Pricing European Options with Stochastic Volatility: Asymptotic Expansions

R. Z. Khasminskii and G. Yin

http://www.math.wayne.edu/~gyin/

Wayne State University

Outline

- Motivation

Outline

- Motivation
- Stochastic Volatility Model

Outline

- Motivation
- Stochastic Volatility Model
- Asymptotic Expansions for Option Price

Outline

- Motivation
- Stochastic Volatility Model
- Asymptotic Expansions for Option Price
- Further Remarks

Motivation

- The celebrated Black-Scholes theory provides an important tool for pricing options.
- However, it has been recognized that the assumption of constant volatility, which is essential in their theory, is a less-than-perfect description of the real world.
- To capture the complicated behavior of stock prices and other derivatives, it is necessary to take into consideration of frequent changes of the volatility.

Motivation (cont.)

- Recently, Fouque, Papanicolaou, and Sircar (FPS) studied a class of volatility models.
- Under mean reversion, two-time-scale methods are used.
- Assume the volatility is a function $f(\cdot)$ of a fast varying diffusion that is mean reverting (or ergodic).
- It was shown that the "slow" component (the leading term or 0th-order outer expansion term in the approximation) of the option prices can be approximated by a Black-Scholes differential equation with constant volatility $\overline{f}$ where $\overline{f}^2$ is the average of $f^2(\cdot)$ w.r.t. the stationary measure of the "fast" component.
- The 1st-order outer expansion term in the asymptotic expansion was also found. The volatility was assumed to be driven by an Ornstein-Uhlenbeck (OU) process.

Motivation (cont.)

Stochastic volatility: Hull and White, Wiggins, Scott (all in 87)

By inverting the Black-Scholes formula, the price of a call is given in terms of its implied volatility (dep. on the strike and the maturity). For fixed maturity and across strikes it is known as the smile or the skew due to the observed asymmetry.

(FSP) mentioned that stochastic volatility can be thought of as a Brownian-type particle (the stock price) moving in an environment where the diffusion coeff. is randomly fluctuating in time according to some mean-reverting diffusion. 1 Brownian motion drives the motion of the particle and the other drives the fluctuations of the medium.

Our Study

Our objectives:

- consider a stochastic volatility model, with the driving volatility being a general diffusion process on a compact set;
- assume that it is a diffusion process on the circle $\mathbb{S}$ of circumference a for some $a > 0$ for simplicity;
- construct full asymptotic expansions.
  It is of interest to find the second term in the asymptotic expansion since it reveals the correction to the shape of the implied volatility skewness.

Features of Our Result

(1) The first two boundary layer terms ($n = 0$ and $n = 1$ in the expansion) are 0, but all higher-order boundary layer terms are nontrivial. These boundary layer terms are essential in characterizing the behavior of the option price as time approaches maturity.

(2) Compared with FPS, in addition to the leading term and the next term in the asymptotic expansions, our asymptotic expansions also include the second-order outer expansion term corresponding to the correction to the implied volatility surface mentioned previously. fixed (x, y), whereas in our paper, *uniform* asymptotic expansions are developed. The *uniform* asymptotic expansion is justified by deriving the error bounds.

Model

- Consider a risky asset price $\widetilde{x}(t)$

$$d\widetilde{x}(t) = \mu\widetilde{x}(t)dt + \sigma(t,\omega)\widetilde{x}(t)dw(t). \qquad (1)$$

- The volatility $\sigma(t,\omega) = f(\widetilde{y}(t/\varepsilon))$ is a function of a diffusion on the circle $\mathbb{S}$ in a time scale t/ε

$$d\widetilde{y}(t) = m(\widetilde{y}(t))dt + \beta(\widetilde{y}(t))dz(t), \qquad (2)$$

  - $m(\cdot)$ and $\beta(\cdot)$ are sufficiently smooth $\beta(y) \neq 0$,
  - $z(\cdot)$ is a standard BM & $f(y) \neq 0$.

Model (cont.)

- Denoting $y^\varepsilon(t) = \widetilde{y}(t/\varepsilon)$ & $\widetilde{z}^\varepsilon(t) = \sqrt{\varepsilon} z(t/\varepsilon)$,

$$dy^\varepsilon(t) = \frac{1}{\varepsilon} m(y^\varepsilon(t))dt + \frac{1}{\sqrt{\varepsilon}} \beta(y^\varepsilon(t))d\widetilde{z}^\varepsilon(t). \qquad (3)$$

- Interpret (2) as

$$\widetilde{y}(t) = y_0 + \int_0^t m(\widetilde{y}(s))ds + \int_0^t \beta(\widetilde{y}(s))dz(s) \pmod{a}. \qquad (4)$$

- Note that $\widetilde{z}^\varepsilon(t)$ & $w(t)$ are dependent.

