Basket CDS Pricing with Contagion Hazard Rates

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Abstract. In this paper we discuss the joint distribution of default times of obligors in a portfolio with the total hazard construction method. We then use it to derive analytic pricing formulas for basket credit default swaps with and without counterparty risk.

1 Introduction

Portfolio credit derivatives have become increasingly popular in financial industry for managing and hedging the risk associated with credit and default events of underlying names in the portfolio. The valuation of portfolio credit derivatives, such as basket credit default swaps and collateralized debt obligations, requires the modelling of not only default processes for individual names but also the correlation of the default processes. The sources of dependence come from the common macro economic factors that affect all firms in the sector and from the direct interaction of firms such as supply chain links and counterparty exposures. In the reduced-form modelling the default time of an obligor is defined as the first jump time of an exogenously given jump process. There are mainly three approaches to model the default correlation in the literature: copula, conditional independence, and contagion.

The copula model constructs the joint distribution of default times by combining marginal distributions of default times of individual names with a copula (a distribution function of multivariate standard uniform random

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variables). The dependence structure is completely determined by the copula. Li (1999) shows that the CreditMetrics essentially uses the normal copula in its default correlation formula.

The conditional independence model assumes that default intensities are influenced by some systematic factors which affect all names in the portfolio and idiosyncratic factors which only affect individual names. Conditional on the realization of the common factor, the default intensities of names are independent to each other. The joint distribution of default times can be expressed semi-analytically, and analytically in the limiting case for homogeneous portfolios, see Gregory (2003) and Schonbucher (2004) for detailed discussion. It is often difficult to compute the joint distribution for large heterogeneous portfolios due to large number of summation elements. Some efficient approximation techniques have been suggested to alleviate the computational burden, such as the tail approximation method (Glasserman (2004)) and the hybrid normal approximation method (Zheng (2006)). One disadvantage of the conditional independence model is that the default correlation is often not sufficiently high, as pointed out by the Basel Committee on Banking Supervision (BCBS (2006)), "Portfolio credit risk models used by the industry do not allow for contagion through business links. In particular, they regularly rely on the assumption of conditional independence; if this is violated and there remains dependence not captured by the model, the loss simulations will produce underestimated measures of economic capital. Recent academic research has raised doubts that the assumption of conditional independence describes the real world sufficiently well".

The contagion model studies the direct interaction of names in which the default probability of one name may increase (or decrease) upon defaults of some other names in the portfolio, and vice versa, and "infectious defaults" may develop, see Davis and Lo (2001). The "looping" dependence of default times of hazard rate processes makes more difficult in characterizing the joint distribution of default times. Jarrow and Yu (2001) suggest the primary secondary framework for the interaction of default intensities, which excludes cyclical default dependence, and derive the joint distribution. Frey and Backhaus (2004) apply the Markov process techniques to analyze in detail a model where the interaction between firms is of the mean-field type and use the Manto Carlo method for the pricing of portfolio related credit products. Leung and Kwok (2005) apply the CGH formula (Collin-Dufresne, et al. (2002)) to derive the joint density function of three names in the contagion model and use it to price single-name CDSs. Yu (2007) applies the total hazard construction method (Shaked and Shanthikumar (1987)) to get the joint density function of three names and uses it to price bonds and single-name CDSs. Yu (2007) also suggests the Monte Carlo method for pricing of basket CDSs.

This paper studies the pricing of basket CDSs in a contagion model setting. The objective is to give an analytic formula for basket default swap rates with defaultable CDS seller. The dependence of names in the portfolio are modelled in a contagion hazard rate process setting. The main contribution of the paper is to provide analytic formulas for density functions of default times and prices of basket CDSs. The paper is organized as follows: Section 2 uses the total hazard construction method to find the analytic formula for the joint density function of default times of all underlying names and the k default time distribution. Section 3 extends the results to include the counterparty default risk and obtain the analytic formula for the pricing of basket CDSs with counterparty risk.

2 Total Hazard Construction and Default Time Distribution

Let $(\Omega, \mathcal{F}, \{\mathcal{F}\}_{t\geq 0}, P)$ be a filtered probability space. Let τ_i be the default time name i and $N_i(t) = 1_{\{\tau_i \leq t\}}$ the default indicator process of name i, $i = 1, \ldots, n$, adapted to the filtration $\{\mathcal{F}\}_{t\geq 0}$. Given $\tau_j = t_j$, $j \in J_k = \{j_1, \ldots, j_k\} \subset \{1, \ldots, n\}$, satisfying $0 = t_{j_0} < t_{j_1} < \ldots < t_{j_k}$ and $\tau_i > t > t_{j_k}$ for $i \notin J_k$, the conditional hazard rate of τ_i at time t is given by

$$\lambda_i(t|t_{J_k}) = \lim_{\Delta t \downarrow 0} \frac{1}{\Delta t} P(t < \tau_i \leq t + \Delta t | \tau_j = t_j, j \in J_k)$$

where t_{J_k} is a short form for $(t_{j_1}, \ldots, t_{j_k})$. When $k = 0$, i.e., no defaults have occurred at time t, then $\lambda_i(t|t_{J_k})$ is the unconditional hazard rate $\lambda_i(t)$

of name i at time t. The total hazard accumulated by name i during time interval $[t_{j_k}, t_{j_k} + s]$, $s \geq 0$, is defined by

$$\Lambda_i(s|t_{J_k}) = \int_{t_{j_k}}^{t_{j_k}+s} \lambda_i(u|t_{J_k})du.$$

When $k = 0$ then $\Lambda_i(s|t_{J_k})$ is the unconditional total hazard accumulated by name i, i.e., $\Lambda_i(s) = \int_0^s \lambda_i(u)du$. The total hazard accumulated by name i by time t, given k observed defaults $\tau_j = t_j, j \in J_k$, is given by

$$\psi_i(t|t_{J_k}) = \sum_{l=0}^{k-1} \Lambda_i(t_{j_{l+1}} - t_{j_l}|t_{J_l}) + \Lambda_i(t - t_{j_k}|t_{J_k}).$$

Define the inverse functions

$$\Lambda_i^{-1}(x|t_{J_k}) = \inf\{s \geq 0 : \Lambda_i(s|t_{J_k}) \geq x\}$$

for $i \notin J_k$ and $x \geq 0$. The total hazard can be constructed as follows: Let $X_1, \ldots, X_n$ be independent standard exponential random variables.

