MONETARY UTILITY FUNCTIONS, BSDE AND QUASI-LINEAR PDE

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Definition of Risk Measures

Notation: $(\Omega, (\mathcal{F}_t)_{0 \le t \le T}, \mathbf{P})$ a filtered probability space with the usual assumptions.

$L^\infty(\Omega, \mathcal{F}, \mathbf{P})$ space of bounded random variables,
$L^1(\Omega, \mathcal{F}, \mathbf{P})$ space of integrable RV.

Liabilities are with $-$ sign!! Wealth is with $+$ sign. Bankruptcy means "under zero".

Utility functions are defined on random variables, not on "lotteries".

Definition. *$u\colon L^\infty \to \mathbb{R}$ is called a monetary utility function if $u(\xi + a) = u(\xi) + a$ for all $a \in \mathbb{R}$.*

Definition. *$u\colon L^\infty \to \mathbb{R}$ is called a (Fatou) monetary concave utility function if*

1. $u(\xi) \geq 0$ *if* $\xi \geq 0$
2. u *is concave*
3. $u(\xi + a) = u(\xi) + a$ *for all* $a \in \mathbb{R}$
4. **Fatou property.** *If* $\sup_n \|\xi_n\|_\infty < \infty$, *if* $\xi_n \to \xi$ *in probability, then* $u(\xi) \geq \limsup u(\xi_n)$.

A utility u is characterised by the acceptance set

$$\mathcal{A} = \{\xi \mid u(\xi) \geq 0\}\,, u(\xi) = \max\{a \in \mathbb{R} \mid \xi - a \in \mathcal{A}\}.$$

In case u is a monetary utility function we define

$$\rho(\xi) = -u(\xi)$$

and call it a **convex risk measure**. It describes the amount of money to be added to become acceptable, i.e. to be in $\mathcal{A}$.

$$\rho(\xi + \rho(\xi)) = 0 \quad \text{and} \quad u(\xi - u(\xi)) = 0.$$

$\mathcal{P} = \{\mathbf{Q} \ll \mathbf{P} \mid \mathbf{Q}$ is a probability$\}$. The Fenchel-Legendre transform of u satisfies (Föllmer-Schied)

$c : \mathcal{P} \to \mathbb{R}_+ \cup \{+\infty\}$ is a convex function, for each $k \in \mathbb{R}_+$ the set $\{\mathbf{Q} \mid c(\mathbf{Q}) \leq k\}$ is convex and closed.

$\inf_{\mathbf{Q}\in\mathcal{P}} c(\mathbf{Q}) = 0$, we will suppose $c(\mathbf{P}) = 0$.

Characterisation of such utility functions. *For given u (Fatou) there is c as above so that*

$$u(\xi) = \inf\{\mathbf{E}_{\mathbf{Q}}[\xi] + c(\mathbf{Q}) \mid \mathbf{Q} \in \mathcal{P}\}.$$

Conversely such a function c defines a Fatou utility function.

Depending on c we get different examples, some of them easy to calculate some are more difficult. Essentially it becomes a linear programme in infinite dimensions.

The proof is essentially the Hahn-Banach theorem together with the Krein-Smulian theorem (needed to get weak* closed sets in L^∞). We also need that on bounded sets of L^∞ the topology of convergence in measure is the Mackey topology, a result that goes back to Grothendieck and which is based on the characterisation of relatively weakly compact sets (in L^1) as the uniformly integrable sets, the so-called Dunford-Pettis theorem.

u is positively homogeneous (coherent) if and only if there is a closed convex set $\mathcal{S} \subset \mathcal{P}$ so that $c(\mathbf{Q}) = 0$ on $\mathcal{S}$ and $c(\mathbf{Q}) = +\infty$ if $\mathbf{Q} \notin \mathcal{S}$.

$$u(\xi) = \inf\{\mathbf{E}_{\mathbf{Q}}[\xi] \mid \mathbf{Q} \in \mathcal{S}\}.$$