Model (cont.)

Write

$$\widetilde{z}^{\varepsilon}(t) = \rho w(t) + \sqrt{1-\rho^2} z_1(t),$$

where $z_1(\cdot)$ is a standard Brownian motion independent of $w(\cdot)$. Consider $\widetilde{x}^{\varepsilon}(t)$ given by

$$d\widetilde{x}^{\varepsilon}(t) = \mu \widetilde{x}^{\varepsilon}(t)dt + f(y^{\varepsilon}(t))\widetilde{x}^{\varepsilon}(t)dw.$$

Take a logarithm transformation $x^{\varepsilon}(t) = \ln \widetilde{x}^{\varepsilon}(t)$.

$$\begin{cases} dx^{\varepsilon}(t) = \left(\mu - \frac{1}{2} f^2(y^{\varepsilon}(t))\right) dt + f(y^{\varepsilon}(t))dw(t), \\ dy^{\varepsilon}(t) = \frac{1}{\varepsilon} m(y^{\varepsilon}(t))dt + \frac{1}{\sqrt{\varepsilon}} \beta(y^{\varepsilon}(t))[\rho dw(t) + \sqrt{1-\rho^2} dz_1(t)]. \end{cases} \tag{5}$$

Model (cont.)

Introduce a risk-neutral measure $P^*(\omega)$ (e.g., FPS) by

$$\frac{dP^*(\omega)}{dP(\omega)} = \exp\Big(- \frac{1}{2}\int_0^T \Big(\frac{(\mu - r)^2}{f(y^\varepsilon(t))} + \gamma^2(y^\varepsilon(t))\Big)dt - \int_0^t \frac{\mu - r}{f(y^\varepsilon(s))}dw(s) - \int_0^t \gamma(y^\varepsilon(s))dz_1(s)\Big),$$

where r is the risk-free interest rate, $\gamma(\cdot)$ is any sufficiently smooth function on $\mathbb{S}$ and is called the risk premium factor. Then

$$w^*(t) = w(t) + \int_0^t \frac{\mu - r}{f(y(s))}ds, \text{ and}$$
$$z^*(t) = z_1(t) + \int_0^t \gamma(y(s))ds$$

are independent standard Brownian motions under P^*.

Model (cont.)

Then,

$$\begin{aligned} dx^\varepsilon(t) &= \left(\mu - \frac{1}{2}f^2(y^\varepsilon(t))\right)dt + f(y^\varepsilon(t))\left(dw^*(t) - \frac{\mu - r}{f(y^\varepsilon(t))}dt\right) \\ &= \left(r - \frac{1}{2}f^2(y^\varepsilon(t))\right)dt + f(y^\varepsilon(t))dw^*(t). \end{aligned} \tag{6}$$

Furthermore,

$$\begin{aligned} dy^\varepsilon(t) &= \left(\frac{1}{\varepsilon}m(y^\varepsilon(t)) + \frac{1}{\sqrt{\varepsilon}}\beta(y^\varepsilon(t))\left(\frac{(r-\mu)\rho}{f(y^\varepsilon(t))} - \sqrt{1-\rho^2}\gamma(y^\varepsilon(t))\right)\right)dt \\ &\quad + \frac{1}{\sqrt{\varepsilon}}\beta(y^\varepsilon(t))\rho dw^*(t) + \frac{\sqrt{1-\rho^2}\beta(y^\varepsilon(t))}{\sqrt{\varepsilon}}dz^*. \end{aligned}$$

Model (cont.)

Denote

$$\Lambda(y) = \gamma(y)\sqrt{1-\rho^2} + \frac{(\mu - r)\rho}{f(y)}, \tag{7}$$

which is a combined market price of risk.