Step 1. Let

$$j_1 = \text{argmin}\{\Lambda_i^{-1}(X_i) : i = 1, \ldots, n\}$$

and define

$$\hat{\tau}_{j_1} = \Lambda_{j_1}^{-1}(X_{j_1}) \tag{1}$$

and set $J_1 = \{j_1\}$.

Step $k+1$ ($k = 1, \ldots, n-1$). Given that Steps $1, \ldots, k$ have resulted in $\hat{\tau}_j$, $j \in J_k$. Let

$$j_{k+1} = \text{argmin}\{\Lambda_i^{-1}[X_i - \psi_i(\hat{\tau}_{j_k}|\hat{\tau}_{J_k})|\hat{\tau}_{J_k}] : i \notin J_k\}$$

and define

$$\hat{\tau}_{j_{k+1}} = \hat{\tau}_{j_k} + \Lambda_{j_{k+1}}^{-1}[X_{j_{k+1}} - \psi_{j_{k+1}}(\hat{\tau}_{j_k}|\hat{\tau}_{J_k})|\hat{\tau}_{J_k}] \tag{2}$$

and set $J_{k+1} = J_k \cup \{j_{k+1}\}$.

Norros (1986) proves that $\hat{\tau} = (\hat{\tau}_1, \ldots, \hat{\tau}_n)$ equals $\tau = (\tau_1, \ldots, \tau_n)$ in distribution. We therefore can generate default times τ by generating $\hat{\tau}$ instead.

We now assume that the hazard rate processes be given by

$$\lambda_i(t) = a_{i0} + \sum_{j=1}^{n} a_{ij} 1_{\{\tau_j \le t\}}$$

for $i = 1, \ldots, n$, where $a_{ii} = 0$. Since default times can be constructed with the total hazard method (1) and (2), we can express default times $\hat{\tau}$ in terms of standard exponential random variables X, and vice versa. For example, if $j_k = k$ for $k = 1, \ldots, n$, i.e., $\hat{\tau}_1$ is the first default time, $\hat{\tau}_2$ is the second default time, etc. Then from (1) and (2) we have

$$\begin{aligned} X_1 &= a_{10}\hat{\tau}_1 \\ X_2 &= a_{20}\hat{\tau}_2 + a_{21}(\hat{\tau}_2 - \hat{\tau}_1) \\ X_k &= a_{k0}\hat{\tau}_k + a_{k1}(\hat{\tau}_k - \hat{\tau}_1) + \cdots + a_{k,k-1}(\hat{\tau}_k - \hat{\tau}_{k-1}) \quad \text{for } k \ge 2 \end{aligned}$$

The Jacobi determinant is given by

$$c = \left| \det \left(\frac{\partial(X_1, \ldots, X_n)}{\partial(\hat{\tau}_1, \ldots, \hat{\tau}_n)} \right) \right| = a_{10}(a_{20} + a_{21}) \cdots (a_{n0} + \cdots + a_{n,n-1}).$$

The density of $\hat{\tau}$ for $\hat{\tau}_1 < \ldots < \hat{\tau}_n$ is therefore given by $f(\hat{\tau}_1, \ldots, \hat{\tau}_n) = ce^{-(X_1 + \cdots + X_n)}$. Substituting X_i into f we get

$$f(t_1, t_2, \ldots, t_n) = ce^{-(w_1 t_1 + \cdots + w_n t_n)} \tag{3}$$

for $0 < t_1 < t_2 < \cdots < t_n$, where

$$\begin{aligned} w_1 &= a_{10} - (a_{21} + \cdots + a_{n1}) \\ w_2 &= (a_{20} + a_{21}) - (a_{32} + \cdots + a_{n2}) \\ w_k &= (a_{k0} + \cdots + a_{k,k-1}) - (a_{k+1,k} + \cdots + a_{n,k}) \quad \text{for } k \ge 2 \end{aligned}$$

The space R_+^n can be divided into $n!$ regions according to the order of $(t_1, t_2, \ldots, t_n)$. The density function f in other regions can be expressed similarly with the permutation method. For example, if $n = 2$ then

$$f(t_1, t_2) = \begin{cases} a_{10}(a_{20} + a_{21})e^{-(a_{10} - a_{21})t_1 - (a_{20} + a_{21})t_2} & \text{if } t_1 < t_2 \\ a_{20}(a_{10} + a_{12})e^{-(a_{20} - a_{12})t_2 - (a_{10} + a_{12})t_1} & \text{if } t_2 < t_1 \end{cases}$$

If $n = 3$ then in the region of $t_1 < t_2 < t_3$ we have

$$\begin{aligned} f(t_1, t_2, t_3) &= a_{10}(a_{20} + a_{21})(a_{30} + a_{31} + a_{32}) \\ &\quad \cdot e^{-[(a_{10} - a_{21} - a_{31})t_1 + (a_{20} + a_{21} - a_{32})t_2 + (a_{30} + a_{31} + a_{32})t_3]}. \end{aligned}$$

