,l)} \mathbb{E}\left[\int_T^{\hat{t}} e^{-\rho(t-T)} U(X(t; T, l, \theta))\, dt + e^{-\rho\left(\hat{t}-T\right)} V\left(X\left(\hat{t}; T, l, \theta\right)\right)\right],$$

so

$$V(l) \geq \int_T^{\hat{t}} e^{-\rho(t-T)} U(X(t;T,l,0))\,dt + e^{-\rho(\hat{t}-T)} V(x),$$

which gives

$$0 \leq V(x) - V(l) \leq -\int_T^{\hat{t}} e^{-\rho(t-T)} U(X(t;T,l,0))\,dt + \left(1 - e^{-\rho(\hat{t}-T)}\right) V(x)$$

(notice that the first inequality is a consequence of the monotonicity of the value function given in Proposition 4.6). Observing that

$$1 - e^{-\rho(\hat{t}-T)} = 1 - \left(\frac{rx - A}{rl - A}\right)^{-\frac{\rho}{r}}$$

and using the integrability of U we get the claim. ■

Remark 4.8 *The continuity of V in l when $rl = A$ and $V(l) > -\infty$ (i.e. when $U(l) > -\infty$ by Proposition 4.4 - (i)) can be proved by using a more refined argument. We do not do it here for brevity (and also because this result in not essential for our purposes) but we will take this result for granted from now on. In the special case of Section 4.4 we will see that V is given explicitly and is continuous in l when $V(l) > -\infty$.* ■

Remark 4.9 *From the above proof it follows that when $U(l)$ and $V(l)$ are finite and $rl > A$ we have, for $x > l$,*

$$\frac{V(x) - V(l)}{x - l} \leq \frac{\frac{1}{r}\ln\frac{rx-A}{rl-A}}{x-l} e^{-\rho T} U(l) + \frac{1 - \left(\frac{rx-A}{rl-A}\right)^{-\frac{\rho}{r}}}{x-l} V(x) + \frac{o(x-l)}{x-l},$$

so recalling that $V'(l^+)$ must exist by the concavity of V it follows

$$V'(l^+) \leq \frac{\rho}{rl - A}\left[\frac{e^{-\rho T}}{\rho} U(l) + V(l)\right].$$

This means, in particular that $V'(l^+)$ is finite. On the other side when $rl = A$, $U'(l^+) = +\infty$ and $U(l) > -\infty$ (hence V is finite and continuous at l, see Remark 4.8), then $V'(l^+)$ is infinite. Indeed in this case $V(l) = \frac{U(l)}{\rho}$ while $V(x) \geq J_T(x;0) \geq \frac{U(x)}{\rho}$, so $V'(l^+) \geq \frac{U'(l^+)}{\rho} = +\infty$. See Section 4.4 for an example. ■

4.2 The HJB equation: viscosity solutions and regularity

Let us consider HJB equation (21) on $[l, +\infty)$. This is a second order PDE which is degenerate elliptic. The concept of viscosity solution we use here is the following (see, e.g., [Crandall, Ishii & Lions, 1992] for a survey on viscosity solution of second order PDE's).

Definition 4.10 *A continuous function $v : (l, +\infty) \to \mathbb{R}$ is a viscosity subsolution (respectively supersolution) of equation (21) in $(l, +\infty)$ if, for any $\psi \in C^2((l, +\infty); \mathbb{R})$ and for any maximum point $x_M \in (l, +\infty)$ (respectively minimum point $x_m \in (l, +\infty)$) of $v - \psi$, we have*

$$\rho\psi(x_M) - H_0\left(x_M, \psi'(x_M), \psi''(x_M)\right) \leq 0 \qquad (\textit{respectively} \geq 0).$$

A continuous function $v : (l, +\infty) \to \mathbb{R}$ is a viscosity solution of equation (21) in $(l, +\infty)$ if it is both a viscosity subsolution and a viscosity supersolution in $(l, +\infty)$.

Definition 4.11 *A continuous function $v : [l, +\infty) \to \mathbb{R}$ is a viscosity subsolution of equation* (21) *on $[l, +\infty)$ if, for any $\psi \in C^2([l, +\infty)); \mathbb{R})$ and for any maximum point $x_M \in [l, +\infty)$ of $v - \psi$, it follows*

$$\rho\psi(x_M) - H_0\left(x_M, \psi'(x_M), \psi''(x_M)\right) \leq 0.$$

A continuous function $v : [l, +\infty) \to \mathbb{R}$ is called a constrained viscosity solution of equation (21) *if it is viscosity subsolution on $[l, +\infty)$ and a viscosity supersolution in $(l, +\infty)$.*

Remark 4.12 *The above definition of constrained viscosity solution comes from [Soner, 1986] (in the deterministic case) and is the same used also by [Zariphopoulou, 1994] to treat a similar HJB equation. Indeed other definitions of viscosity solutions can be used (see, e.g., [Ishii & Loreti, 2002]) that differs in the boundary conditions. The definition we use was introduced for cases where it is possible to find a control that brings the state from the boundary to the interior of the state space, as in the case $rl > A$ (see also [Katsoulakis, 1994] on this subject).*■

Now we can state and prove the following result.

Theorem 4.13 *The value function V defined in* (18) *is a viscosity solution of HJB equation* (21) *in $(l, +\infty)$. If U is finite in l then V is a constrained viscosity solution of HJB equation* (21) *on $[l, +\infty)$.*

Proof. We have to show that V is both (i) viscosity supersolution in $(l + \infty)$, (ii) viscosity subsolution in $(l + \infty)$, (iii) viscosity subsolution in l when U is finite in l.
(i) Let us consider $\psi \in C^2((l, +\infty); \mathbb{R})$ and a minimum point $x_m \in (l, +\infty)$ for the function $V - \psi$. We can assume without loss of generality that

$$V(x_m) = \psi(x_m) \qquad \text{and} \qquad V(x) \geq \psi(x), \qquad \forall x \in (l, +\infty). \tag{28}$$

Let $\varepsilon > 0$ such that $x_m - \varepsilon > l$. For $\theta(\cdot) \in \Theta(T, x_m)$ we write for simplicity $X(t)$ in place of $X(t; T, x_m, \theta)$. Consider the stopping time $\tau_\varepsilon = \inf\{t \geq T \ : \ |X(t) - x_m| \geq \varepsilon\}$. By (28) we get, for any $t \geq T$,

$$e^{-\rho t}V(X(t)) - V(x_m) \geq e^{-\rho t}\psi(X(t)) - \psi(x_m).$$

Let $h > T$. Calling $\tau_{\varepsilon,h} = \tau_\varepsilon \wedge h$, by the dynamic programming principle (20) we get

$$\begin{aligned} 0 &= \sup_{\theta \in \Theta(T, x_m)} \mathbb{E}\left[\int_T^{\tau_{\varepsilon,h}} e^{-\rho(t-T)}U(X(t))\,dt + e^{-\rho\left(\tau_{\varepsilon,h}-T\right)}V(X(\tau_{\varepsilon,h})) - V(x_m)\right] \\ &\geq \sup_{\theta \in \Theta(T, x_m)} \mathbb{E}\left[\int_T^{\tau_{\varepsilon,h}} e^{-\rho(t-T)}U(X(t))\,dt + e^{-\rho\left(\tau_{\varepsilon,h}-T\right)}\psi(X(\tau_{\varepsilon,h})) - \psi(x_m)\right]. \end{aligned} \tag{29}$$

Applying the Dynkin formula to the function $(t, x) \to e^{-\rho(t-T)}\psi(x)$ with the process $X(t)$, we get ($\mathcal{L}^\theta$ is defined in (22))

$$\mathbb{E}\left[e^{-\rho\left(\tau_{\varepsilon,h}-T\right)}\psi(X(\tau_{\varepsilon,h}))\right] - \psi(x_m) = \mathbb{E}\int_T^{\tau_{\varepsilon,h}} e^{-\rho(t-T)}\left[-\rho\psi(X(t)) + \left(\mathcal{L}^{\theta(t)}\psi\right)(X(t))\,dt\right],$$

and thus by (29) we have

$$\begin{aligned} 0 &\geq \sup_{\theta \in \Theta(T, x_m)} \mathbb{E}\int_T^{\tau_{\varepsilon,h}} e^{-\rho(t-T)}\left[-\rho\psi(X(t)) + \left(\mathcal{L}^{\theta(t)}\psi\right)(X(t)) + U(X(t))\right]dt \\ &= \sup_{\theta \in \Theta(T, x_m)} \mathbb{E}\int_T^{\tau_{\varepsilon,h}} e^{-\rho(t-T)}\left[-\rho\psi(X(t)) + H_{0,cv}\left(X(t), \psi'(X(t)), \psi''(X(t)); \theta(t)\right)\right]dt. \end{aligned}$$