Then

$$\begin{aligned} dy^\varepsilon(t) \; &= \left(\frac{1}{\varepsilon} m(y^\varepsilon(t)) - \frac{1}{\sqrt{\varepsilon}} \beta(y^\varepsilon(t))\Lambda(y^\varepsilon(t))\right) dt \\ &\quad + \frac{1}{\sqrt{\varepsilon}} \beta(y^\varepsilon(t))\rho dw^*(t) + \frac{\sqrt{1-\rho^2}\beta(y^\varepsilon(t))}{\sqrt{\varepsilon}} dz^*(t). \end{aligned} \tag{8}$$

Kolmogorov Equation

- Suppose payoff at maturity T is $H(x)$ (≥ 0) & denote $h(x) = H(e^x)$.
- Denote the option price at $t < T$ of the stock with present value of $x^\varepsilon(t) = x$ & current value of the process driving the volatility $y^\varepsilon(t) = y$ by $P^\varepsilon(t, x, y)$.
- Then with E^* denoting exp. w.r.t. risk-neutral measure P^*,

$$P^\varepsilon(t, x, y) = E^*[\exp(-r(T-t))h(x^\varepsilon(T))|x^\varepsilon(t) = x, y^\varepsilon(t) = y]. \tag{9}$$

- P^ε satisfies the Kolmogorov eq.

$$\begin{aligned} &\frac{\partial P^\varepsilon}{\partial t} + \frac{1}{2} f^2(y) \frac{\partial^2 P^\varepsilon}{\partial x^2} + \frac{1}{\sqrt{\varepsilon}} \beta(y) \rho f(y) \frac{\partial^2 P^\varepsilon}{\partial x \partial y} + \frac{1}{2\varepsilon} \beta^2(y) \frac{\partial^2 P^\varepsilon}{\partial y^2} \\ &\quad + \left(r - \frac{1}{2} f^2(y) \right) \frac{\partial P^\varepsilon}{\partial x} - r P^\varepsilon + \left(\frac{1}{\varepsilon} m(y) - \frac{1}{\sqrt{\varepsilon}} \beta(y) \Lambda(y) \right) \frac{\partial P^\varepsilon}{\partial y} = 0, \\ &P^\varepsilon(T, x, y) = h(x). \end{aligned} \tag{10}$$

Kolmogorov Equation (cont.)

Introduce

$$
\begin{aligned}
\mathcal{L}_0 &= \frac{1}{2}\beta^2(y)\frac{\partial^2}{\partial y^2} + m(y)\frac{\partial}{\partial y},\\
\mathcal{L}_1 &= \beta(y)\rho f(y)\frac{\partial^2}{\partial x \partial y} - \beta(y)\Lambda(y)\frac{\partial}{\partial y},\\
\mathcal{L}_2 &:= \mathcal{L}_{\text{BS}}(f^2(y)) = \frac{\partial}{\partial t} + \frac{1}{2}f^2(y)\frac{\partial^2}{\partial x^2} + \left(r - \frac{1}{2}f^2(y)\right)\frac{\partial}{\partial x} - r\cdot.
\end{aligned} \tag{11}
$$

Then (10) can be written as

$$
\left(\frac{1}{\varepsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\varepsilon}}\mathcal{L}_1 + \mathcal{L}_2\right)P^{\varepsilon}(t,x,y) = 0. \tag{12}
$$

Asymptotic Expansions

We seek asymptotic expansions of the form

$$\begin{aligned} P_0(t,x,y) \; & + \sqrt{\varepsilon} P_1(t,x,y) + \varepsilon P_2(t,x,y) + \varepsilon^{3/2} P_3(t,x,y) + \cdots \\ & + Q_0\left(\frac{T-t}{\varepsilon}, x, y\right) + \sqrt{\varepsilon} Q_1\left(\frac{T-t}{\varepsilon}, x, y\right) \\ & + \varepsilon Q_2\left(\frac{T-t}{\varepsilon}, x, y\right) + \varepsilon^{3/2} Q_3\left(\frac{T-t}{\varepsilon}, x, y\right) + \cdots , \end{aligned} \tag{13}$$

where $P_0 + \sqrt{\varepsilon} P_1 + \cdots$ is an outer expansion, and $Q_0 + \sqrt{\varepsilon} Q_1 + \cdots$ is an inner expansion (near maturity time T).

$$\tau = (T-t)/\varepsilon. \tag{14}$$

Asymptotic Expansions (cont.)