If we want to find the density function in the region of $t_3 < t_1 < t_2$, we only need to change the indexes of the above equation from $1 \to 3$, $2 \to 1$, and $3 \to 2$, and we have

$$\begin{aligned} f(t_1, t_2, t_3) &= a_{30}(a_{10} + a_{13})(a_{20} + a_{23} + a_{21}) \\ &\quad \cdot e^{-[(a_{30} - a_{13} - a_{23})t_3 + (a_{10} + a_{13} - a_{21})t_1 + (a_{20} + a_{23} + a_{21})t_2]}. \end{aligned}$$

In the subsequent discussion we need to compute the following integral

$$A_k(t, w_1, \ldots, w_k) := \int_0^t \int_{t_1}^t \cdots \int_{t_{k-1}}^t e^{-(w_1 t_1 + \cdots + w_k t_k)} dt_1 \ldots dt_k.$$

Integration with respect to dt_k gives the recursive formula

$$\begin{aligned} A_k(t, w_1, \ldots, w_k) &= -\frac{1}{w_k} e^{-w_k t} A_{k-1}(t, w_1, \ldots, w_{k-1}) \\ &\quad + \frac{1}{w_k} A_{k-1}(t, w_1, \ldots, w_{k-2}, w_{k-1} + w_k) \end{aligned} \tag{4}$$

We can use the induction method to show that

$$A_k(t, w_1, \ldots, w_k) = \sum_{j=0}^{k} \gamma_{k,j}(w_1, \ldots, w_k) e^{-(\sum_{l=j+1}^{k} w_l)t} \tag{5}$$

where

$$\gamma_{k,j}(w_1, \ldots, w_k) = \frac{(-1)^{k-j}}{\left\{\prod_{m=j+1}^{k}(\sum_{l=j+1}^{m} w_l)\right\} \cdot \left\{\prod_{m=1}^{j}(\sum_{l=m}^{j} w_l)\right\}}$$

with the convention that $\prod_{i=j}^{k} a_i = 1$ and $\sum_{i=j}^{k} a_i = 0$ if $k < j$. In fact, (5) trivially holds for $k = 1$. Assume (5) holds for $k - 1$ where $k \geq 2$. Substituting (5) into (4) and combining the first $k - 1$ (i.e., $j = 0, \ldots, k - 2$) terms together, we can show that (5) holds for k too.

Note that by change of variables we have

$$\begin{aligned}
&\int_{\alpha}^{t}\int_{t_1}^{t}\cdots\int_{t_{k-1}}^{t} e^{-(w_1t_1+\cdots+w_kt_k)}dt_1\ldots dt_k \\
= \quad & e^{-(w_1+\cdots+w_k)\alpha}A_k(t-\alpha, w_1,\ldots,w_k) \\
= \quad & \sum_{j=0}^{k}\gamma_{k,j}(w_1,\ldots,w_k)e^{-(\sum_{l=j+1}^{k}w_l)t-(\sum_{l=1}^{j}w_l)\alpha}
\end{aligned}$$

Differentiating with respect to t and simplifying, we have

$$\frac{d}{dt}A_k(t,w_1,\ldots,w_k) = e^{-w_kt}A_{k-1}(t,w_1,\ldots,w_{k-1}).$$

3 Basket CDS Pricing without Counterparty Risk

Denote τ^k the kth default time and $N(t)$ the number of defaults by time t, i.e., $N(t) = 1_{\{\tau_1\le t\}} + \cdots + 1_{\{\tau_n\le t\}}$. The probabilities of $N(t)$ and τ^k can be computed as follows:

$$P(N(t)=k) = P(\tau^k \le t < \tau^{k+1})$$

and

$$P(\tau^k > t) = \sum_{i=0}^{k-1}P(\tau^i \le t < \tau^{i+1}).$$

To compute the probability $P(\tau^i \le t < \tau^{i+1})$ we may order default times $\{\tau_1,\ldots,\tau_n\}$ in such a way that the first i default times are less than or equal to t and the rest greater than t, then compute the probability for each case, e.g., $P(\tau_1,\ldots,\tau_i \le t, \tau_{i+1},\ldots t_n > t)$, and finally sum all the probabilities. Since there are $n!$ different combinations this is computationally infeasible for general a_{i0} and a_{ij}. We assume from now on that

$$\lambda_i(t) = a + \sum_{j=1,j\ne i}^{n} b1_{\{\tau_j\le t\}}. \tag{6}$$

The next result characterizes the probability distributions of $N(t)$ and τ^k.

Lemma 1 *The probability $P(N(t)=k)$ is given by*

$$P(N(t)=k)=\sum_{j=0}^{k}\frac{c_k}{d_{k,j}}e^{-\beta_j t}$$

and the density function of τ^k is given by

$$f_{\tau^k}(t)=\sum_{j=0}^{k-1}\alpha_{k,j}e^{-\beta_j t}$$

where $\alpha_{k,j}=c_k/d_{k-1,j}$, $\beta_j=(n-j)(a+jb)$, and

$$\begin{aligned} c_k &= \frac{n!}{(n-k)!}\prod_{m=0}^{k-1}(a+mb) \\ d_{k,j} &= (-1)^{k-j}j!(k-j)!\prod_{m=0,m\neq j}^{k}(a-(n-m-j)b). \end{aligned}$$