Taking any constant control $\bar{\theta}(\cdot) \equiv \bar{\theta} \in [0,1]$ we get

$$0 \geq \mathbb{E} \int_T^{\tau_{\varepsilon,h}} e^{-\rho(t-T)} \left[-\rho\psi\left(X\left(t\right)\right) + H_{0,cv}\left(X\left(t\right), \psi'\left(X\left(t\right)\right), \psi''\left(X\left(t\right)\right); \bar{\theta}\right) \right] dt.$$

Divide now by $\tau_{\varepsilon,h} - T$ and let $h \to T$. By continuity of $H_{0,cv}$ we obtain

$$0 \geq -\rho\psi\left(x_m\right) + H_{0,cv}\left(x_m, \psi'\left(x_m\right), \psi''\left(x_m\right); \bar{\theta}\right).$$

By the arbitrariness of $\bar{\theta}$ the claim follows.

(ii) Let $\psi \in \mathrm{C}^2\left((l,+\infty);\mathbb{R}\right)$ and let $x_M \in (l,+\infty)$ be a maximum point of $V - \psi$ in $(l,+\infty)$. Let us assume without loss of generality that

$$V(x_M) = \psi(x_M) \qquad \textit{and} \qquad V(x) \leq \psi(x), \qquad \forall x \in (l,+\infty). \tag{30}$$

We must prove that

$$\rho\psi(x_M) - H_0\left(x_M, \psi'(x_M), \psi''(x_M)\right) \leq 0.$$

Let us suppose by contradiction that this relation is false. Then a strictly positive number ν exists such that

$$0 < \nu < \rho\psi(x_M) - H_0\left(x_M, \psi'(x_M), \psi''(x_M)\right).$$

Since the functions U and ψ are continuous, there exists $\varepsilon \in (0, x_M - l)$ such that, for any $x \in (x_M - \varepsilon, x_M + \varepsilon)$, we have

$$0 < \frac{\nu}{2} < \rho\psi(x) - H_0\left(x, \psi'(x), \psi''(x)\right). \tag{31}$$

We consider any admissible control strategy θ for the initial point x_M; the associated state trajectory is $X(t) = X(t;T,x_M,\theta)$. Define the stopping time $\tau'_\varepsilon = \inf\left\{t \geq T \;:\; \left|X\left(t\right) - x_M\right| \geq \varepsilon\right\}$ and note that, $\mathbb{P}$-a.s., $T < \tau'_\varepsilon < +\infty$. Now we take (31) for $x = X\left(t\right)$, we multiply it by $e^{-\rho(t-T)}$, we integrate it on $[T, \tau'_\varepsilon]$, and we calculate its expected value obtaining

$$0 < \frac{\nu}{2}\mathbb{E}\int_T^{\tau'_\varepsilon} e^{-\rho(t-T)}dt < \mathbb{E}\int_T^{\tau'_\varepsilon} e^{-\rho(t-T)}\left[\rho\psi(X\left(t\right)) - \max_{\theta\in[0,1]}\left\{\left(\mathcal{L}^\theta\psi\right)\left(X\left(t\right)\right)\right\} - U\left(X\left(t\right)\right)\right]dt,$$

from which follows

$$\begin{aligned} 0 &< \mathbb{E}\int_T^{\tau'_\varepsilon} e^{-\rho(t-T)}U(X\left(t\right))dt + \frac{\nu}{2}\mathbb{E}\int_T^{\tau'_\varepsilon} e^{-\rho(t-T)}dt \\ &< \mathbb{E}\int_T^{\tau'_\varepsilon} e^{-\rho(t-T)}\left[\rho\psi(X\left(t\right)) - \left(\mathcal{L}^{\theta(t)}\psi\right)\left(X\left(t\right)\right)\right]dt. \end{aligned} \tag{32}$$

Similarly to what we have done in (i), we apply the Dynkin formula to the function $(t,x) \to e^{-\rho(t-T)}\psi\left(x\right)$ with the process $X\left(t\right)$. We get

$$\mathbb{E}\left[e^{-\rho(\tau'_\varepsilon-T)}\psi(X\left(\tau'_\varepsilon\right))\right] - \psi(x_M) = \mathbb{E}\int_T^{\tau'_\varepsilon} e^{-\rho(t-T)}\left[\left(\mathcal{L}^{\theta(t)}\psi\right)\left(X\left(t\right)\right) - \rho\psi(X\left(t\right))\right]dt, \tag{33}$$

and from (32) and (30) it follows, rearranging the terms,

$$V(x_M) > \mathbb{E}\left[\int_T^{\tau'_\varepsilon} e^{-\rho t}U(X\left(t\right))dt + e^{-\rho(\tau'_\varepsilon-T)}V\left(X\left(\tau'_\varepsilon\right)\right)\right] + \frac{\nu}{2}\mathbb{E}\int_T^{\tau'_\varepsilon} e^{-\rho(t-T)}dt. \tag{34}$$

Observe that there exists a constant $\alpha > 0$ independent of $\theta(\cdot)$ such that

$$\mathbb{E}\int_T^{\tau'_\varepsilon} e^{-\rho(t-T)}dt > \alpha.$$

Indeed, for suitable $K > 0$, set

$$w(x) = K\left(|x - x_M|^2 - \varepsilon^2\right), \qquad x \geq l.$$

Since $w'(x) = 2K(x - x_M)$ and $w''(x) = 2K$, for every $\theta \in [0,1]$ we have

$$\left(\mathcal{L}^\theta w\right)(x) - \rho w(x) = K\left[2(x - x_M)([\theta\sigma\lambda + r]x - A) + \theta^2\sigma^2 x^2 - \rho\left(|x - x_M|^2 - \varepsilon^2\right)\right].$$

Choosing

$$K = \frac{1}{\sup_{|x-x_M|<\varepsilon,\ \theta\in[0,1]}\left\{2(x - x_M)([\theta\sigma\lambda + r]x - A) + \theta^2\sigma^2 x^2 - \rho\left(|x - x_M|^2 - \varepsilon^2\right)\right\}},$$

we obtain, for $|x - x_M| < \varepsilon$ and $\theta \in [0,1]$,

$$\left(\mathcal{L}^\theta w\right)(x) - \rho w(x) \leq 1$$

Now, arguing as we did above to get (33), we have

$$\begin{aligned}\mathbb{E}&\left[e^{-\rho(\tau'_\varepsilon - T)} w\left(X\left(\tau'_\varepsilon\right)\right)\right] - w(x_M) \\ &= \mathbb{E}\int_T^{\tau'_\varepsilon} e^{-\rho(t-T)}\left[\left(\mathcal{L}^{\theta(t)} w\right)(X(t)) - \rho w(X(t))\right]dt \leq \mathbb{E}\int_T^{\tau'_\varepsilon} e^{-\rho(t-T)}dt,\end{aligned}$$

from which, since $w(X(\tau'_\varepsilon)) = 0$ and $w(x_M) = -K\varepsilon^2$, we obtain

$$\mathbb{E}\int_T^{\tau'_\varepsilon} e^{-\rho(t-T)}dt \geq K\varepsilon^2.$$

Therefore by (34) we get

$$V(x_M) > \mathbb{E}\left[\int_T^{\tau'_\varepsilon} e^{-\rho(t-T)}U(X(t))dt + e^{-\rho(\tau'_\varepsilon - T)}V\left(X\left(\tau'_\varepsilon\right)\right)\right] + \frac{\nu}{2}K\varepsilon^2$$

contradicting

$$V(x_M) = \sup_{\theta(\cdot)\in\Theta(T,x_M)} \mathbb{E}\left[\int_T^{\tau'_\varepsilon} e^{-\rho(t-T)}U(X(t))dt + e^{-\rho(\tau'_\varepsilon - T)}V\left(X\left(\tau'_\varepsilon\right)\right)\right]$$

which comes from the dynamic programming principle (20).
(iii) Let U be finite in l. If $rl = A$ then V is continuous in l (Remark 4.8) and $\rho V(l) = U(l)$; so the subsolution inequality is immediate from the fact that $H_1(x,p,Q)$ is always nonnegative for $x \geq l$, $p \geq 0$, $Q \leq 0$.

