Optimal Funding of a Defined Benefit Pension Plan

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Outline

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- Individual savings
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  - Promoted by the state via tax exemptions

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  - inflation,
  - investment performance,
  - salary development,
  - death and withdrawal of members.

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  - Optimisation of contribution adjustments in order to minimize the total unanticipated cost for the sponsor.

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- Consider a 2-dimensional standard Brownian motion $W_t^{P^f} = \left(W_t^{r,P^f}, W_t^{S,P^f} \right)$ defined on a complete probability space $(\Omega^f, \mathcal{F}^f, P^f)$.

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- Under Q^f, the risk free rate is the solution of the following SDE:

$$dr_t = a.(\underbrace{b - \sigma_r.\frac{\lambda_r}{a}}_{b^Q} - r_t).dt + \sigma_r.\underbrace{\left(dW_t^{r,P^f} + \lambda_r.dt \right)}_{dW_t^{r,Q^f}},$$

where W_t^{r,Q^f} is a Wiener process under Q^f,
and with a, b, σ_r and λ_r constants.

- Consider a rolling bond of maturity K whose price is denoted R_t^K. This bond is a zero coupon bond continuously rebalanced in order to keep a constant maturity and its price obeys to the dynamics:

$$\begin{aligned}\frac{dR_t^K}{R_t^K} &= r_t.dt - \sigma_r.n(K).\left(dW_t^{r,P^f} + \lambda_r.dt\right) \\ &= r_t.dt - \sigma_r.n(K).dW_t^{r,Q^f}\end{aligned}$$

where $n(K)$ is a function of the maturity K :

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where $n(K)$ is a function of the maturity K :

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- A stock with price process S_t is modelled by a geometric Brownian motion and is correlated with the interest rates fluctuations:

$$\begin{aligned} \frac{dS_t}{S_t} &= r_t.dt + \sigma_{Sr}.\left(dW_t^{r,P^f} + \lambda_r.dt\right) + \sigma_S.\left(dW_t^{S,P^f} + \lambda_S.dt\right) \\ &= r_t.dt + \sigma_{Sr}.dW_t^{r,Q^f} + \sigma_S.dW_t^{S,Q^f} \end{aligned}$$

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where $\mu_A(t)$ is the average growth of the salary and W_t^{A,P^a} is a Wiener process defined on a probability space $(\Omega^a, \mathcal{F}^a, P^a)$, that represents the intrinsic randomness of the salary and is independent of W_t^{r,P^f} and W_t^{S,P^f}.

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- All members retire at the age $x + T$ and in case of death, no benefits are paid.
- Each pensioner will receive a continuous annuity whose rate B is a fraction, α, of the last wage:

$$B = A_T.\alpha.$$

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- N_t points out the total number of deaths observed till time t and is given by

$$N_t = \sum_{i=1}^{n_x} I(T_i \leq t)$$

where I is an indicator function, $T_1, T_2, \ldots, T_{n_x}$ are exponentially distributed random variables modelling the remaining lifetimes of the affiliates and where the mortality rate of this jump process is denoted by μ_{x+t}.

Expected number of survivors

- The expected number of survivors under P^m is equal to the current number of survivors times a survival probability:

$$\mathbb{E}\left((n_x - N_s)|\mathcal{F}_t^m\right) = (n_x - N_t).\underbrace{\exp\left(-\int_t^s \mu(x+u).du\right)}_{{}_{s-t}p_{x+t}}.$$

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${}_{s-t}p_{x+t}$ is the actuarial notation for the probability that an individual of age $x+t$ survives till age $x+s$.

Deflator

- Let $(\Omega, \mathcal{F}, P)$ be the probability space resulting from the product of the financial, wage and mortality probability spaces:

  $$\Omega = \Omega^f \times \Omega^a \times \Omega^m \qquad \mathcal{F} = \mathcal{F}^f \otimes \mathcal{F}^a \otimes \mathcal{F}^m \vee \mathcal{N} \qquad P = P^f \times P^a \times P^m$$

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- The pricing of pension fund liabilities is hence done under a probability measure Q which is equal to the product of Q^f, Q^a and Q^m.
- An insurer's deflator is here composed of three elements called abusively the financial, wage and actuarial deflators, and is an extension of the deflators used in Hainaut and Devolder (2006b).