In continuous time we need to add an extra assumption: time consistency (Koopmans 1960). This means that we have the following decomposition property (for each stopping time σ):

$$
\begin{aligned}
&(1)\ \ \mathcal{A}^\sigma = \{\xi \in \mathcal{A} \mid \text{ for all } A \in \mathcal{F}_\sigma : \ \mathbf{1}_A \xi \in \mathcal{A}\} \\
&(2)\ \ \mathcal{A}_\sigma = \mathcal{A} \cap L^\infty(\mathcal{F}_\sigma) \\
&(3)\ \ \mathcal{A} = \mathcal{A}^\sigma + \mathcal{A}_\sigma
\end{aligned}
$$

For each stopping we can define a utility function

$$u_\sigma(\xi) = \text{ess.sup}\{\eta \in L^\infty(\mathcal{F}_\sigma) \mid \xi - \eta \in \mathcal{A}^\sigma\}$$

This means that time consistency and u_0 completely define the process u_t.

This gives, in the usual way (duality theory) a penalty function $c_\sigma(\mathbf{Q})$. The process c_t can be made càdlàg as well as the process $u_t(\xi)$. (Jocelyne Bion-Nadal)

The proof uses the decomposition property.

Time consistency is usually defined as:

if $\xi, \eta \in L^\infty$, if for stopping times $\sigma \leq \tau$ we have $u_\tau(\xi) \leq u_\tau(\eta)$ then also $u_\sigma(\xi) \leq u_\sigma(\eta)$.

On finite time intervals the time consistency is equivalent to the Bellman dynamic programming principle. On infinite time intervals this is wrong!!

Duffie-Epstein

Epstein-Schneider

Frittelli, Scandolo, Biagini

Maccheroni-Marinacci-Rustichini

Berlin-school, Detlefsen

Cheridito

Kupper

...

We take the case of $d-$dimensional Brownian Motion B, with the usual filtration. Finite time interval $[0, T]$. Time consistent (with Fatou property) utility functions can be defined via convex optimisation.

There is a function $f(t, \omega, x)$ such that

(1) for all $x \in \mathbb{R}^{\mathrm{d}}$, the function $f(., ., x)$ is predictable

(2) for (t, ω) the function is convex takes values in $\mathbb{R}_+ \cup \{+\infty\}$ and is proper

(3) $f(., ., 0) = 0$

(4) g is the Fenchel-Legendre transform of f

For $\mathbf{Q} \sim \mathbf{P}$ and $\frac{d\mathbf{Q}}{d\mathbf{P}} = \mathcal{E}(q \cdot B)_T$ we have

$$c_t(\mathbf{Q}) = \mathbf{E}_\mathbf{Q}\left[\int_t^T f(q_u)\,du \mid \mathcal{F}_t\right] \leq +\infty$$

Uses previous characterisations of Chen, Peng, El Karoui-Quenez-Peng. However we do not suppose any dominance of u by a $g-$expectation (some kind of hidden weak compactness)

$$u_t(\xi) = \text{ess.inf}_{\mathbf{Q}\sim\mathbf{P}}\,\mathbf{E}_\mathbf{Q}\left[\xi + \int_t^T f(q_u)\,du \mid \mathcal{F}_t\right]$$

We will suppose (for simplicity) that g is real valued ($< +\infty$) and that f and g do not depend on (t, ω). In this case we have precise results

Theorem. *For all* $\mathbf{Q} \ll \mathbf{P}$ *we have that* $u_t(\xi) + \int_0^{\tau \wedge t} f(q_u)\, du$ *is a* $\mathbf{Q}-$*submartingale,* $\tau = \inf\{t \mid L_t = \mathbf{E_P}\left[\frac{d\mathbf{Q}}{d\mathbf{P}} \mid \mathcal{F}_t\right] = 0\}$.
If there is $\mathbf{Q} \ll \mathbf{P}$ *with* $u_0(\xi) = \mathbf{E_Q}\left[\xi + \int_0^{\tau} f(q_u)\, du\right]$, *then* $u_t(\xi) + \int_0^{\tau \wedge t} f(q_u)\, du$ *is a* $\mathbf{Q}-$*martingale.*

For the Doob-Meyer decomposition (under $\mathbf{P}$) we get

$$u_t(\xi) = u_0(\xi) + A_t - \int_0^t Z_u \, dB_t$$

It is easy to see that $Z \cdot B$ is BMO and that A_T has exponential moments.