Let now $rl > A$. Take $\psi \in C^2([l,+\infty);\mathbb{R})$ such that l is a maximum point of $V - \psi$ in $[l,+\infty)$. Then we can argue exactly as in point (ii) to get the claim taking right neighborhoods of l instead of whole neighborhoods. ■

Remark 4.14 *The above proof is similar to the one of [Choulli, Taksar & Zhou, 2003], Theorem 1, pp. 1954–1958. We have provided it here since, as far as we know, our problem does not fit exactly into the results contained in [Choulli, Taksar & Zhou, 2003] or in other papers. In Theorem 3.1 of [Zariphopoulou, 1994], pp. 65–69, a different proof of the existence results is given for an HJB equation similar to ours (featuring state constraints and unboudedness of the data).* ■

Remark 4.15 *We are not proving here a comparison theorem. This should be possible, e.g., arguing as in [Zariphopoulou, 1994] Theorem 4.1, p. 69–74, even if our HJB equation is different (see also [Ishii & Loreti, 2002] for uniqueness results in presence of state constraints). We do not do it here for brevity since the comparison result is not essential for our applications.* ■

We now prove the smoothness of V.

Theorem 4.16 *The value function V defined in* (18) *belong to the class* $\mathrm{C}^2((l,+\infty);\mathbb{R})$ *(*$\mathrm{C}([l,+\infty);\mathbb{R}) \cap \mathrm{C}^2((l,+\infty);\mathbb{R})$ *if $V(l)$ is finite).*

To prove this theorem we need the following simple lemmata.

Lemma 4.17 *Suppose g is a concave function on $\mathbb{R}$ such that $g(x) = g(x_0) + a(x - x_0)$ for $x \le x_0$ and $g(x) = g(x_0) + b(x - x_0)$ for $x \ge x_0$, where $a > b$. Then for each sufficiently small $\varepsilon > 0$ there exists a concave $C^2(\mathbb{R};\mathbb{R})$ function $f \ge g$ such that $f(x_0) = g(x_0)$, $f'(x) = a$ for $x \le x_0 + \varepsilon$, $f'(x) = b$ for $x \ge x_0 + \varepsilon$, $f'(x_0) = \frac{(a+b)}{2}$, and $f''(x_0) \le -\varepsilon^{-1}$.*

Proof. This is Lemma 2, p. 1958, of [Choulli, Taksar & Zhou, 2003]. ■

Lemma 4.18 *Let I be a given interval in $\mathbb{R}$, $g \in C^0(I;\mathbb{R})$ and let x_0 be an interior point of I. Assume that there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ such that $x_n < x_0$, $\exists g'(x_n)$ for every $n \in \mathbb{N}$, and $g'(x_n) \to -\infty$ as $x_n \to x_0$. Then $D^+g(x_0) = \emptyset$, where $D^+g(x_0)$ is the superdifferential of g at x_0.*

Proof. If $p \in D^+g(x_0)$ then, for every x in a given neighborhood of x_0, we have

$$g(x) - g(x_0) \le p(x - x_0) + o(x - x_0),$$

so

$$\liminf_{x \to x_0^-} \frac{g(x) - g(x_0)}{x - x_0} \ge p. \tag{35}$$

On the other hand, for every $n \in \mathbb{N}$ we have

$$g(x_n) - g(x_0) = -\left[g'(x_n)(x_0 - x_n) + o(x_0 - x_n)\right],$$

so

$$\lim_{n\to+\infty} \frac{g(x_n) - g(x_0)}{x_n - x_0} = \lim_{n\to+\infty} g'(x_n) = -\infty$$

which contradicts (35). ■

Proof of Theorem 4.16. We first prove that V is differentiable. Since V is concave by the Alexandrov Theorem we know that for a.e. $x \in (l,+\infty)$ there exists $V'(x)$ and $V''(x)$. Let

$x_0 \in (l, +\infty)$ be such that $\nexists V'(x_0)$. Then by concavity the right and left derivatives $V'\left(x_0^+\right)$ and $V'\left(x_0^-\right)$ exist and $V'\left(x_0^-\right) > V'\left(x_0^-\right)$. Moreover the subdifferential $D^- V(x_0)$ is empty and the superdifferential $D^+ V(x_0)$ is the interval $\left[V'\left(x_0^+\right), V'\left(x_0^-\right)\right]$.

Now, using Lemma 4.17 with

$$g(x) = \begin{cases} V(x_0) + V'(x_0^+)(x - x_0) & \text{when} \quad x \geq x_0, \\ V(x_0) + V'(x_0^-)(x - x_0) & \text{when} \quad x < x_0, \end{cases}$$

we get that for every ε there exists $f_\varepsilon \in C^2(\mathbb{R};\mathbb{R})$ such that $f_\varepsilon(x_0) = V(x_0)$, $f_\varepsilon(x) \geq g(x) \geq V(x)$ for $x \in (l, +\infty)$, $f'_\varepsilon(x) = V'(x^+)$ for $x \geq x_0 + \varepsilon$, $f'_\varepsilon(x) = V'(x^-)$ for $x \geq x_0 + \varepsilon$, $f'_\varepsilon(x_0) = \frac{V'(x_0^-)+V'(x_0^+)}{2}$, and $f''_\varepsilon(x_0) \leq \varepsilon^{-1}$. Since V is a viscosity solution of HJB equation (21) (in the form (23)) then

$$\rho V(x_0) \leq (rx_0 - A) f'_\varepsilon(x_0) + U(x_0) + H_1\left(x_0, f'_\varepsilon(x_0), f''_\varepsilon(x_0)\right).$$

For ε sufficiently small the above implies

$$\rho V(x_0) < (rx_0 - A) V'\left(x_0^-\right) + U(x_0). \tag{36}$$

On the other hand, let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence such that, for every $n \in \mathbb{N}$, $x_n > x_0$, $\exists V'(x_n)$, $V''(x_n)$ and $V'(x_n) \to V'\left(x_0^-\right)$, $V''(x_n) \to Q$ for some $Q \in [-\infty, 0]$ when $x_n \to x_0$. Therefore we have

$$\rho V(x_n) = (rx_n - A) V'(x_n) + U(x_n) + H_1\left(x_n, V'(x_n), V''(x_n)\right).$$

Passing to the limit for $n \to +\infty$ we get, if $Q > -\infty$

$$\rho V(x_0) = (rx_0 - A) V'\left(x_0^-\right) + U(x_0) + H_1\left(x_0, V'\left(x_0^-\right), Q\right), \tag{37}$$

if $Q = -\infty$

$$\rho V(x_0) = (rx_0 - A) V'\left(x_0^-\right) + U(x_0). \tag{38}$$

Both equalities (37) and (38) are not compatible with (36), so a contradiction arise and V must be differentiable at x_0.

We now prove the twice differentiability. Again by the Alexandrov Theorem, there exists a set $\mathcal{A} \subseteq (l, +\infty)$ such that the Lebesgue measure of $\mathcal{A}^c = (l, +\infty) - \mathcal{A}$ is zero and V is twice differentiable at every point of $\mathcal{A}$. Let $x_0 \in (l, +\infty)$. Take any sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq \mathcal{A}$ such that $x_n \to x_0$. Then, by the continuous differentiability of V, we get that $V(x_n) \to V(x_0)$ and $V'(x_n) \to V'(x_0)$.

To prove the claim it is enough to prove that the sequence $V''(x_n)$ has a finite limit Q that is the same for every sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq \mathcal{A}$ such that $x_n \to x_0$.

First of all we observe that each element of the sequence $V''(x_n)$ belongs to $(-\infty, 0]$, so there exists at least a subsequence converging either to $-\infty$ or to a finite nonpositive limit.