Deflator and bond price

- The deflator used to price liabilities, written $H(t,s)$ is in our settings the product of the financial, wage and actuarial deflators:

$$H(t,s) \quad = \quad \frac{\exp\left(-\int_0^s r_u.du\right)}{\exp\left(-\int_0^t r_u.du\right)} . \frac{\left(\frac{dQ^f}{dP^f}\right)_s}{\left(\frac{dQ^f}{dP^f}\right)_t} . \frac{\left(\frac{dQ^{a,\lambda_a}}{dP^a}\right)_s}{\left(\frac{dQ^{a,\lambda_a}}{dP^a}\right)_t} . \frac{\left(\frac{dQ^{m,h}}{dP^m}\right)_s}{\left(\frac{dQ^{m,h}}{dP^m}\right)_t} .$$

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- Remark that the expectation of the deflator $H(t,s)$ is equal to the price of a zero coupon bond, denoted $B(t,s)$:

$$\begin{aligned} B(t,s) &= \mathbb{E}\left(H(t,s)|\mathcal{F}_t\right) = \mathbb{E}^Q\left(e^{-\int_t^s r_u.du}|\mathcal{F}_t\right) \\ &= \exp\left(-\beta.(s-t) + n(s-t).(\beta - r_t) - \frac{\sigma_r^2}{4.a}.n(s-t)^2\right) \end{aligned}$$

where

$$\beta = b^Q - \frac{\sigma_r^2}{2.a^2} = b - \sigma_r.\frac{\lambda_r}{a} - \frac{\sigma_r^2}{2.a^2}$$

Financial and wage deflator

- The financial deflator $H^f(t,s)$ at time t for a cash flow paid at time $t \leq s$ is equal to the product of the discount factor and of the change of measure:

$$H^f(t,s) = \frac{\exp\left(-\int_0^s r_u.du\right).\left(\frac{dQ^f}{dP^f}\right)_s}{\exp\left(-\int_0^t r_u.du\right).\left(\frac{dQ^f}{dP^f}\right)_t}$$

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- The wage deflator at instant t, for a payment occurring at time $s \geq t$:

$$H^a(t,s) = \frac{\left(\frac{dQ^{a,\lambda_a}}{dP^a}\right)_s}{\left(\frac{dQ^{a,\lambda_a}}{dP^a}\right)_t} = \exp\left(-\frac{1}{2}.\int_t^s |\lambda_{a,u}|^2.du - \int_t^s \lambda_{a,u}.dW_u^{A,P^a}\right)$$

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- By the incompleteness caused by the salary risk, $\lambda_{a,u}$ will be chosen in the sequel to be some (arbitrary) constant.

Actuarial Deflator

- The second source of incompleteness is the mortality risk. For any $\mathcal{F}^m$-predictable process h_s, such that $h_s > -1$, an equivalent actuarial measure $Q^{m,h}$ is defined by the random variable solution of the SDE:

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\begin{aligned}
d\left(\frac{dQ^{m,h}}{dP^m}\right)_t &= \left(\frac{dQ^{m,h}}{dP^m}\right)_t .h_t.d\left(N_t - \int_0^t \left(n_x - N_{u-}\right)\mu(x+u)du\right) \\
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- We adopt the notation $\lambda_{N,u} = (n_x - N_{u-})\,.\mu(x+u)$ for the intensity of jumps.

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- The actuarial deflator at instant t, for a payment occurring at time $s \geq t$, is defined by:

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- Under $Q^{m,h}$, the expected number of survivors at time s is equal to the number of survivors at time t multiplied by a modified probability of survival ${}_{s-t}p^h_{x+t}$:

$$\mathbb{E}^{Q^{m,h}}\left((n_x - N_s)|\mathcal{F}^m_t\right) = (n_x - N_t).\underbrace{\exp\left(-\int_t^s \mu(x+u).(1+h_u).du\right)}_{{}_{s-t}p^h_{x+t}}$$

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- In the sequel of this work, we restrict our field of research to a constant process $h_u = h$.

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- If T^m is the maximum time horizon of the insurer's commitments, L_t is equal to:

$$L_t = \mathbb{E}\left(-\int_t^T H(t,s).c_s.ds + \int_T^{T^m} H(t,s).\left(n_x - N_s\right).B.ds|\mathcal{F}_t\right).$$

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- The fair value at the retirement date T of the liabilities is given by:

$$\begin{aligned} L_T &= \mathbb{E}\left(\int_T^{Tm} H(T,s).\left(n_x - N_s\right).B.ds|\mathcal{F}_T\right) \\ &= \left(n_x - N_T\right).\alpha.A_T.\int_T^{Tm} {}_{s-T}p^h_{x+T}.B(T,s).ds. \end{aligned}$$

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- One of the pension fund manager's goal is to maintain c_t as close as possible to NC.