If k is small, $\alpha_{k,j}$ can be computed directly. If k is large, to avoid large number of multiplications, we can first compute

$$\alpha_{k,0}=(-1)^{k-1}\frac{n!}{(k-1)!(n-k)!}(a-nb)\prod_{m=0}^{k-1}\frac{(a+mb)}{(a-(n-m)b)}$$

then compute $\alpha_{k,j}$, $j=1,\ldots,k-1$, with the recursive formula

$$\alpha_{k,j}=c_{k,j}\alpha_{k,j-1}$$

where

$$c_{k,j}=\frac{(-1)(k-j)(a-(n-2j)b)(a-(n-j+1)b)}{j(a-(n-2j+2)b)(a-(n-k-j+1)b)}.$$

For $k=1,2$, we have $\alpha_{1,0}=na$, $\alpha_{2,0}=-n(n-1)a(a+b)/(a-(n-1)b)$, $c_{2,1}=-1$, and $\alpha_{2,1}=-\alpha_{2,0}$, therefore

$$\begin{aligned} f_{\tau^1}(t) &= nae^{-nat} \\ f_{\tau^2}(t) &= \frac{n(n-1)a(a+b)}{(a-(n-1)b)}(-e^{-nat}+e^{-(n-1)(a+b)t}). \end{aligned}$$

The contagion has no effect on the first default time, but affects all subsequent default times.

We can now price kth default basket CDS without counterparty risk. Assume X_k is the annualized swap rate, paid at time t_i, $i = 1, \ldots, N$, where $0 = t_0 < t_1 < \ldots < t_N = T$ and T is the maturity time of the contract, $\Delta_i = t_i - t_{i-1}$, R is recovery rate, r is the riskless interest rate, and δ is the settlement period. The present value of the contingent leg is equal to

$$C_k = (1-R)e^{-r(\tau^k+\delta)}1_{\{\tau^k \le T\}}$$

and the present value of the fee leg is equal to

$$F_k = \sum_{i=1}^{N} \left[X_k \Delta_i e^{-rt_i} 1_{\{\tau^k > t_i\}} + X_k(\tau^k - t_{i-1})e^{-r\tau^k} 1_{\{t_{i-1} < \tau^k \le t_i\}} \right].$$

The swap rate X_k is characterized in the next result.

Theorem 1 *Let the harzard rate processes $\lambda_i(t)$ be given by (6). Then*

$$\begin{aligned} E(C_k) &= (1-R)e^{-r\delta} \sum_{j=0}^{k-1} \frac{\alpha_{k,j}}{r+\beta_j}(1 - e^{-(r+\beta_j)T}) \\ E(F_k) &= X_k \sum_{i=1}^{N} \sum_{j=0}^{k-1} \frac{\alpha_{k,j}}{(r+\beta_j)^2} \left[e^{-(r+\beta_j)t_{i-1}} + \left(\frac{\Delta_i r(r+\beta_j)}{\beta_j} - 1 \right) e^{-(r+\beta_j)t_i} \right] \end{aligned}$$

where $\alpha_{k,j}$ and β_j are given in Lemma 1. The kth default swap rate X_k is obtained by equating $E(C_k)$ and $E(F_k)$.

Theorem 1 is the special case of Theorem 2 with counterparty risk.

4 Basket CDS Pricing with Counterparty Risk

Since $\tau^1 < \tau^2 < \ldots < \tau^n$ the joint density function of $\tau = (\tau^1, \ldots, \tau^n)$ is given by

$$f_\tau(t_1, \ldots, t_n) = n! c e^{-(w_1 t_1 + \cdots + w_n t_n)}$$

for $0 < t_1 < \ldots < t_n$ and 0 for all other $t_1 \ldots, t_n$. Suppose the default time τ^B is modelled by the intensity process

$$\lambda_B(t) = \lambda_0 + \sum_{i=1}^{n} \lambda_i 1_{\{\tau^i \le t\}}. \tag{7}$$

Note that the hazard rate process $\lambda_B(t)$ of basket CDS seller B are influenced by the defaults of underlying names i, but not vice versa. This follows the observations in Leung and Kwok (2005) and Yu (2007) that the contagion of seller B on underlying names i does not have impact on CDS pricing.

We now study the joint distribution of (τ^k, τ^B) for $k = 1, \ldots, n$ and have the following result.

Lemma 2 *The joint density functions of (τ^k, τ^B), $k = 1, \ldots, n$, are given by*

$$f_{\tau^k,\tau^B}(t_k, t_B) = \sum_{i=k}^{n} \sum_{j=0}^{i-k} \sum_{j_1=0}^{k-1} \delta_{i,k,j,j_1} e^{-(\sum_{l=0}^{k+j} \lambda_l + \sum_{l=k+j+1}^{n} w_l) t_B - (\sum_{l=j_1+1}^{k+j} \tilde{w}_l) t_k} \tag{8}$$

for $t_k < t_B$ and

$$f_{\tau^k,\tau^B}(t_k, t_B) = \sum_{i=0}^{k-1} \sum_{j=0}^{k-1-i} \sum_{j_1=0}^{i} \tilde{\delta}_{i,k,j,j_1} e^{-(\sum_{l=0}^{j_1} \lambda_l + \sum_{l=j_1+1}^{i+j} w_l) t_B - (\sum_{l=i+j+1}^{n} w_l) t_k} \tag{9}$$

for $t_k > t_B$, where

$$\begin{aligned}
\delta_{i,k,j,j_1} &= c_i \gamma_{i-k,j}(\tilde{w}_{k+1}, \ldots, \tilde{w}_i) \gamma_{k-1,j_1}(\tilde{w}_1, \ldots, \tilde{w}_{k-1}) (\sum_{l=0}^{k+j} \lambda_l + \sum_{l=k+j+1}^{n} w_l) \\
\tilde{\delta}_{i,k,j,j_1} &= c_k \gamma_{k-1-i,j}(w_{i+1}, \ldots, w_{k-1}) \gamma_{i,j_1}(\tilde{w}_1, \ldots, \tilde{w}_i) (\sum_{l=0}^{j_1} \lambda_l + \sum_{l=j_1+1}^{i+j} w_l)
\end{aligned}$$