Now we prove that the limit exists and does not depend on the sequence. Let $\{y_n\}_{n\in\mathbb{N}}$ and $\{z_n\}_{n\in\mathbb{N}}$ two sequences in $\mathcal{A}$ such that $y_n \to x_0$, $z_n \to x_0$ and $V''(y_n) \to Q_1$, $V''(z_n) \to Q_2$ with $Q_1, Q_2 \in [-\infty, 0]$, $Q_1 \neq Q_2$. Therefore, by HJB equation (23), we have

$$\rho V(y_n) = (ry_n - A) V'(y_n) + U(y_n) + H_1\left(y_n, V'(y_n), V''(y_n)\right),$$

$$\rho V(z_n) = (rz_n - A) V'(z_n) + U(z_n) + H_1\left(z_n, V'(z_n), V''(z_n)\right),$$

so passing to the limit we get for $i = 1, 2$

$$\rho V(x_0) = (rx_0 - A) V'(x_0) + U(x_0) + H_1\left(x_0, V'(x_0), Q_i\right)$$

with the formal agreement that $H_1(x_0, V'(x_0), -\infty) = 0$. Since in this way $H_1(x_0, V'(x_0), Q)$ is injective as function of $Q \in [-\infty, 0]$ then we get the claim.

Finally we prove that such limit Q can never be $-\infty$. Assume by contradiction that for a given $x_0 \in (l, +\infty)$ we have $V''(x_n) \to -\infty$, for every sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq \mathcal{A}$ such that $x_n \to x_0$. Take the function defined in $(l, +\infty)$

$$g(x) = \frac{1}{\rho}\left[U(x) + V'(x)(rx - A)\right].$$

For every $x \in (l, +\infty)$ we have $g(x) \le V(x)$. Indeed, arguing as above and calling Q the limit of $V''(x_n)$ for every $x_n \to x$, $\{x_n\} \subseteq \mathcal{A}$, we get

$$\rho V(x) = (rx - A) V'(x) + U(x) + H_1\left(x, V'(x), Q\right) \ge (rx - A) V'(x) + U(x),$$

where the inequality is strict on all points of $\mathcal{A}$ and for points x such that $Q > -\infty$. Moreover $g(x_0) = V(x_0)$ because in such case $Q = -\infty$. Since V is differentiable at x_0 we have

$$g(x) \le V(x) \qquad \text{and} \qquad g(x_0) = V(x_0) \qquad \Longrightarrow \qquad V'(x_0) \in D^+ g(x_0).$$

In particular this means that $D^+ g(x_0) \neq \emptyset$. However, for every sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq \mathcal{A}$ such that $x_n \to x_0$, we have that $\exists g'(x_n)$ and

$$g'(x_n) = \frac{1}{\rho}\left[U'(x_n) + V''(x_n)(rx_n - A) + rV'(x_n)\right],$$

so that

$$\lim_{n\to+\infty} g'(x_n) = -\infty.$$

This is a contradiction thanks to Lemma 4.18. ■

Remark 4.19 *The proof above uses some arguments taken from the proof of Theorem 2, pp. 1958–1960, in [Choulli, Taksar & Zhou, 2003] even if it needs new ideas since* here we do not have uniform ellipticity *of the second order term. In Theorem 5.1 of [Zariphopoulou, 1994], pp. 78–82, a similar regularity result is proved for a similar HJB equation with a different technique (restriction to bounded intervals where the equation is proved to be uniformly elliptic). In any case the result of [Zariphopoulou, 1994] cannot be applied as it is to this case.*

We also notice that the above proof indeed states the C^2 interior regularity for every concave viscosity solution of HJB equation (23) *in* $(l, +\infty)$. ■

Remark 4.20 *We can get more regularity of the value function. In particular observe that from the HJB equation, for any $x_0 \in (l, +\infty)$, if $-\frac{\lambda V'(x_0)}{\sigma x_0 V''(x_0)} < 1$ then in a suitable neighborhood of x_0*

$$V''(x) = -\frac{\lambda^2 \left[V'(x)\right]^2}{\rho V(x) - (rx - A) V'(x) - U(x)}, \tag{39}$$

so V'' is differentiable at x_0. Similarly if $\frac{\lambda V'(x_0)}{\sigma x_0 V''(x_0)} > 1$ (or even when $V''(x_0) = 0$) then in a suitable neighborhood of x_0

$$V''(x) = \frac{2}{\sigma^2 x^2}\left[\rho V(x) - (rx + \sigma\lambda x - A) V'(x) - U(x)\right],$$

so V'' is differentiable at x_0. ■

Now we give a Corollary which will be very useful in proving the verification theorem for the case $rl > A$ in next subsection.

Corollary 4.21 *The value function is strictly concave and satisfies*

$$V'(x) > 0, \qquad V''(x) < 0,$$

for $x \in (l, +\infty)$. Moreover if $rl > A$ and $U(l) > -\infty$ we have

$$V''(x) \longrightarrow -\infty \qquad \textit{when} \quad x \longrightarrow l^+, \tag{40}$$

and if $U'(l^+)$ is finite

$$(x-l)\left[V''(x)\right]^2 \longrightarrow \frac{\lambda^2 \left[V'(l^+)\right]^2}{2(rl-A)} \qquad \textit{when} \quad x \longrightarrow l^+. \tag{41}$$

Proof. The fact that $V'(x) > 0$ for $x > l$ comes from concavity and strict monotonicity. Moreover if $V''(x) = 0$ in an interval $[a,b] \subseteq (l, +\infty)$ then on this interval V' is constant (say equal to $c > 0$) and $V(x) = cx + d$. Therefore for $x \in [a,b]$ HJB equation (23) becomes

$$\rho(cx+d) = [(r+\lambda\sigma)x - A]c + U(x),$$

which is impossible since U is strictly concave. Now let $x_0 > l$ such that $V''(x_0) = 0$. In this case the maximum of $H_{1,cv}(x_0, V'(x_0), V''(x_0); \theta)$ is reached for $\theta = 1$, and so we have taking HJB equation (23) for x in sufficiently small neighborhood $I(x_0)$ of x_0

$$\rho V(x) - (rx - A)V'(x) - U(x) = \sigma\lambda x V'(x) + \frac{1}{2}\sigma^2 x^2 V''(x).$$

Call now, for $x \in I(x_0)$,

$$h(x) = \frac{1}{2}\sigma^2 x^2 V''(x) = \rho V(x) - (rx + \sigma\lambda x - A)V'(x) - U(x).$$

Clearly h has a local maximum at x_0 and is twice differentiable at x_0 thanks to Remark 4.20. So it must be $h'(x_0) = 0$ and $h''(x_0) \le 0$. Now

$$\begin{aligned} h'(x) &= (\rho - r - \lambda\sigma)V'(x) - U'(x) - V''(x)(rx + \sigma\lambda x - A) \\ h''(x) &= (\rho - 2r - 2\lambda\sigma)V''(x) - U''(x) - V'''(x)(rx + \sigma\lambda x - A) \end{aligned}$$

and therefore, using that $V''(x_0) = 0$, we obtain

$$\begin{aligned} h'(x_0) &= (\rho - r - \lambda\sigma)V'(x_0) - U'(x_0) \\ h''(x_0) &= -U''(x_0) - V'''(x_0)(rx_0 + \sigma\lambda x_0 - A). \end{aligned}$$

Since x_0 is also a maximum for V'', it is clearly $V'''(x_0) = 0$ and consequently $h''(x_0) = -U''(x_0) > 0$ by Hypothesis 3.4-(i), a contradiction.