- The second objective pursued by the pension plan manager is to obtain a value of the assets as close as possible to L_T, the market value of the liabilities at the time of retirement. The target total asset value is denoted $\tilde{X}_T$.

The optimisation problem

Optimisation problem and value function

$$V(t,x,n,a) = \min_{c_t,\tilde{X}_T \in \mathcal{A}_t(x)} \mathbb{E}\left[\int_t^T u_1.\left(c_s - NC\right)^2 .ds + u_2.(\tilde{X}_T - L_T)^2 \mid \mathcal{F}_t, \tilde{X}_t = x\right]$$

where u_1 and u_2 are constant weights

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Budget constraint

$$\mathcal{A}_t(x) = \Big\{\Big(\left(c_s\right)_{s\in[t,T]}, \tilde{X}_T\Big) \quad \textit{such that} \quad \mathbb{E}\left(-\int_t^T H(t,s).c_s.ds + H(t,T).\tilde{X}_T|\mathcal{F}_t\right) \leq x\Big\} \quad (1)$$

Literature

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- As the market is incomplete, the fact that $\tilde{X}_T$ belongs to $\mathcal{A}_t(x)$ doesn't guarantee that this process is replicable by an adapted investment policy.
- We inspire us upon the approach of Brennan and Xia (2002), see also Hainaut and Devolder (2006a, b).

Martingale method of Cox-Huang

- Following Brennan and Xia (2002), we will use the Martingale method and minimize first with respect to the contributions and the associated terminal target wealth.

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- Following Brennan and Xia (2002), we will use the Martingale method and minimize first with respect to the contributions and the associated terminal target wealth.
- Let $y_t \in \mathbb{R}^+$ be the Lagrange multiplier associated to the budget constraint at instant t and define the Lagrangian by:

$$\mathcal{L}\left(t, x, n, a, (c_s)_s\,,\ \tilde{X}_T,\ y_t\right) = \tag{2}$$

$$\mathbb{E}\left(\int_t^T u_1.\left(c_s - NC\right)^2.ds + u_2.(\tilde{X}_T - L_T)^2)|\mathcal{F}_t\right) -$$

$$y_t.\left(x - \mathbb{E}\left(-\int_t^T H(t,s).c_s.ds + H(t,T).\tilde{X}_T|\mathcal{F}_t\right)\right).$$

Optimal contribution rate and target wealth

- Under technical conditions, the optimal contribution rate and target wealth are:

$$c_s^* = y_t^*.H(t,s).\frac{1}{2.u_1} + NC$$

$$\tilde{X}_T^* = -y_t^*.H(t,T).\frac{1}{2.u_2} + L_T.$$

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- The optimal Lagrange multiplier, y_t^*, is such that the budget constraint (1) is binding:

$$y_t^* \quad = \quad \frac{\mathbb{E}\left(H(t,T).L_T|\mathcal{F}_t\right) - x - NC.\int_t^T \mathbb{E}\left(H(t,s)|\mathcal{F}_t\right) ds}{\frac{1}{2.u_1}.\int_t^T \mathbb{E}\left(H(t,s)^2|\mathcal{F}_t\right) ds + \frac{1}{2.u_2}.\mathbb{E}\left(H(t,T)^2|\mathcal{F}_t\right)}. \qquad (3)$$

Unfunded liabilities

- The numerator of (3) represents precisely the unfunded liabilities, denoted by

$$UL_t = \mathbb{E}\left(H(t,T).L_T|\mathcal{F}_t\right) - x - NC.\underbrace{\int_t^T \mathbb{E}\left(H(t,s)|\mathcal{F}_t\right) ds}_{\bar{a}_{t,T}}, \quad (4)$$

namely the part of the benefits that are not yet financed:
the expected fair value of reserves less the current asset value and less the normal cost times a financial annuity $\bar{a}_{t,T}$ of maturity $T - t$.