The complicated expressions for the density function f in Lemma 2 is due to the assumption that the hazard rate process $\lambda_B(t)$ depends on default times of name i, $i = 1, \ldots, n$. For example, to compute the joint probability $P(\tau^k > t_k, \tau^B > t_B)$ for $t_k > t_B$ we need to know how many defaults have occurred before time t_B to determine uniquely the hazard rate process $\lambda_B(t)$ and we have to compute probabilities of k mutually independent events $\{\tau^k > t_k, \tau^i < t_B, \tau^{i+1} > t_B, \tau^B > t_B\}$, $i = 0, \ldots, k-1$, with $\tau^0 = 0$. If the hazard rate process of name B does not depend on default times of all names i, then we may simplify the computation.

Assume the default time τ^B is modelled by the intensity process

$$\lambda_B(t) = \lambda_0 + \lambda_k 1_{\{\tau^k \le t\}}. \tag{10}$$

For $t_k < t_B$ we have

$$\begin{aligned} P(\tau^k < t_k, \tau^B > t_B) &= E_{\tau^k}(1_{\{\tau^k < t_k\}} E(1_{\{\tau^B > t_B\}} | \tau^k)) \\ &= \int_0^{t_k} f_{\tau^k}(s) E(1_{\{\tau^B > t_B\}} | \tau^k = s) ds \\ &= \int_0^{t_k} f_{\tau^k}(s) e^{-(\lambda_0+\lambda_k)t_B + \lambda_k s} ds \end{aligned}$$

For $t_B < t_k$ we have

$$\begin{aligned} P(\tau^k > t_k, \tau^B < t_B) &= E_{\tau^k}(1_{\{\tau^k > t_k\}} E(1_{\{\tau^B < t_B\}} | \tau^k)) \\ &= \int_{t_k}^{\infty} f_{\tau^k}(s) E(1_{\{\tau^B < t_B\}} | \tau^k = s) ds \\ &= \int_{t_k}^{\infty} f_{\tau^k}(s)(1 - e^{-\lambda_0 t_B}) ds. \end{aligned}$$

The density function of (τ^k, τ^B) is given by

$$f_{\tau^k,\tau^B}(t_k, t_B) = \begin{cases} f_{\tau^k}(t_k)(\lambda_0 + \lambda_k) e^{\lambda_k t_k - (\lambda_0+\lambda_k)t_B} & \text{if } t_k < t_B \\ f_{\tau^k}(t_k)\lambda_0 e^{-\lambda_0 t_B} & \text{if } t_k > t_B \end{cases}$$

We can now price kth default basket CDS. Assume X_k is the annualized swap rate, paid at time t_i, $i = 1, \ldots, N$, where $0 = t_0 < t_1 < \ldots < t_N = T$ and T is the maturity time of the contract, $\Delta_i = t_i - t_{i-1}$, R is recovery rate, r is the riskless interest rate, and δ is the settlement period. The present value of the contingent leg is equal to

$$C_k = (1-R) e^{-r(\tau^k+\delta)} 1_{\{\tau^k \le T, \tau^B \ge \tau^k + \delta\}}$$

and the present value of the fee leg is equal to

$$F_k = \sum_{i=1}^{N} \left[X_k \Delta_i e^{-r t_i} 1_{\{\tau^k > t_i, \tau^B > t_i\}} + X_k(\tau^k - t_{i-1}) e^{-r\tau^k} 1_{\{t_{i-1} < \tau^k \le t_i, \tau^B \ge t^k\}} \right] x.$$

The swap rate X_k is characterized in the next result.

Theorem 2 *Let the harzard rate processes λ_t^i and $\lambda_B(t)$ be given by (6) and (10) respectively. Then*

$$
\begin{aligned}
E(C_k) &= (1-R)e^{-(r+\lambda_0+\lambda_k)\delta}\sum_{j=0}^{k-1}\frac{\alpha_{k,j}}{\gamma_j}(1-e^{-\gamma_j T}) \\
E(F_k) &= X_k\sum_{i=1}^{N}\sum_{j=0}^{k-1}\frac{\alpha_{k,j}}{\gamma_j^2}\left[e^{-\gamma_j t_{i-1}}+\left(\frac{\Delta_i(r+\lambda_0)\gamma_j}{\beta_j}-1\right)e^{-\gamma_j t_i}\right]
\end{aligned}
$$

where $\alpha_{k,j}$ and β_j are given in Lemma 1 and $\gamma_j = r+\lambda_0+\beta_j$. The kth default swap rate X_k is obtained by equating $E(C_k)$ and $E(F_k)$.

Note that if the settlement period $\delta = 0$ then there is no contagion effect of the kth default on the swap rate X_k. We can similarly derive the kth default swap rate X_k for the general hazard rate process $\lambda_B(t)$ of (7).