We now prove (40). Observe that, for $x > l$

$$\rho V(x) - (rx - A)V'(x) - U(x) = H_1\left(x, V'(x), V''(x)\right).$$

Take any sequence $x_n \to l^+$ such that $V''(x_n) \to Q \in [-\infty, 0]$. Then, passing to the limit for $n \to +\infty$ in the HJB equation above, we get

$$\rho V(l) - (rl - A) V'\left(l^+\right) - U(l) = H_1\left(l, V'\left(l^+\right), Q\right). \tag{42}$$

On the other hand by concavity we know that, for $x \geq l$,

$$V(x) \leq V(l) + V'\left(l^+\right)(x - l).$$

Applying Lemma 4.17 with (δ is a given positive number)

$$g(x) = \begin{cases} V(l) + V'(l^+)(x - l) & \text{when} \quad x \geq l, \\ V(l) + (V'(l^+) + \delta)(x - l) & \text{when} \quad x < l, \end{cases}$$

we find f_ε defined on $\mathbb{R}$ such that $f_\varepsilon(x) \geq V(l) + V'(l^+)(x - l)$ for $x \geq l$, $f_\varepsilon(l) = V(l)$, $f_\varepsilon'(x) = V'(l^+)$ for $x \geq l + \varepsilon$, $f_\varepsilon'(l) \in [V'(l^+), V'(l^+) + \delta]$, $f_\varepsilon''(l) \leq -\varepsilon^{-1}$. Then we have, for $x \geq l$

$$0 = V(l) - f_\varepsilon(l) \geq V(x) - f_\varepsilon(x),$$

so that, being V a subsolution of HJB equation (23) at $x = l$,

$$\rho V(l) - (rl - A) f_\varepsilon'(l) - U(l) \leq H_1\left(l, f_\varepsilon'(l), f_\varepsilon''(l)\right).$$

This gives

$$\rho V(l) - (rl - A) V'\left(l^+\right) - U(l) \leq \varepsilon(rl - A) + \varepsilon\lambda^2\left(V'\left(l^+\right) + \varepsilon\right)^2,$$

and by the arbitrariness of ε

$$\rho V(l) - (rl - A) V'\left(l^+\right) - U(l) \leq 0. \tag{43}$$

This means, using (42), that (43) holds with $=$ and

$$H_1\left(l, V'\left(l^+\right), Q\right) = 0 \quad \Longrightarrow \quad Q = -\infty.$$

The claim follows by a standard argument on subsequences.

Finally we prove (41). First observe that, for x in a suitable right neighborhood of l, we must have as a consequence of (40)

$$\frac{\lambda V'(x)}{\sigma x V''(x)} < 1,$$

so that by (39) we obtain

$$(x - l)\left[V''(x)\right]^2 = \lambda^4\left[V'(x)\right]^4 \cdot \frac{(x - l)}{\left[\rho V(x) - (rx - A) V'(x) - U(x)\right]^2}$$

To calculate the limit of the second factor we use the Bernoulli-De l'Hôpital rule. The ratio of the derivatives is (using (39) to rewrite it)

$$\frac{1}{2\left[\rho V(x) - (rx - A) V'(x) - U(x)\right]\left[(\rho - r) V'(x) - (rx - A) V''(x) - U'(x)\right]}$$
$$= \frac{1}{2\lambda^2\left[V'(x)\right]^2} \cdot \frac{-V''(x)}{(\rho - r) V'(x) - (rx - A) V''(x) - U'(x)}.$$

Since $U'(l^+)$ is finite the limit of the second factor is clearly $\frac{1}{rl - A}$, so the claim is proved. ■

Remark 4.22 *Equation* (40) *(and also* (41)*) is a consequence of the boundary condition, i.e. of the subsolution inequality at the boundary (which depends on the structure of the second order superdifferential at the boundary). In particular we can say that if* $rl > A$ *and* $U(l) > -\infty$ *the value function* V *solves HJB equation* (23) *on* $[l, +\infty)$ *with the usual agreement that* $H_1(l, V'(l^+), -\infty) = 0$.

Similar results can be proved in the case when $rl = A$ *but we will not need them since in that case we will only study a special case where explicit solutions are available (Subsection 4.4).*

Finally the claim of Corollary 4.21 holds for every concave constrained viscosity solution of equation (23) *on* $[l, +\infty)$ *when* $rl > A$ *and* $U(l)$, $U'(l)$ *are finite.* ■

4.3 The verification theorem and the optimal policies when $rl > A$

We now prove a verification theorem and the existence of optimal feedbacks when $rl > A$ and $U(l)$, $U'(l^+)$ are finite. We start by a lemma.

Lemma 4.23 *Let* $rl > A$ *and* $U(l), U'(l)$ *be finite. Set*

$$G(x) = \begin{cases} -\frac{\lambda V'(x)}{\sigma x V''(x)} \wedge 1 & \text{when} \quad x > l, \\ 0 & \text{when} \quad x \le l. \end{cases} \tag{44}$$

For every $x \ge l$*, the closed loop equation*

$$\begin{cases} dX(t) = \left([\sigma\lambda G(X(t)) + r] X(t) - A\right) dt + \sigma X(t) G(X(t))\, dB(t), \quad t \ge T, \\ X(T) = x \ge l, \end{cases} \tag{45}$$

has a unique strong solution $X_G(\cdot; T, x)$*. Moreover for every* $t \ge T$ *we have* $X_G(t; T, x) \ge l$ $\mathbb{P}-a.s.$.

Proof. First of all we apply the Girsanov Theorem as, e.g., in the proof of Proposition 4.4. Under the probability $\tilde{\mathbb{P}}$ for every $t \ge T$ we have

$$\begin{cases} dX(t) = (rX(t) - A)\, dt + G(X(t))\, \sigma X(t)\, d\tilde{B}(t), \\ X(T) = x. \end{cases}$$

Such equation has a weak solution under $\tilde{\mathbb{P}}$. Indeed, G is continuous since V, V', V'' are continuous and since $V''(l^+) = -\infty$ while $V'(l^+) < +\infty$. Moreover G is clearly bounded. Thus applying Theorem 2.3, p. 159, and Theorem 2.4, p. 163, of [Ikeda & Watanabe, 1981] we get the existence of a weak solution.

Now we want to apply Theorem 3.5 (ii), p. 390, of [Revuz & Yor, 1999] (see also Proposition 2.13, p. 291, of [Karatzas & Shreve, 1991]). Since the drift is Lipschitz it is enough to prove that the diffusion coefficient is $\frac{1}{2}$-Holder continuous. This is guaranteed by (41) since, by a straightforward calculations, G is locally Lipschitz out of a right neighborhood of l (that we call $I(l^+)$) while in $I(l^+)$ we have

$$G(x) = \frac{\lambda V'(x)}{\sigma x V''(x)} = \frac{2}{\lambda\sigma x V'(x)} H_1(x, V'(x), V''(x)) = \frac{2}{\lambda\sigma x V'(x)} \left[\rho V(x) - (rx - A)V'(x) - U(x)\right].$$

Using that $G(l) = 0$ and $\rho V(l) - (rl - A)V'(l^+) - U(l) = 0$, we have in $I(l^+)$

$$|G(x) - G(l)| \le K_1 |x - l| + K_2 \left|V'(x) - V'\left(l^+\right)\right| \le K_3 \int_l^x V''(y)\, dy,$$

so by (41) we get

$$|G(x) - G(l)| \leq K_4 |x-l|^{\frac{1}{2}}.$$

This proves pathwise uniqueness which implies by Yamada-Watanabe theory existence and uniqueness of a strong solution (see, e.g., Section 5.3.D, pp. 308–311, of [Karatzas & Shreve, 1991]). The claim for the original equation follows simply applying the Girsanov transform.

To prove that $X_G(t) \geq l \ \mathbb{P}-a.s.$, it is enough to argue by contradiction using that $G(x)=0$ when $x \leq l$. ■

Theorem 4.24 *Let $rl > A$ and $U(l), U'(l)$ be finite. Then, for every $x \geq l$, the control strategy $\theta^*(\cdot) \in \Theta(T,x)$ such that*

$$\theta^*(t) = G(X_G(t;T,x)),$$

where G is given by (44) *and $X_G(\cdot;T,x)$ is the unique strong solution of* (45)*, is the unique optimal strategy at (T,x).*

Proof. We cannot proceed with the standard proof of the verification theorem in the regular case (see on this, e.g., [Yong & Zhou, 1999], p. 268) since the function V is not C^2 up to the boundary. Thus we use an approximation procedure. Given any $\varepsilon > 0$ we define a function $V_\varepsilon \in C^2(\mathbb{R})$ such that

- $V_\varepsilon(x) = V(x)$ in $(l+\varepsilon, +\infty)$;
- $V_\varepsilon(x) = a_1 + b_1 x + c_1 x^2$ in $\left(\frac{l+\frac{A}{r}}{2}, l+\varepsilon\right)$, where

$$\begin{aligned} c_1 &= \frac{1}{2} V''(l+\varepsilon), \\ b_1 &= V'(l+\varepsilon) - V''(l+\varepsilon)(l+\varepsilon), \\ a_1 &= V(l+\varepsilon) - V'(l+\varepsilon)(l+\varepsilon) + \frac{1}{2} V''(l+\varepsilon)(l+\varepsilon)^2; \end{aligned}$$

- $V'_\varepsilon(x) \geq 0$ in $\mathbb{R}$ and $V'_\varepsilon(x) = 0$ for $x \leq \frac{A}{r}$. To define V'_ε on $\left[\frac{A}{r}, \frac{l+\frac{A}{r}}{2}\right]$ it is enough to take a suitable third degree polynomial.

