Bond Prices Via Nuclear Space Valued Semi-Martingales

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1 Introduction

One of the most recent and advanced techniques employed for modeling term structure dynamics is using the infinite dimensional framework, referred most of the time as random field or string models. The basic motivation behind using infinite dimensional settings is their ability to capture the correlation structure of zero-coupon bonds with different maturities in a parsimonious and accurate manner. Using finite dimensional models leads to certain inconsistencies not only in practical implementations such as hedging contingent claims and calibration of term structure but also in statistical descriptions of the bond prices [1]. Although infinite dimensional models are not a panacea for all the complications inherited with finite dimensional counterparts, they become popular in term structure modeling especially with the introduction of random field models by Kennedy [2, 3] and Goldstein [4]. In this work, we extend the infinite-dimensional framework by placing the stochastic components, both continuous and discontinuous, of the forward rate on a pair of nuclear spaces in duality and finding the martingale condition of discounted zero-coupon bond prices to preclude the arbitrage.

We place this work in the continuation of the pioneering works by Bjork et al. [5, 6]. Since then much has been said on the topic of interest rate modeling. We ask the authors to be indulgent with us if we have omitted to put their names in the list of the bibliography we have added at the end of this paper. According to us, the origin of all these works is the framework of Heath, Jarrow and Morton [7], in which the price of a zero coupon bond is represented as

$$P(t,T) = exp\left[-\int_t^T f(t,s)ds\right]$$

where the forward interest rate is a semi-martingale

$$df(t,s) = \mu(t,s)dt + \sigma(t,s)dW(t)$$

driven by a Brownian motion $W(t)$. Their interesting result concerns the fact that, under the risk neutral probability, μ is entirely dependent on σ.

$$\mu(t,T) = \sigma(t,T)\int_t^T \sigma(t,u)du$$

In the literature this result has been called the HJM drift condition. The majority of publications have dealt with the extension of this condition under various representations of forward interest rates. As examples, we can give Bjork et al.[5] work with Marked Point Process and Kennedy [2, 3] and Goldstein's studies [4] where the noise process is a random field depending on the present time t and time of maturity T. The work presented by Korezlioglu [8] at the first AMAMEF conference can be associated with Kennedy and Goldstein's studies where the random field is generated by a Brownian Sheet. The idea of using the random field approach has been developed further by Hamza et al. [9] using semi-martingales, by Santa-Clara and Sornette [10] using string models, and by Ozkan and Schmidt [11] using Levy random fields. We would also like to mention the works by De Donno and Pratelli [12] who extend the original idea of Bjork et al. [5, 6]

In this work, we introduce a noise process taking values in the dual F' of a nuclear space F, which is also supposed to be nuclear. Such examples of nuclear dual pairs are well known items of mathematical analysis. For example, any finite dimensional vector space is nuclear, since any operator on a finite dimensional vector space is nuclear. Additionally and more relevant to our interest, the space of smooth functions on any compact manifold and the Schwartz space of smooth functions on $\mathbb{R}^n$, for which the derivatives of all orders are rapidly decreasing can be given as examples of nuclear spaces. This would stress the importance of our approach. The stochastic integration with respect to F'- valued square integrable martingales have almost all trajectories in a Hilbertian subspace of F'. Hence, this observation reduces the stochastic integration with respect to F' valued square integrable martingales to the stochastic integration on a Hilbert space. This idea is fully used in [13] where the method of Metivier and Pistone [14] was revised. This same approach has been used here as follows. The forward interest rate is represented as

$$f(t,s) = f(0,s) + \int_0^t \mu(u, s-u)du + \int_0^t \sigma(u, s-u)dM(u)$$

This type of model was also considered by Ozkan and Schmidt [11] for the case of square integrable Hilbert space valued Levy Processes where σ was considered as a uniformly bounded functional. In our approach, the stochastic integral with respect to F'-valued square integrable martingales has an extended version to not necessarily continuous functional valued processes. Instead of giving a result concerning the global representation of $f(t,T)$, given above, we wanted to separate the continuous and discontinuous parts of M. However, as a remark, we can not use the same σ for both the continuous and discontinuous parts of M because they do not guarantee the same space of integrable processes.

In the following, we give first the necessary mathematical preliminaries regarding the nuclear space valued martingales and their stochastic integral. After giving foundations, the next section is devoted to the case of a general square integrable martingale and the corresponding extended HJM condition. The second model that we consider concerns the case where the discontinuous part of M is generated by a F' valued Markov Jump Process. Finally we consider the case of a square integrable Levy Process with values in F'. Our result is very similar to that of [11]. The difference lies only in our extended definition of stochastic integrals.

2 Preliminaries and Construction of Stochastic Integral

2.1 Nuclear Spaces

The topological vector spaces considered here are over the field $\mathbb{R}$. Given two locally convex vector spaces in duality (E, E'), where E' denotes the dual of E, $e'(e)$ or (e', e) or, if more precision is needed, $(e', e)_{E'E}$ will represent the value of $e' \in E'$ at $e \in E$. For any absolutely convex set $A \subset E$, p_A will denote its gauge. For two locally convex spaces E and F, the space of continuous linear mappings of E into F

is denoted by $L(E, F)$. We refer to Schaefer's book [15] for the general properties of topological vector spaces used in this work.

Definition 2.1 (Nuclear Space)**.** A nuclear space is a locally convex topological vector space V such that for any seminorm p we can find a larger seminorm q so that the natural map from V_q, Banach space given by completing V using the seminorm q, to V_p is a nuclear operator.