Then $Q_i(\tau, x, y)$ must be chosen so that

$$Q_i(\tau, x, y) \to 0 \text{ as } \tau \to \infty. \tag{15}$$

Substituting the outer expansion into (12) leads to the formal expansion

$$\begin{aligned} \frac{1}{\varepsilon}\mathcal{L}_0 P_0 \; &+ \frac{1}{\sqrt{\varepsilon}}(\mathcal{L}_0 P_1 + \mathcal{L}_1 P_0) + \varepsilon^0(\mathcal{L}_0 P_2 + \mathcal{L}_1 P_1 + \mathcal{L}_2 P_0) + \cdots \\ &+ \cdots + \varepsilon^{n/2}(\mathcal{L}_0 P_{n+2} + \mathcal{L}_1 P_{n+1} + \mathcal{L}_2 P_n) + \cdots = 0. \end{aligned} \tag{16}$$

Asymptotic Expansions (cont.)

Choose P_i's to satisfy

$$\begin{cases} \mathcal{L}_0 P_0 = 0, \\ \mathcal{L}_0 P_1 + \mathcal{L}_1 P_0 = 0, \\ \mathcal{L}_0 P_2 + \mathcal{L}_1 P_1 + \mathcal{L}_2 P_0 = 0, \\ \cdots\ \cdots\ \cdots, \\ \mathcal{L}_0 P_{n+2} + \mathcal{L}_1 P_{n+1} + \mathcal{L}_2 P_n = 0. \end{cases} \tag{17}$$

To ensure the matching of the outer expansion and boundary layer terms, use the boundary conditions

$$\begin{aligned} Q_0(0,x,y) + P_0(T,x) &= h(x), \\ Q_i(0,x,y) + P_i(T,x,y) &= 0, \quad i \geq 1. \end{aligned} \tag{18}$$

Remark

The leading term in the asymptotic expansions can be obtained using the "slow" component (6) in the system from general results on averaging principle of diffusion processes. Using the results of (Khasminskii 68), we can show $x^\varepsilon(\cdot)$ converges weakly to $x^0(\cdot)$ with

$$dx^0(t) = \left(r - \frac{1}{2}\overline{f}^2\right)dt + \sqrt{\overline{f}^2}\,dw_0(t). \tag{19}$$

This enables us to determine the 0th-order approximation $P_0(t,x)$. In FPS, the problem of obtaining the next term in the asymptotic expansions of P^ε was considered. Here, we develop a method of obtaining asymptotic expansions of all orders, and demonstrate the details for the case $n = 2$.

Assumptions

(A1) $f(\cdot)$, $\gamma(\cdot)$, $\beta(\cdot)$, and $m(\cdot)$ are sufficiently smooth on $\mathbb{S}$ s.t. $f^2(y) > 0$, $\gamma(y) > 0$, and $\beta^2(y) > 0$.

(A2) $h(\cdot)$ is smooth enough and bounded. Its derivative is zero outside a compact set.

Remark 1

- Similar to FPS, we have assumed $h(x)$ is bounded & smooth.
- For European options, $h(x) = (x - K)^+$ (call) where K is the exercise price ($h(\cdot)$ is not smooth & is unbounded).
- In a paper FPSS, asymptotic expansions for $P^\varepsilon(t, x, y)$ up to $O(\varepsilon)$ was obtained for nonsmooth payoff by regularization or smoothing techniques for fixed x, y and $t < T$.
- We confine our attention to smooth & bounded payoff $h(x)$, and prove *uniform* in x, y & $t < T$ accuracy of approximation.
- It appears that the unbounded and nonsmooth payoff function $h(x)$ can be handled and formal asymptotic expansions can be obtained, but such expansions are not uniform in x.

Remark 2

Under (A1), $\widetilde{y}(t)$ has a unique stationary distribution with stationary density $p(y)$

$$\mathcal{L}_0^* p(y) = 0,\ \int_0^a p(y)dy = 1,\ p(y+a) = p(y),\ y \in \mathbb{R}, \tag{20}$$

where $\mathcal{L}_0^*$ is the adjoint of $\mathcal{L}_0$. It is easy to see that

$$p(y) = C\frac{V(y)}{\beta^2(y)}\left(1 + \frac{1 - V(a)}{\widetilde{z}(a)\int_0^a V^{-1}(z)dz}\int_0^y V^{-1}(z)dz\right), \tag{21}$$

where

$$V(y) = \exp\left(2\int_0^y \frac{m(u)}{\beta^2(u)}du\right),$$

and C is chosen so that $\int_0^a p(y)dy = 1$.

Remark 2 (cont.)