Since, for $x \in [l, l+\varepsilon]$

$$\begin{gathered} V_\varepsilon(x) - V(x) = V(l+\varepsilon) - V(x) - V'(l+\varepsilon)(l+\varepsilon-x) + \frac{1}{2} V''(l+\varepsilon)(l+\varepsilon-x)^2 \\ V'_\varepsilon(x) - V'(x) = V'(l+\varepsilon) - V'(x) - V''(l+\varepsilon)(l+\varepsilon-x) \\ V''_\varepsilon(x) - V''(x) = V''(l+\varepsilon) - V''(x), \end{gathered}$$

using that $\varepsilon V''(l+\varepsilon) \to 0$ as $\varepsilon \to 0$ (see (41)), we have

$$V_\varepsilon \longrightarrow V, \qquad V'_\varepsilon \longrightarrow V', \qquad \text{uniformly in } [l, +\infty), \tag{46}$$

and

$$H_1\left(x, V'_\varepsilon(x), V''_\varepsilon(x)\right) \longrightarrow H_1\left(x, V'(x), V''(x)\right), \qquad \text{uniformly in } [l, +\infty). \tag{47}$$

We claim that V_ε solves in $\mathbb{R}$ the HJB equation

$$\rho V_\varepsilon(x) - (rx - A) V_\varepsilon'(x) - H_1\left(x, V_\varepsilon'(x), V_\varepsilon''(x)\right) = g_\varepsilon(x) \tag{48}$$

where $g_\varepsilon \to U$ uniformly in $[l, +\infty)$ as $\varepsilon \to 0$ while $g_\varepsilon(x) \to -\infty$ for every $x < l$.

Indeed, (46), (47), and Remark 4.22 imply immediately that $g_\varepsilon \to U$ uniformly in $[l, +\infty)$. Moreover it is clear by its definition that $V_\varepsilon(x) \to -\infty$ for every $x < l$ and that $V_\varepsilon'(x) = 0$ for every $x < \frac{A}{r}$. Since $H_1(x, V_\varepsilon'(x), V_\varepsilon''(x)) \geq 0$ then we have

$$g_\varepsilon(x) \leq \rho V_\varepsilon(x), \qquad \forall x < l,$$

and so the claim.

Take $x \geq l$ and $\theta : [T, +\infty) \times \Omega \to [0,1]$ adapted to $\{\mathcal{F}_t^B\}_{t \geq T}$. Here we do not require that $X(t; T, x, \theta) \geq l$. Consider the function $(t, x) \to e^{-\rho(t-T)} V_\varepsilon(x)$ and apply to it the Dynkin formula for the process $X(\cdot) = X(\cdot; T, x, \theta)$. We have, for $t_1 \geq T$

$$\mathbb{E}\left[e^{-\rho(t_1-T)} V_\varepsilon(X(t_1)) - V_\varepsilon(x)\right] = \mathbb{E}\int_T^{t_1} e^{-\rho(t_1-T)} \left[-\rho V_\varepsilon(X(t)) + \mathcal{L}^{\theta(t)} V_\varepsilon(X(t))\right] dt,$$

so by (48)

$$\begin{aligned}
&\mathbb{E}\left[e^{-\rho(t_1-T)} V_\varepsilon(X(t_1)) - V_\varepsilon(x)\right] \\
&\qquad = \mathbb{E}\int_T^{t_1} e^{-\rho(t-T)} \Big[- g_\varepsilon(X(t)) - H_1\left(X(t), V_\varepsilon'(X(t)), V_\varepsilon''(X(t))\right) \\
&\qquad\qquad - (rX(t) - A) V_\varepsilon'(X(t)) + \mathcal{L}^{\theta(t)} V_\varepsilon(X(t)) \Big] dt
\end{aligned}$$

which implies

$$\begin{aligned}
V_\varepsilon(x) = {}& \mathbb{E}\left[\int_T^{t_1} e^{-\rho(t-T)} g_\varepsilon(X(t))\, dt + e^{\rho(t_1-T)} V_\varepsilon(X(t_1))\right] \\
&+ \mathbb{E}\int_T^{t_1} e^{-\rho(t-T)} \big[H_1\left(X(t), V_\varepsilon'(X(t)), V_\varepsilon''(X(t))\right) \\
&\qquad - H_{1,cv}\left(X(t), V_\varepsilon'(X(t)), V_\varepsilon''(X(t)); \theta(t)\right) \big] dt.
\end{aligned}$$

Sending $t_1 \to +\infty$ we get $e^{-\rho(t_1-T)} V_\varepsilon(X(t_1)) \to 0$ by using (7), the last statement of Proposition 4.4, and estimating $\mathbb{E}\left[X(t)^\beta\right]$ as in its proof. Therefore

$$\begin{aligned}
V_\varepsilon(x) = {}& \mathbb{E}\int_T^{+\infty} e^{-\rho(t-T)} g_\varepsilon(X(t))\, dt \\
&+ \mathbb{E}\int_T^{+\infty} e^{-\rho(t-T)} \big[H_1\left(X(t), V_\varepsilon'(X(t)), V_\varepsilon''(X(t))\right) \\
&\qquad - H_{1,cv}\left(X(t), V_\varepsilon'(X(t)), V_\varepsilon''(X(t)); \theta(t)\right) \big] dt.
\end{aligned} \tag{49}$$

Now we take $\theta(\cdot) \in \Theta(T, x)$ and send $\varepsilon \to 0^+$ in the above formula. We have by the proof above

$$\begin{aligned}
V_\varepsilon(x) &\longrightarrow V(x), \\
\mathbb{E}\int_T^{+\infty} e^{-\rho(t-T)} g_\varepsilon(X(t))\, dt &\longrightarrow J_T(x; \theta(\cdot)), \\
\mathbb{E}\int_T^{+\infty} e^{-\rho(t-T)} H_1\left(X(t), V_\varepsilon'(X(t)), V_\varepsilon''(X(t))\right) dt &\longrightarrow \mathbb{E}\int_T^{+\infty} e^{-\rho(t-T)} H_1\left(X(t), V'(X(t)), V''(X(t))\right) dt.
\end{aligned}$$

This means that also the limit

$$\lim_{\varepsilon\to 0^+} \mathbb{E}\int_T^{+\infty} e^{-\rho(t-T)} H_{1,cv}\left(X(t), V_\varepsilon'(X(t)), V_\varepsilon''(X(t)); \theta(t)\right) dt$$

exists. Take now the closed loop strategy

$$\theta^*(t) = G(X_G(t;T,x)),$$

where G is given by (44) and $X_G(\cdot;T,x)$ is the unique strong solution of (45). If we prove that (setting $X_G(t;T,x) = X_G(t)$ for brevity and recalling that we have set $H_1(l, V'(l^+), -\infty) = 0$)

$$\lim_{\varepsilon\to 0^+} \mathbb{E}\int_T^{+\infty} e^{-\rho(t-T)} H_{1,cv}\left(X_G(t), V_\varepsilon'(X_G(t)), V_\varepsilon''(X_G(t)); \theta^*(t)\right) dt$$
$$= \mathbb{E}\int_T^{+\infty} e^{-\rho(t-T)} H_1\left(X_G(t), V'(X_G(t)), V''(X_G(t))\right) dt \quad (50)$$

then, passing to the limit in (49), we obtain

$$V(x) = J(T, x; \theta^*(\cdot)),$$

and so the optimality of $\theta^*(\cdot)$. To prove (50) it is enough to observe that

$$H_{1,cv}\left(x, V_\varepsilon'(x), V_\varepsilon''(x); G(x)\right) \longrightarrow$$
$$\longrightarrow H_1\left(x, V'(x), V''(x)\right) = \begin{cases} H_{1,cv}(x, V'(x), V''(x); G(x)) & \text{if } x > l, \\ 0 & \text{if } x = l, \end{cases} \quad (51)$$

uniformly as $\varepsilon \to 0^+$. Indeed, the convergence for $x = l$ is obvious. Moreover, for $x \in (l, +\infty)$,

$$H_{1,cv}\left(x, V_\varepsilon'(x), V_\varepsilon''(x); G(x)\right) - H_{1,cv}\left(x, V'(x), V''(x); G(x)\right)$$
$$= G(x)\lambda\sigma x\left[V_\varepsilon'(x) - V'(x)\right] + \frac{1}{2}G^2(x)\sigma^2 x^2\left[V_\varepsilon''(x) - V''(x)\right].$$