Optimal contribution rate and target wealth

The optimal contribution process c_t^* and the terminal target wealth $\tilde{X}_T^*$ depend on the unfunded liabilities UL_t:

Optimal contribution and target wealth

$$c_s^* = UL_t \underbrace{\frac{F(t,s)}{2u_1}}_{\text{amortisation rate}} + NC$$

$$\tilde{X}_T^* = -UL_t \frac{F(t,T)}{2u_2} + L_T$$

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where

$$F(t,s) = \frac{H(t,s)}{\frac{1}{2.u_1}.\int_t^T \mathbb{E}\left(H(t,v)^2|\mathcal{F}_t\right) dv + \frac{1}{2.u_2}.\mathbb{E}\left(H(t,T)^2|\mathcal{F}_t\right)} > 0$$

Value function

- The value function depends on the square of unfunded liabilities:

$$V(t,x,n,a) = \frac{UL_t^2}{\frac{1}{u_1}.\int_t^T \mathbb{E}\left(H(t,s)^2|\mathcal{F}_t\right) ds + \frac{1}{u_2}.\mathbb{E}\left(H(t,T)^2|\mathcal{F}_t\right)} \quad (5)$$

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- Our reasoning in this paper is based on dynamic programming (see e.g. Fleming and Rishel 1975 for details) and is also applied in Hainaut and Devolder (2006b).

The set of replicable processes

- Let (π_t^S, π_t^R) denote respectively the fraction of the wealth invested in stocks and rolling bonds and define

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$$\begin{aligned}\mathcal{A}_t^{\pi}(x) &= \Big\{ \Big((c_s)_{s\in[t,T]}, X_T \Big) \quad | \quad \exists\, (\pi_t^S)_t\ (\pi_t^R)_t \quad F_t - adapted \quad : \\ & e^{-\int_t^T r_s.ds}.X_T = x + \int_t^T e^{-\int_t^s r_u.du}.c_s.ds \\ & + \int_t^T e^{-\int_t^s r_u.du}.\pi_s^S.X_s.dS_s + \int_t^T e^{-\int_t^s r_u.du}.\pi_s^R.X_s.dR_s^K \Big\}.\end{aligned}$$

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- By definition, the set $\mathcal{A}_t^{\pi}(x)$ is included in $\mathcal{A}_t(x)$ and

$$\begin{aligned} dX_t &= \Big(\Big(r_t + \pi_t^S.\nu_S + \pi_t^R.\nu_R \Big).X_t + c_t \Big).dt + \pi_t^S.\sigma_S.X_t.dW_t^{S,P^f} \\ & + \Big(\pi_t^S.\sigma_{Sr} - \pi_t^R.\sigma_r.n(K) \Big).X_t.dW_t^{r,P^f}. \end{aligned}$$

Dynamic programming principle

- For a small step of time Δt, the dynamic programming principle states that:

$$\begin{aligned} V(t,x,n,a) &= \mathbb{E}\left[\int_t^{t+\Delta t} u_1.\left(c_s^* - NC\right)^2 .ds + \right. \\ &\left. + V\left(t+\Delta t, \tilde{X}^*_{t+\Delta t}, N_{t+\Delta t}, A_{t+\Delta t}\right) \mid \mathcal{F}_t \right]. \end{aligned}$$

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- Given that $(\tilde{X}^*_t)_t$ is the process minimizing the value function, any other process $(X_t)_t \in \mathcal{A}^\pi_t(x) \subset \mathcal{A}_t(x)$ verifies the inequality:

$$\begin{aligned} V(t,x,n,a) &\leq \mathbb{E}\left[\int_t^{t+\Delta t} u_1.\left(c_s^* - NC\right)^2 .ds + \right. \\ &\left. + V\left(t+\Delta t, X_{t+\Delta t}, N_{t+\Delta t}, A_{t+\Delta t}\right) \mid \mathcal{F}_t\right]. \end{aligned}$$

Itô's lemma and generator

- Using Ito's lemma for jump processes:

$$\mathbb{E}\left(V(t+\Delta t, X_{t+\Delta t}, N_{t+\Delta t}, A_{t+\Delta t})|\mathcal{F}_t\right) =$$
$$V(t,x,n,a) + \mathbb{E}\left(\int_t^{t+\Delta t} G^{\pi}(s, X_s, N_s, A_s).ds|\mathcal{F}_t\right) +$$
$$\mathbb{E}\left(\int_t^{t+\Delta t} \left(V(s, X_s, N_s, A_s) - V(s, X_s, N_{s-}, A_s)\right) dN_s|\mathcal{F}_t\right)$$

where $G^{\pi}(s, X_s, N_s, A_s)$ is the generator of the value function.