It is known $p(y_1,t,y) \to p(y)$ exponentially fast. That is, for some $K_0 > 0$ and $\kappa_0 > 0$,

$$|p(y_1,t,y) - p(y)| \le K_0 \exp(-\kappa_0 t). \tag{22}$$

Let $\widetilde{y}^y(t)$ be the diffusion given by (2) with $\widetilde{y}^y(0) = y$. Let $\varphi(\cdot)$ be any continuous function on $\mathbb{S}$. Denote

$$\overline{\varphi} = \int_{\mathbb{S}} \varphi(y) p(y) dy.$$

It follows from (22) that

$$|E\varphi(\widetilde{y}^y(t)) - \overline{\varphi}| \le K_0 \exp(-\kappa_0 t). \tag{23}$$

Lemmas

Lemma 1. *Let $f(x,y)$ be bounded for $x \in \mathbb{R}$ and $y \in \mathbb{S}$ together with derivatives up to order ι w.r.t. x and y.*

(1) *The Poisson equation on $\mathbb{S}$*

$$\mathcal{L}_0 u = f(x,y), \tag{24}$$

has a solution iff

$$\langle f(x,\cdot)\rangle \stackrel{\text{def}}{=} \int_{\mathbb{S}} f(x,y)p(y)dy = 0, \tag{25}$$

where $p(y)$ is the stationary density associated with $\mathcal{L}_0$.

(2) *Any soln of* (24) *satisfying* (21) *can be written as*

$$u(x,y) = u_0(x,y) + C(x). \tag{26}$$

(3) *The solutions above are bounded and have bounded derivatives up to order ι w.r.t. both $x \in \mathbb{R}$ and $y \in \mathbb{S}$.*

Lemmas

Corollary 2. *Consider the Poisson equation on* $\mathbb{S}$*,*

$$\mathcal{L}_0 V(y) = \psi(y). \tag{27}$$

Eq. (27) *has a solution iff*

$$\overline{\psi} = \langle \psi(\cdot) \rangle = \int \psi(y) p(y) dy = 0, \tag{28}$$

where $p(y)$ *is the stationary density. Any solution of* (27) *satisfying* (28) *can be written as*

$$V(y) = V_0(y) + C,$$

where $V_0(y)$ *is the unique solution of* (27) *satisfying the condition* $\overline{V}_0 = 0$*, and* C *is an arbitrary constant.*

Lemmas

Lemma 3. *The solution of the problem*

$$\mathcal{L}_{\scriptscriptstyle BS}(\overline{f}^2)u(t,x) = \varphi(t,x),\ u(T,x) = \psi(x) \qquad (29)$$

is bounded for $x \in \mathbb{R}$ and $0 \leq t \leq T$, and has bounded derivatives in x up to the ιth order, if $\varphi(\cdot)$ and $\psi(\cdot)$ are bounded and have bounded derivatives in x up to the ιth order.

Corollary 4. *The functions $P_0(t,x)$ and $(\partial^\iota/\partial x^\iota)P_0(t,x)$ are bounded uniformly in $x \in \mathbb{R}$ and $0 \leq t \leq T$ if $h(x)$ and $(d^\iota/dx^\iota)h(x)$ are bounded for a positive integer ι.*

Lemmas

Lemma 5. *Denote*

$$L^{\varepsilon} = \frac{1}{\varepsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\varepsilon}}\mathcal{L}_1 + \mathcal{L}_2.$$

Assume conditions (A1) *and* (A2). *Denote* $L^{\varepsilon} e_4^{\varepsilon}(t,x,y) = \eta^{\varepsilon}(t,x,y)$. *Then*

$$\sup_{(t,x,y)\in[0,T]\times\mathbb{R}\times\mathbb{S}} |\eta^{\varepsilon}(t,x,y)| = O(\varepsilon^{3/2}). \qquad (30)$$

Lemma 6. *Under the conditions of Lemma* 5*,*

$$\sup_{(t,x,y)\in[0,T]\times\mathbb{R}\times\mathbb{S}} |e_2^{\varepsilon}(t,x,y))| = O(\varepsilon^{3/2}).$$

Theorem

Under conditions (A1) *and* (A2), *we can construct* $P_k(t,x,y)$ *and* $Q_k(\tau,x,y)$ *for* $k=0,1,2$ *such that*