The first term goes to 0 uniformly as $\varepsilon \to 0^+$ thanks to (46) while the second is, for ε sufficiently small and $x \in (l, l+\varepsilon)$ (for $x \geq l+\varepsilon$ it is zero),

$$\frac{1}{2}\lambda^2\left(V'(x)\right)^2 \frac{V_\varepsilon''(x) - V''(x)}{\left[V''(x)\right]^2}.$$

Since $\frac{V_\varepsilon''(x)-V''(x)}{[V''(x)]^2}$ is negative and greater than $[V''(l+\varepsilon)]^{-1}$ the convergence (51) follows, and so (50) and the optimality of $\theta^*(\cdot)$. The uniqueness follows from the strict concavity of U arguing as in the proof of Proposition 4.5: one takes two different optimal strategies θ_1 and θ_2 with corresponding trajectories X_1 and X_2 and one proves that for any $\eta \in [0,1]$ there exists an admissible strategy θ_η whose associated trajectory is $\eta X_1 + (1-\eta)X_2$. Then the strict concavity of U implies that $J_T(x, \theta_\eta(\cdot)) < J_T(x, \theta_1(\cdot)) = J_T(x, \theta_2(\cdot)) = V(x)$, a contradiction. ■

Remark 4.25 *If $rl > A$ and $U(x) = \gamma^{-1}\left(x - \frac{A}{r}\right)^\gamma$ then, arguing as in the proof of Proposition 4.26, one can see that the function*

$$v(x) = \gamma^{-1}\left(\rho - \gamma r - \frac{\lambda^2\gamma}{2(1-\gamma)}\right)^{-1}\left(x - \frac{A}{r}\right)^\gamma$$

is a regular solution of HJB equation (23) *in* $(l, +\infty)$ *when* $\lambda \le \sigma(1-\gamma)$*. However this function* v *is not a constrained viscosity solution since it does not satisfy* (40) *that comes from the boundary condition (see on this Remark 4.22), and so* v *is not the value function.*

In next section we will see that we have $v = V$ *when* $rl = A$*. Therefore, from the arguments of next section, it follows that when* $rl > A$ *the function* v *is the value function if the state constraint* $x \ge l$ *is replaced by* $x \ge \frac{A}{r}$*, so we clearly have* $v \ge V$*.*

Finally we observe that the proof of the above Theorem 4.24 works if V *is replaced by any concave constrained viscosity solution of equation* (23) *in* $[l, +\infty)$*. So as a byproduct of it we get that the value function is the unique concave constrained viscosity solution of equation* (23) *in* $[l, +\infty)$*.* ■

4.4 An example when $rl = A$ with explicit solution

In the case of $rl = A$ and $U'(l^+) = +\infty$ it is possible to prove a general verification theorem on the line of Theorem 4.24. We do not do it here for brevity but we study a special case where, differently from the case $rl > A$, the explicit form of the value function and of the optimal couples is available. The utility function is given by

$$U(x) = \frac{(x-l)^\gamma}{\gamma}, \qquad \gamma \in (-\infty, 0) \cup (0, 1)\,. \tag{52}$$

This utility function is defined for any $x \ge l$ if $\gamma \in (0,1)$, and for any $x > l$ if $\gamma \in (-\infty, 0)$; therefore the set of admissible strategies is never empty thanks to Lemma 3.9. Moreover it always satisfies Hypothesis 3.4. Notice that, considering the utility as a function of $x - l$, the above specification represents constant relative risk aversion preferences. The case of logarithmic utility may be treated in the same way but we do not do it for brevity.

We look for a solution of HJB equation (23) of the form

$$v\,(x) = C\frac{(x-l)^\gamma}{\gamma}, \qquad \gamma \in (-\infty, 0) \cup (0, 1), \tag{53}$$

for a suitable constant C. Substituting into HJB equation (23) we see that it must be

$$C = \left(\rho - \gamma r - \frac{\lambda^2\gamma}{2\,(1-\gamma)}\right)^{-1}, \tag{54}$$

under the conditions

$$\rho > \gamma r + \frac{\lambda^2\gamma}{2(1-\gamma)}, \qquad \text{and} \qquad \lambda \le \sigma\,(1-\gamma)\,. \tag{55}$$

The first condition is necessary in order to grant the finiteness of the value function. It is guaranteed by (7) for $\gamma \in (0,1)$, and is always true for $\gamma < 0$. The second condition guarantees that the maximum point in the Hamiltonian is ≤ 1, so the no borrowing constraint is never active: this allows to keep H_1 in the form which is suitable to find the explicit solution[15].

The main result of this section is the following.

[15] Indeed, when $\lambda > \sigma(1-\gamma)$ it is not difficult to see that $V(x) < v(x)$ for any $x > l$ using the fact that v is the value function of a problem with larger control set whose optimal trajectory is not admissible for our problem.

Proposition 4.26 *Let conditions* (55) *be verified and the utility function be given by* (52) *with* $\gamma \in (-\infty, 0) \cup (0, 1)$*. Then* v *given in* (53)*, with* C *given by* (54)*, is the value function, i.e.*

$$V(x) = \gamma^{-1} \left[\rho - \gamma r - \frac{\lambda^2 \gamma}{2(1-\gamma)} \right]^{-1} (x-l)^{\gamma}, \qquad x \geq l \quad (x > l \text{ when } \gamma < 0).$$

Proof. We need to prove that the solution of HJB equation (23) in this case is the value function. To do this we argue as in the standard verification theorem. We know that the function v given in (53) satisfies the following HJB equation for $x > l$

$$\rho v(x) - \frac{(x-l)^{\gamma}}{\gamma} - v'(x)(rx - A) - H_1\left(x, v'(x), v''(x)\right) = 0. \tag{56}$$

Let us take $x > l$ and $\theta(\cdot) \in \Theta_{ad}(T, x)$ with the associated state trajectory $X(\cdot)$. Assume that, for every $t \geq T$, $X(t) > l$ $\mathbb{P}-$a.s., and apply the Dynkin formula to the function $(t, x) \to (e^{-\rho(t-T)} v(x))$ with the process $X(\cdot)$. For any $t_1 \geq T$, we obtain

$$\mathbb{E}\left[e^{-\rho(t_1-T)} v(X(t_1)) - v(x)\right] = \mathbb{E}\int_T^{t_1} e^{-\rho(t-T)} \left(-\rho v(X(t)) + \mathcal{L}^{\theta(t)} v(X(t))\right) dt.$$