5 Appendix: Proofs of Theorems.

Proof of Lemma 1:

$$
\begin{aligned}
& P(\tau_1 < \ldots < \tau_k < t < \tau_{k+1} < \ldots < \tau_n) \\
= & \int_0^t\int_{t_1}^t\cdots\int_{t_{k-1}}^t\int_t^\infty\int_{t_{k+!}}^\infty\cdots\int_{t_{n-1}}^\infty f(t_1,\ldots,t_n)dt_1\ldots dt_n \\
= & c\left(\int_0^t\int_{t_1}^t\cdots\int_{t_{k-1}}^t e^{-w_1t_1-\cdots-w_kt_k}dt_1\ldots dt_k\right) \\
& \cdot\left(\int_t^\infty\int_{t_{k+1}}^\infty\cdots\int_{t_{n-1}}^\infty e^{-w_{k+1}t_{k+1}-\cdots-w_nt_n}dt_{k+1}\ldots dt_n\right) \\
= & c\left(\sum_{j=0}^k\gamma_{k,j}(w_1,\ldots,w_k)e^{-(\sum_{l=j+1}^k w_l)t}\right)\left(\frac{1}{\prod_{m=k+1}^n(\sum_{l=m}^n w_l)}e^{-(\sum_{l=k+1}^n w_l)t}\right) \\
= & \sum_{j=0}^k(-1)^{k-j}\frac{(a-(n-2j)b)\prod_{m=0}^{k-1}(a+mb)}{(k-j)!j!(n-k)!\prod_{m=0}^k(a-(n-m-j)b)}e^{-(n-j)(a+jb)t}.
\end{aligned}
$$

Here we have used the relations

$$
c = \prod_{m=0}^{n-1}(a+mb)
$$

$$
\begin{aligned}
w_l &= a-(n-2l+1)b \\
\sum_{l=n_1}^{n_2} w_l &= (n_2-n_1+1)(a-(n+1-n_1-n_2)b) \qquad \text{for } n_1 \le n_2 \\
\prod_{m=j+1}^{k}(\sum_{l=j+1}^{m} w_l) &= (k-j)!\prod_{m=j+1}^{k}(a-(n-m-j)b) \\
\prod_{m=1}^{j}(\sum_{l=m}^{j} w_l) &= j!\prod_{m=0}^{j-1}(a-(n-m-j)b) \\
\prod_{m=k+1}^{n}(\sum_{l=m}^{n} w_l) &= (n-k)!\prod_{m=k}^{n-1}(a+mb) \\
\frac{c\gamma_{k,j}(w_1,\ldots,w_k)}{\prod_{m=k+1}^{n}(\sum_{l=m}^{n} w_l)} &= (-1)^{k-j}\frac{(a-(n-2j)b)\prod_{m=0}^{k-1}(a+mb)}{(k-j)!j!(n-k)!\prod_{m=0}^{k}(a-(n-m-j)b)}
\end{aligned}
$$

The homogeneous and symmetric property implies that

$$P(\tau^k \le t < \tau^{k+1}) = n!P(\tau_1 < \ldots < \tau_k < t < \tau_{k+1} < \ldots < \tau_n)$$

which gives the probability $P(N(t)=k)$. We now compute the kth default time distribution

$$
\begin{aligned}
& P(\tau^k > t) \\
= & \sum_{i=0}^{k-1} P(\tau^i < t, \tau^{i+1} > t) \\
= & \sum_{i=0}^{k-1} n! \sum_{j=0}^{i} \frac{(-1)^{i-j}(a-(n-2j)b)\prod_{m=0}^{i-1}(a+mb)}{(i-j)!j!(n-i)!\prod_{m=0}^{i}(a-(n-m-j)b)} e^{-(n-j)(a+jb)t} \\
= & \sum_{j=0}^{k-1}\left\{\sum_{i=j}^{k-1}\frac{(-1)^{i-j}\prod_{m=0}^{i-1}(a+mb)}{(i-j)!(n-i)!\prod_{m=0}^{i}(a-(n-m-j)b)}\right\} \\
& \cdot n!\frac{(a-(n-2j)b)}{j!}e^{-(n-j)(a+jb)t} \\
= & \sum_{j=0}^{k-1}\left\{\sum_{i=0}^{k-1-j}\frac{(-1)^{i}\prod_{m=0}^{i+j-1}(a+mb)}{i!(n-i-j)!\prod_{m=0}^{i+j}(a-(n-m-j)b)}\right\} \\
& \cdot n!\frac{(a-(n-2j)b)}{j!}e^{-(n-j)(a+jb)t}
\end{aligned}
$$

$$= \sum_{j=0}^{k-1} \left\{ \frac{(-1)^{k-1-j}}{(k-1-j)!(n-j)(n-k)!(a+jb)} \prod_{m=0}^{k-1} \frac{(a+mb)}{(a-(n-m-j)b)} \right\}$$
$$\cdot n! \frac{(a-(n-2j)b)}{j!} e^{-(n-j)(a+jb)t}.$$

Here we have used the relation

$$\sum_{i=0}^{l} \frac{(-1)^i \prod_{m=1}^{i+j}(a+(m-1)b)}{i!(n-i-j)! \prod_{m=0}^{i+j}(a-(n-m-j)b)}$$
$$= \frac{(-1)^l}{l!(n-j)(n-j-l-1)!(a+jb)} \prod_{m=0}^{l+j} \frac{(a+mb)}{(a-(n-m-j)b)}$$

which can be proved with the induction method. Differentiating $P(\tau^k > t)$ with respect to t leads to the density function of τ^k. □