- $P_0(t,x,y)=P_0(t,x)$, $P_1(t,x,y)=P_1(t,x)$, $Q_0(\tau,x,y)=Q_1(\tau,x,y)=0$ *and* $|Q_2(\tau,x,y)|\le C\exp(-\kappa_0\tau)$;
- $P_0(t,x)$ *is a solution of the Black-Scholes equation*

$$\begin{aligned}\big\langle \mathcal{L}_{BS}(f^2(\cdot))P_0(t,x)\big\rangle &= \big\langle \mathcal{L}_{BS}(f^2(\cdot))\big\rangle P_0(t,x)\\ &= \frac{\partial P_0}{\partial t}+\frac{1}{2}\overline{f}^2\frac{\partial^2 P_0}{\partial x^2}+r\left(\frac{\partial P_0}{\partial x}-P_0\right)-\frac{1}{2}\overline{f}^2\frac{\partial P_0}{\partial x}=0,\end{aligned} \tag{31}$$

where

$$\overline{f^2}=\langle f^2\rangle=\int f^2(y)p(y)dy$$

with the condition $P_0(T,x)=h(x)$*;*

Theorem (cont.)

- $P_1(t,x)$ *is given by*

$$\mathcal{L}_0\psi = f^2(y) - \overline{f^2}, \;\; \langle\psi\rangle = 0 \tag{32}$$

and

$$P_1(t,x) = -\frac{(T-t)}{2}\Big[\rho\langle\beta(\cdot)f(\cdot)\psi'(\cdot)\rangle\Big(\frac{\partial^3 P_0(t,x)}{\partial x^3} - \frac{\partial^2 P_0(t,x)}{\partial x^2}\Big) - \langle\beta(\cdot)\Lambda(\cdot)\psi'(\cdot)\rangle\Big(\frac{\partial^2 P_0(t,x)}{\partial x^2} - \frac{\partial P_0(t,x)}{\partial x}\Big)\Big]; \tag{33}$$

Theorem (cont.)

- $P_2(t,x,y)$ *is given by* (32),

$$P_2(t,x,y) = -\frac{1}{2}\left(\frac{\partial^2 P_0(t,x)}{\partial x^2} - \frac{\partial P_0}{\partial x}\right)\psi(y) + A_1(t,x), \qquad (34)$$

$$\mathcal{L}_{BS}(\overline{f^2})A_1(t,x) + \frac{1}{4}\langle f^2(\cdot)\psi(\cdot)\rangle G(t,x) = F(t,x), \qquad (35)$$

where ψ *is the unique solution of*

$$\mathcal{L}_0\psi = f^2(y) - \overline{f^2},\ \overline{\psi} = 0,$$

$$G(t,x) = -\left(\frac{\partial^4 P_0(t,x)}{\partial x^4} - 2\frac{\partial^3 P_0(t,x)}{\partial x^3} + \frac{\partial^2 P_0(t,x)}{\partial x^2}\right), \quad \textit{and}$$

$$A_1(T,x) = 0;$$

Theorem (cont.)

- $Q_2(\tau, x, y)$ *is given by*

$$Q_2(\tau, x, y) = \frac{1}{2}(h''(x) - h'(x))\phi(\tau, y), \tag{36}$$

where $\phi(\tau, y)$ *is a solution of*

$$\widetilde{\mathcal{L}}_0(y)\phi(\tau, y) = 0, \;\; \phi(0, y) = \psi(y). \tag{37}$$

Moreover,

$$\sup_{(t,x,y)\in[0,T]\times\mathbb{R}\times\mathbb{S}} \left| e^{\varepsilon}(t, x, y) \right| = O(\varepsilon^{3/2}), \tag{38}$$

where $e_2^{\varepsilon}(t, x, y) =$

$$P^{\varepsilon}(t, x, y) - \left[P_0(t, x) + \sqrt{\varepsilon} P_1(t, x) + \varepsilon P_2(t, x, y) + \varepsilon Q_2\left(\tfrac{T-t}{\varepsilon}, x, y \right) \right].$$

Further Remarks

- Asymptotic expansions for option prices are obtained and justified. For the derivative pricing problem considered, we worked out the asymptotic expansions of the first few terms. In addition to the leading term, the first-order term, the second-order term in the asymptotic expansion, and the boundary layer correction terms, we can use the same technique to obtain the full asymptotic expansion of the option prices.
- Compared with the previous work, for a fixed $t < T$ and fixed x, y, an important aspect our result is that the asymptotic error bound is *uniform* in $t \leq T$ and x, y.
- It is interesting to exam multi-scale counter part.
- It is interesting to investigate stochastic volatility in conjunction with regime switching.
- It is interesting to study jump diffusion models.

Thank you.