As $t_1 \to +\infty$ we get, as in the proof of Theorem 4.24,

$$v(x) = \mathbb{E}\int_T^{+\infty} e^{-\rho(t-T)} \left(\rho v(X(t)) - \mathcal{L}^{\theta(t)} v(X(t))\right) dt$$

that is, recalling (56),

$$v(x) = \mathbb{E}\int_T^{+\infty} e^{-\rho(t-T)} \left[\frac{(X(t)-l)^{\gamma}}{\gamma} + v'(X(t)) + (rX(t) - A) \right.$$
$$\left. + H_1\left(X(t), v'(X(t)), v''(X(t))\right) - \mathcal{L}^{\theta}(t) v(X(t)) \right] dt,$$

and thus

$$v(x) = J_T(x; \theta(\cdot)) + \mathbb{E}\int_T^{+\infty} e^{-\rho(t-T)} \left[H_1\left(X(t), v'(X(t)), v''(X(t))\right) \right.$$
$$\left. - H_{1,cv}\left(X(t), v'(X(t)), v''(X(t)); \theta(t)\right) \right] dt, \tag{57}$$

which implies

$$v(x) \geq J_T(x; \theta(\cdot)).$$

Assume now that the trajectory $X(\cdot)$ touches the boundary in finite time with positive probability (hence it remain there forever since $rl = A$). If $\gamma < 0$ then this trajectory has payoff $-\infty$, and so it cannot be optimal. Therefore in this case we can restrict to strategies that keep the trajectory strictly above l. If $\gamma \in (0, 1)$ we argue as follows. Define

$$\tau_l = \inf\{t \geq T : X(t) = l\},$$

and arguing as above, for every $\varepsilon > 0$, we get

$$v(x) = \mathbb{E}\left[\int_T^{\tau_l - \varepsilon} e^{-\rho(t-T)} U(X(t))\, dt + e^{-\rho(\tau_l - \varepsilon - T)} v(X(\tau_l - \varepsilon))\right]$$
$$+ \mathbb{E}\int_T^{\tau_l - \varepsilon} e^{-\rho(t-T)} \left[H_1\left(X(t), v'(X(t)), v''(X(t))\right) - H_{1,cv}\left(X(t), v'(X(t)), v''(X(t)); \theta(t)\right) \right] dt.$$

Letting $\varepsilon \to 0$ we see that the term in the first line have a limit and so also the term in the second line has a limit. Since the term in the second line is always positive we obtain

$$v(x) \geq \mathbb{E}\left[\int_T^{\tau_l} e^{-\rho(t-T)} U(X(t))\,dt + e^{-\rho(\tau_l - T)} v(X(\tau_l))\right]. \tag{58}$$

Now

$$v(X(\tau_l)) = v(l) = \frac{U(l)}{\rho} = \int_{\tau_l}^{+\infty} e^{-\rho(t-\tau_l)} U(l)\,dt,$$

so, by (58), $v(x) \geq J_T(x;\theta(\cdot))$. This implies $v(x) \geq V(x)$. Taking the feedback map

$$G(x) = \frac{\lambda}{\sigma(1-\gamma)} \frac{x-l}{x}, \tag{59}$$

we see that the associated closed loop equation is

$$\begin{cases} dX(t) = \left(\dfrac{\lambda^2}{1-\gamma} + r\right)(X(t)-l)\,dt + \frac{\lambda}{1-\gamma}(X(t)-l)\,dB(t), \\ X(T) = x. \end{cases} \tag{60}$$

Such equation is linear and it has a unique strong solution $X_G(\cdot;T,x)$ for every $x \geq l$. Moreover $X_G(\cdot;T,l) \equiv l$ $\mathbb{P}-$a.s. while, for $x > l$, $X_G(\cdot;T,x) > l$ $\mathbb{P}-$a.s.. Choosing

$$\theta^*(t) = \frac{\lambda}{\sigma(1-\gamma)} \frac{X_G(t)-l}{X_G(t)} \tag{61}$$

we see that it is optimal since the second term of the right hand side of (57) is zero, so also $v(x) = V(x)$, the claim. ■

An immediate consequence of the above proof is the following result.

Corollary 4.27 *The optimal risky asset investment strategy under conditions* (55) *for any $x > l$ is given by* (61)*, where $X_G(\cdot)$ is the unique strong solution of the closed loop equation* (60)*.*

5 Analysis of the optimal policies

We discuss now the properties of the optimal policies described in Subsections 4.3 and 4.4 in the cases

- $rl > A$, $U(l)$ and $U'(l)$ finite;
- $rl = A$, $U'(l) = +\infty$.

First of all observe that in both cases the optimal feedback map is given by the function G of (44) which, when < 1, is (similarly to the Merton model) the product of the payoff for every unit of risk $\frac{\lambda}{\sigma}$ and of the quantity $-\frac{V'}{x V''}$, i.e. the Arrow-Pratt measure of risk tolerance of the indirect utility function V (the value function).

This implies that the optimal feedback map is increasing with the payoff per unit of risk and with the relative risk aversion of V, while the relation between the optimal policy and the level of wealth is known only implicitly, unless we know the explicit expression of V.

In the case of $rl > A$, $U(l)$ and $U'(l)$ finite, even taking a CRRA utility function the explicit form of the value function is not available. As seen in Remark 4.25 the natural candidate solution of HJB equation (23) does not satisfy the required boundary condition. This comes from the presence of the state constraints $x \geq l$ and from the fact that the control $\theta \equiv 0$ bring the state from the boundary $x = l$ in the interior of the state region.

So, even starting from initial wealth equal to the solvency level l, the set of admissible strategies does not reduce to the trivial one (investment in the riskless asset forever) but allows to the fund manager to reinvest in the risky asset. Moreover, starting from an initial wealth $x > l$, the boundary is always reached with positive probability.

The possibility to exit from the boundary l (if the wealth process starts from or reaches it) is given by the fact that the capital amount l invested in the riskless asset will generate a return per unit of time rl. Hence the accrued return will produce disposable wealth to be invested in the risky asset and the wealth process can exit from the trivial state $X(\cdot) \equiv l$.

When $rl = A$, $U'(l) = +\infty$, and the utility function is in CRRA form (under constraints on the parameters), an explicit form of V is available and it is exactly the natural candidate solution of HJB equation (23). Indeed, here the situation at the boundary is different. The control $\theta \equiv 0$ leaves forever the state in the boundary $x = l$, so when the initial wealth x equals l the unique admissible allocation strategy is given by investing all the wealth in the riskless asset forever, and no risky investment is allowed. On the contrary, when initial wealth x is strictly greater than l the fund wealth will never reach the solvency level.

Concerning the case $rl = A$ treated in Subsection 4.4 the explicit form of the value function allows us to make a further consideration. According to the common sense, the portfolio selection rule (59) suggests to increase the fraction invested in the risky asset if the wealth level grows, and diminish the share invested in it if the fund level decreases. Indeed, this kind of policy seems to be reasonable with the social target of a pension fund, whose manager must be interested in protecting the wealth level and in caring about the risk the portfolio strategies involve.

Finally, we observe that within our model (whether the case of $rl > A$, $U(l)$ and $U'(l)$ finite, or the case of $rl = A$ and $U'(l) = +\infty$) we have similar results if we assume that the portfolio strategy $\theta(\cdot)$ belong to $[0, \theta_0]$ with $\theta_0 < 1$, i.e. if the pension fund is forced not to invest the total amount of its wealth in the risky asset. Sometimes this constraint is imposed by the supervisory authority.

6 Conclusions

We have formulated and studied a model for the optimal management of a defined contribution pension fund with a minimum guarantee formulated as a stochastic control problem. Our emphasis has been put on the constraints faced by the fund manager: we model the inflows and outflows generated by contributions and benefits under demographic stationarity, the requirement of having a solvency level on the fund wealth, and the borrowing and short selling constraints on the allocation strategies.

We have developed the dynamic programming approach to characterize the optimal policies when the solvency level is deterministic. We have proved that the value function is a smooth solution of the associated HJB equation and then we have used this result to write the optimal policies in feedback form.

In the case of $rl > A$ we have seen, for generic utility functions with $U(l)$ and $U'(l)$ finite, that the solvency level l can be reached with positive probability but the optimal portfolio strategy does not become trivial, i.e. the fund manager can still reinvest, later, in the risky asset. Also starting from $x = l$ would give rise to nontrivial investment strategies.

In the specific case of $rl = A$ we have seen, for a special class of utility functions with $U'(l)$ infinite, that the solvency level is never reached when the initial wealth is $x > l$. Moreover in this setting an explicit solution is provided and discussed.

Future developments concern the study of the model described in Section 2 in a more realistic framework: in particular when the interest rate and the wage rate are stochastic and when benefits includes the surplus as represented in (4). Also the introduction of demographic risk (relaxing the hypothesis of demographic stationarity) is currently under study.

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