Deriving $G^\pi(t, X_t, N_t, A_t)$ with respect to π_t^S and π_t^R leads to:

The best replicating strategy

$$\pi_t^{S*} = \underbrace{\left(\frac{\nu_R.\sigma_{Sr}}{\sigma_S^2.\sigma_r.n(K)} + \frac{\nu_S}{\sigma_S^2}\right)}_{constant} . \frac{UL_t}{X_t} + \frac{\sigma_{AS}}{\sigma_S} . \frac{\mathbb{E}\left(H(t,T).L_T|\mathcal{F}_t\right)}{X_t} \qquad (6)$$

$$\begin{aligned}\pi_t^{R*} &= \underbrace{\left(\frac{\nu_S.\sigma_{Sr}}{\sigma_S^2.\sigma_r.n(K)} + \frac{\nu_R}{\sigma_r^2.n(K)^2}.\left(1+\frac{\sigma_{Sr}^2}{\sigma_S^2}\right)\right)}_{constant} . \frac{UL_t}{X_t} \\ &- \underbrace{\left(\frac{\sigma_{Ar}}{\sigma_r.n(K)} - \frac{\sigma_{AS}.\sigma_{Sr}}{\sigma_S.\sigma_r.n(K)}\right)}_{constant} . \frac{\mathbb{E}\left(H(t,T).L_T|\mathcal{F}_t\right)}{X_t} \\ &+ \underbrace{\frac{1}{n(K)} . \frac{V_{Xr}}{V_{XX}} . \frac{1}{X_t}}_{correction\ term} . \qquad (7)\end{aligned}$$

Parameter values

- We consider a male population, age 50 of $n_{50} = 10.000$ affiliates.

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- Normal cost 2.676.300
- The mortality rates are given by:

$$\mu(x) = a_\mu + b_\mu . c_\mu^x \qquad a_\mu = -\ln(s_\mu) \qquad b_\mu = \ln(g_\mu) . \ln(c_\mu)$$

where the parameters s_μ, g_μ, c_μ take the values showed in the table:

s_μ:	0.999441703848
g_μ:	0.999733441115
c_μ :	1.116792453830

Other parameters:

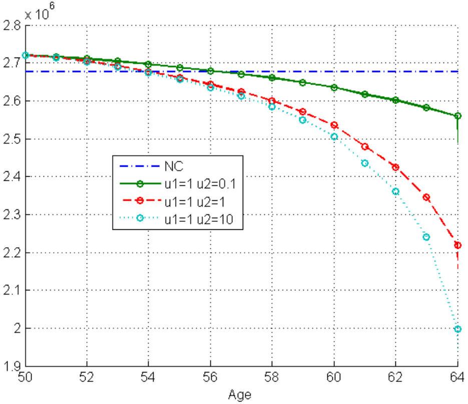
a	12.72%	σ_{SR}	-0.10%
b	3.88%	ν_S	5.35%
σ_r	1.75%	μ_A	2.00%
λ_r	-2.36%	σ_{Ar}	2.00%
$r_{t=0}$	2.00%	σ_{AS}	2.00%
K	8 years	μ_A^Q	2.00%
ν_R	2.77%	σ_A	5.00%
λ_S	34.94%	λ_a	-4.54%
σ_S	15.24%	h	0.0

Table: Parameters.

Contribution rates

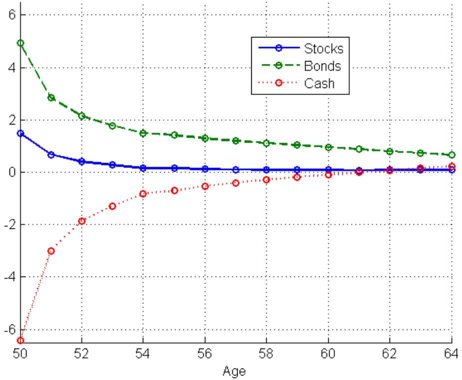
- Monte Carlo simulation: 5000 scenarios generated.

Figure: Contribution rates.



Investment proportions

Figure: Asset mix for $u_1 = 1$ and $u_2 = 10$.



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- The amortization of those unfunded liabilities is a function of the weights u_1 and u_2 defining the employer's preferences.
- Positions in risky assets decrease when we approach to the maturity.
- A quadratic utility penalizes without distinction positive and negative spreads

Thank you for your attention!