Proof of Lemma 2. For $t_k < t_B$ we have

$$P(\tau^k < t_k, \tau^B > t_B) = \sum_{i=k}^{n} P(\tau^k < t_k, \tau^i < t_B, \tau^{i+1} > t_B, \tau^B > t_B)$$

where $\tau^{n+1} = \infty$. Since

$$E(1_{\{\tau^B > t_B\}} | \tau^i < t_B, \tau^{i+1} > t_B) = e^{\lambda_1 \tau^1 + \cdots + \lambda_i \tau^i - (\lambda_0 + \cdots + \lambda_i) t_B}$$

and

$$\int_{t_B}^{\infty} \int_{s_{i+1}}^{\infty} \cdots \int_{s_{n-1}}^{\infty} n! c e^{-(\sum_{l=i+1}^{n} w_l s_l)} ds_{i+1} \dots ds_n = c_i e^{-(\sum_{l=i+1}^{n} w_l) t_B}$$

where $c_i = \frac{n!}{(n-i)!} \prod_{m=0}^{i-1}(a+mb)$, we have (and denote $\tilde{w}_l = w_l - \lambda_l$)

$$P(\tau^k < t_k, \tau^i < t_B, \tau^{i+1} > t_B, \tau^B > t_B)$$
$$= E_\tau \left[1_{\{\tau^1 < \dots < \tau^k < t_k\}} 1_{\{\tau^k < \dots < t^i < t_B\}} 1_{\{t_B < \tau^{i+1} < \dots < \tau^n\}} E(1_{\{\tau^B > t_B\}} | \tau) \right]$$
$$= \int_0^{t_k} \int_{s_1}^{t_k} \cdots \int_{s_{k-1}}^{t_k} \int_{s_k}^{t_B} \int_{s_{k+1}}^{t_B} \cdots \int_{s_{i-1}}^{t_B} \int_{t_B}^{\infty} \int_{s_{i+1}}^{\infty} \cdots \int_{s_{n-1}}^{\infty}$$
$$n! c e^{-(w_1 s_1 + \cdots + w_n s_n)} E(1_{\{\tau^B > t_B\}} | \tau^1 = s_1, \dots, \tau^n = s_n) ds_1 \dots ds_n$$
$$= c_i e^{-(\sum_{l=0}^{i} \lambda_l + \sum_{l=i+1}^{n} w_l) t_B}$$

$$
\begin{aligned}
& \cdot \int_0^{t_k} \int_{s_1}^{t_k} \cdots \int_{s_{k-1}}^{t_k} \int_{s_k}^{t_B} \int_{s_{k+1}}^{t_B} \cdots \int_{s_{i-1}}^{t_B} e^{-(\tilde{w}_1 s_1 + \cdots \tilde{w}_i s_i)} ds_1 \ldots ds_i \\
= \; & c_i e^{-(\sum_{l=0}^{i} \lambda_l + \sum_{l=i+1}^{n} w_l) t_B} \int_0^{t_k} \int_{s_1}^{t_k} \cdots \int_{s_{k-1}}^{t_k} e^{-(\tilde{w}_1 s_1 + \cdots + \tilde{w}_k s_k)} \\
& \cdot \sum_{j=0}^{i-k} \gamma_{i-k,j}(\tilde{w}_{k+1}, \ldots, \tilde{w}_i) e^{-(\tilde{w}_{k+j+1} + \cdots + \tilde{w}_i) t_B - (\tilde{w}_{k+1} + \cdots + \tilde{w}_{k+j}) s_k} ds_1 \ldots ds_k \\
= \; & \sum_{j=0}^{i-k} c_i \gamma_{i-k,j}(\tilde{w}_{k+1}, \ldots, \tilde{w}_i) e^{-(\sum_{l=0}^{k+j} \lambda_l + \sum_{l=k+j+1}^{n} w_l) t_B} \\
& \cdot A_k(t_k, \tilde{w}_1, \ldots, \tilde{w}_{k-1}, \tilde{w}_k + \cdots + \tilde{w}_{k+j})
\end{aligned}
$$

Differentiating with respect to t_k and t_B gives the joint density function f in (8).

Similarly, for $t_k > t_B$ we have

$$
P(\tau^k > t_k, \tau^B > t_B) = \sum_{i=0}^{k-1} P(\tau^k > t_k, \tau^i < t_B, \tau^{i+1} > t_B, \tau^B > t_B)
$$

where $\tau^0 = 0$. Next we compute these probabilities

$$
\begin{aligned}
& P(\tau^k > t_k, \tau^i < t_B, \tau^{i+1} > t_B, \tau^B > t_B) \\
= \; & E_\tau \left[1_{\{\tau^1 < \ldots < \tau^i < t_B\}} 1_{\{t_B < \tau^{i+1} < \ldots < \tau^{k-1} < \tau^k\}} 1_{\{t_k < \tau^k\}} 1_{\{\tau^k < \tau^{k+1} < \ldots < \tau^n\}} E(1_{\{\tau^B > t_B\}} | \tau) \right] \\
= \; & \int_0^{t_B} \int_{s_1}^{t_B} \cdots \int_{s_{i-1}}^{t_B} n! c e^{-(\sum_{l=1}^{i} \tilde{w}_l s_l) - (\sum_{l=0}^{i} \lambda_l) t_B} ds_1 \ldots ds_i \\
& \cdot \int_{t_k}^{\infty} \left(\int_{t_B}^{s_k} \int_{s_{i+1}}^{s_k} \cdots \int_{s_{k-2}}^{s_k} e^{-\sum_{l=i+1}^{k-1} w_l s_l} ds_{i+1} \ldots ds_{k-1} \right) \\
& \quad \cdot \left(\int_{s_k}^{\infty} \int_{s_{k+1}}^{\infty} \cdots \int_{s_{n-1}}^{\infty} e^{-\sum_{l=k+1}^{n} w_l s_l} ds_{k+1} \ldots ds_n \right) e^{-w_k s_k} ds_k \\
= \; & c_k e^{-(\sum_{l=0}^{i} \lambda_l) t_B} \sum_{j_1=0}^{i} \gamma_{i,j_1}(\tilde{w}_1, \ldots, \tilde{w}_i) e^{-(\sum_{l=j_1+1}^{i} \tilde{w}_l) t_B} \\
& \cdot \int_{t_k}^{\infty} \left(\sum_{j=0}^{k-1-i} \gamma_{k-1-i,j}(w_{i+1}, \ldots, w_{k-1}) e^{-(\sum_{l=i+j+1}^{k-1} w_l) s_k - (\sum_{l=i+1}^{i+j} w_l) t_B)} \right) \\
& \quad \cdot e^{-(\sum_{l=k}^{n} w_l) s_k} ds_k
\end{aligned}
$$