Furthermore, duality shows that $\quad dA_t \geq g(Z_t)\, dt.$

Theorem. *Suppose that for $\xi \in L^\infty$ there is $\mathbf{Q} \sim \mathbf{P}$ with $u_0(\xi) = \mathbf{E}_{\mathbf{Q}}\left[\xi + \int_0^T f(q_u)\, du\right]$, then $dA_t = g(Z_t)\, dt$.*

$$u_t(\xi) = u_0(\xi) + \int_0^t g(Z_t)\, dt - \int_0^t Z_u \, dB_t$$

Theorem. *Are equivalent*

(1) *g has at most quadratic growth* $g(x) \le k(1 + |x|^2)$.

(2) $f(x) \ge c\,x^2 - c$ *for some* $c > 0$

(3) *for all* $\xi \in L^\infty$ *there is* $\mathbf{Q} \ll \mathbf{P}$ *with* $u_0(\xi) = \mathbf{E}_{\mathbf{Q}}\left[\xi + \int_0^\tau f(q_u)\,du\right]$

(4) *for all* $\xi \in L^\infty$ *there is* $\mathbf{Q} \sim \mathbf{P}$ *with* $u_0(\xi) = \mathbf{E}_{\mathbf{Q}}\left[\xi + \int_0^\tau f(q_u)\,du\right]$

(5) *for all* $\xi \in L^\infty$ *the BSDE* $dY_t = g(Z_t) - Z_t\,dB_t$ *has a (unique) bounded solution with* $Y_T = \xi$

(6) *for all* $k \ge 0$, *the set* $\{\mathbf{Q} \mid c_0(\mathbf{Q}) \le k\}$ *is weakly compact.*

This is a combination of James' theorem, Jouini-Schachermayer-Touzi and the "martingale" theorem. Results of Barrieu-El Karoui are also used. The new part is the equivalence of 5 and 1. There is a relation with entropy.

BSDE with subquadratic driver were considered by Kobylanski, Briand-Cocquet-Hu, Imkeller, ...

One can easily see (convexity theory) that bounded solutions Y of the BSDE satisfy

$$Y_t \leq u_t(\xi)$$

Theorem. *Are equivalent*

1. (1) $\limsup_{x\to\infty} \frac{g(x)}{x^2} = \infty$ *or* $\liminf_{x\to\infty} \frac{f(x)}{x^2} = 0$
2. (2) *there is* $\xi \in L^\infty$ *so that the BSDE has no bounded solution. Moreover the set of* ξ *for which there is a solution is not norm dense in* L^∞.
3. (3) *if* $\xi \in L^\infty$ *is so that there is a bounded solution for the BSDE, then for each* $y < Y_0$*, there are infinitely many bounded solutions with* $Y'_0 = y$.
4. (4) *for some* ξ*, there are infinitely many bounded solutions with* $Y_0 = u_0(\xi)$.
5. (5) *the utility function* u_0 *is NOT strictly monotone.*

Theorem. *If* ξ *is minimal, i.e.* $\eta \le \xi$ *and* $\mathbf{P}[\eta < \xi] > 0$ *imply* $u_0(\eta) < u_0(\xi)$, *then for* ξ *there is a bounded solution* Y.

The converse is not true.

Remark. The fact that there is $\mathbf{Q} \sim \mathbf{P}$ with $c(\mathbf{Q}) = 0$ is equivalent to 0 being minimal, some kind of relevance axiom. It says that for all A with $\mathbf{P}[A] > 0$, we must have $u(-\mathbf{1}_A) < 0$.

What happens if $\xi = \phi(B_T)$, i.e. the Markov case. In this case u_t is a function of B_t.

In all the "bad" examples, ξ depends on the history of B.

Itô calculus leads to

$$\partial_t u + \frac{1}{2}\partial_{xx} u - g(-\partial_x u) = 0, \quad u(T, x) = \phi(x),$$

The related quasi-linear PDE has a bounded solution.

$$\partial_t u + \frac{1}{2}\partial_{xx} u - g(-\partial_x u) = 0, \quad u(T,x) = \phi(x),$$

for bounded ϕ with $\phi' \in L^\infty(\mathbb{R})$.

The proof uses BSDE techniques. As an example look at ($\beta > 2$)

$$\partial_t u + \frac{1}{2}\partial_{xx} u - |\partial_x u|^\beta = 0, \quad u(T,x) = j(x)$$