$$= \sum_{j=0}^{k-1-i} \sum_{j_1=0}^{i} \frac{c_k \gamma_{k-1-i,j}(w_{i+1},\ldots,w_{k-1})\gamma_{i,j_1}(\tilde{w}_1,\ldots,\tilde{w}_i)}{(\sum_{l=i+j+1}^{n} w_l)}$$
$$\cdot e^{-(\sum_{l=0}^{j_1}\lambda_l + \sum_{l=j_1+1}^{i+j} w_l)t_B - (\sum_{l=i+j+1}^{n} w_l)t_k}$$

Differentiating with respect to t_k and t_B gives the joint density function f in (9). $\square$

Proof of Theorem 2. The expected present value of the contingent leg is equal to

$$\begin{aligned}
E(C_k) &= E\Big((1-R)e^{-r(\tau^k+\delta)}1_{\{\tau^k\le T,\tau^B\ge\tau^k+\delta\}}\Big) \\
&= (1-R)\int_0^T \int_{s+\delta}^{\infty} e^{-r(s+\delta)} f_{\tau^k,\tau^B}(s,t)dsdt \\
&= (1-R)\int_0^T \int_{s+\delta}^{\infty} e^{-r(s+\delta)} f_{\tau^k}(s)(\lambda_0+\lambda_k)e^{\lambda_k s-(\lambda_0+\lambda_k)t}dsdt \\
&= (1-R)\int_0^T e^{-r(s+\delta)} f_{\tau^k}(s)e^{-\lambda_0 s-(\lambda_0+\lambda_k)\delta}dt \\
&= (1-R)\sum_{j=0}^{k-1}\frac{\alpha_{k,j}e^{-(r+\lambda_0+\lambda_k)\delta}}{r+\lambda_0+\beta_j}(1-e^{-(r+\lambda_0+\beta_j)T})
\end{aligned}$$

and the expected present value of the fee leg is equal to

$$E(F_k) = E\left(\sum_{i=1}^{N}\Big[X_k\Delta_i e^{-rt_i}1_{\{\tau^k>t_i,\tau^B>t_i\}} + X_k(\tau^k-t_{i-1})e^{-r\tau^k}1_{\{t_{i-1}<\tau^k\le t_i,\tau^B\ge t^k\}}\Big]\right).$$

We now compute

$$\begin{aligned}
&E(1_{\{\tau^k>t_i,\tau^B>t_i\}}) \\
&= \int_{t_i}^{\infty}\int_{t_i}^{\infty} f_{\tau^k,\tau^B}(s,t)dsdt \\
&= \int_{t_i}^{\infty}\int_{t_i}^{s} f_{\tau^k}(s)\lambda_0 e^{-\lambda_0 t}dsdt + \int_{t_i}^{\infty}\int_{s}^{\infty} f_{\tau^k}(s)(\lambda_0+\lambda_k)e^{\lambda_k s-(\lambda_0+\lambda_k)t}dsdt \\
&= \int_{t_i}^{\infty} f_{\tau^k}(s)(e^{-\lambda_0 t_i}-e^{-\lambda_0 s})ds + \int_{t_i}^{\infty} f_{\tau^k}(s)e^{-\lambda_0 s}ds \\
&= \sum_{j=0}^{k-1}\frac{\alpha_{k,j}}{\beta_j}e^{-(\lambda_0+\beta_j)t_i}
\end{aligned}$$

and

$$\begin{aligned}
& E((\tau^k - t_{i-1})e^{-r\tau^k}1_{\{t_{i-1}<\tau^k\le t_i, \tau^B \ge t^k\}}) \\
= & \int_{t_{i-1}}^{t_i}\int_s^\infty (s-t_{i-1})e^{-rs} f_{\tau^k,\tau^B}(s,t)dsdt \\
= & \int_{t_{i-1}}^{t_i}\int_s^\infty (s-t_{i-1})e^{-rs} f_{\tau^k}(s)(\lambda_0+\lambda_k)e^{\lambda_k s-(\lambda_0+\lambda_k)t}dsdt \\
= & \int_{t_{i-1}}^{t_i} (s-t_{i-1})e^{-rs} f_{\tau^k}(s)e^{-\lambda_0 s}ds \\
= & \sum_{j=0}^{k-1}\frac{\alpha_{k,j}}{\gamma_j^2}\left[e^{-\gamma_j t_{i-1}} - (\Delta_i\gamma_j+1)e^{-\gamma_j t_i}\right]
\end{aligned}$$

where $\gamma_j = r + \lambda_0 + \beta_j$. Therefore

$$E(F_k) = X_k \sum_{i=1}^{N}\sum_{j=0}^{k-1}\frac{\alpha_{k,j}}{\gamma_j^2}\left[e^{-\gamma_j t_{i-1}} + \left(\frac{\Delta_i(r+\lambda_0)\gamma_j}{\beta_j} - 1\right)e^{-\gamma_j t_i}\right].$$

□

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