Let E be a complete nuclear space. If U is an absolutely convex neighborhood of zero in E, $E(U)$ is the completion of the normed space $(E/p_U^{-1}(0), p_U)$ and $k(U)$ the canonical mapping of E into $E(U)$. For two absolutely continuous convex neighborhoods of 0, U and V in E such that $U \subset V$, the canonical mapping of $E(U)$ into $E(V)$ is denoted by $k(V, U)$ and satisfies the relation: $k(V, U) \circ K(U) = k(V)$. Since E is nuclear there exists a neighborhood base $\mathcal{U}_h(E)$ such that $\forall\ U \in \mathcal{U}_h(E)$, $E(U)$ is a separable Hilbert space and for all $U, V \in \mathcal{U}_h(E)$ such that $U \subset V$ the canonical mappings $k(U)$ and $k(V, U)$ are nuclear operators.

If B is any non-empty closed, bounded and absolutely convex subset of E, then $E[B]$ denotes the Banach subspace of E generated by B and equipped with the norm p_B. The canonical injection of $E[B]$ into E is denoted by $i(B)$. For two bounded and absolutely convex closed subsets A and B of E such that $A \subset B$, the canonical injection of $E[A]$ into $E[B]$ is denoted by $i(B, A)$.

2.2 Construction of the Stochastic Integral

The stochastic integral on nuclear spaces was introduced by Ustunel [16] for semimartingales, by Korezlioglu and Martias [13] for square integrable martingales and by Bojdecki and Jakubowski [18] and Korezlioglu and Martias [13] for Brownian motion.

In this work F represents a nuclear space which is separable and complete. Its strong topological dual F' is also supposed to be complete and nuclear.The fact that F and F' are complete nuclear spaces implies their reflexivity.

For $U \in \mathcal{U}_h(F)$, U^0 denotes its polar and $F'[U^0]$ is shown to be isometric to $F(U)'$, the topological dual of $F(U)$.

All random variables and processes considered here are supposed to be defined on the same probability space $(\Omega, \mathcal{A}, \mathbb{P})$, equipped with the filtration $(\mathcal{F}_t)_t \in \mathbb{R}_+$ satisfying the usual conditions.

We put $\Omega' = \mathbb{R}_+ \times \Omega$. A mapping $X : \Omega' \to F'$ is called a weakly measurable process, if for all $\phi \in F'$ and all $t \in \mathbb{R}_+$, $X_t(\phi)$ is a real random variable. We refer to Ustunel [16] for an introduction to nuclear space valued semi-martingales. A real-valued semi-martingale is a right-continuous adapted process $\{X_t, t \in \mathbb{R}_t\}$ having the following decomposition.

$$x_t = m_t + a_t, \qquad a_0 = 0$$

where m is a local martingale and a is a finite(bounded) variation process. The space S denotes the Banach space of real-semimartingales having finite norms

$$\|x\|_1 = inf\left[\mathbb{E}\left([x,x]_{\infty}^{\frac{1}{2}} + \int_0^{\infty} |da_s|\right)\right]$$

where the infimum is taken over all decompositions $x = m + a$ and where $[x,x]_{\infty}$ denotes the total quadratic variation of x. If x is a special semi-martingale, i.e., if a is predictable, then the decomposition $x = m + a$ is unique. Now we give the definition of a F' valued semi-martingale.

Definition 2.2. Let X be a weakly measurable process on $(\mathbb{R}_t \times \Omega, \mathcal{B}(\mathbb{R}_t) \bigotimes \mathcal{A})$ with values in F'. Then X is called a (F'-valued) semi-martingale if for any $\phi \in F$ the stochastic process $(t,\omega) \mapsto (\phi, X(t,\omega))_{F,F'}$ has a modification $(t,\omega) \mapsto \tilde{X}_t(\phi)(\omega)$ which is a semi-martingale in S.

Ustunel [16] defined a F'-valued semi-martingale as a projective system of semi-martingales and gave the above class of semi-martingales as a particular class. We consider here only this type of semi-martingales because of their interesting property expressed in the following.

Proposition 2.1. *There is a neighborhood $G \in \mathcal{U}_h(F)$ such that almost all trajectories of X are in $F'[G^0]$.*

Definition 2.3. A weakly measurable F'-valued process X is called a square integrable martingale if for all $\phi \in F$, $X(\phi) := ((X_t(\omega), \phi)_{F',F}; (t,\omega) \in \Omega)$, has a modification in $\mathfrak{M}^2(\mathbb{R})$, the space of real valued square integrable martingales. Similarly, M is said to be a square integrable martingale if for all $\phi \in F$, $M(\phi)$ has a modification in $\mathfrak{M}_c^2(\mathbb{R})$, the space of continuous real valued square integrable martingales.

Remark 2.1. We rather adopted here the above definition for possible future applications. In this paper we only deal with F'-valued square integrable martingales.

Therefore we could give the characterization of square integrable F'-valued martingales by replacing the space S of semi-martingales by the space S of real-valued square-integrable martingales as in [13].The property we are interested in is that there is a neighborhood $G \in \mathcal{U}_h(F)$ such that almost all trajectories of a F'-valued square integrable martingale are in $F'[G^0]$.

In what follows $\mathfrak{M}^2(F, F')$ will represent the space of F'-valued square integrable martingales and M a particular element of this space. We denote by $\mathcal{U}_h(F, M)$ the set of all neighborhoods $U \in \mathcal{F}_h(F)$ such that M is the injection of an $F'[U^0]$-valued square integrable martingales according to the above proposition.

Now we fix a neighborhood $G \in \mathcal{U}_h(F, M)$. We identify $F(G)$ with $F'[G^0]$ and we denote both of them by H. This is a separable Hilbert space and M is a H-valued square-integrable martingale. At this point we need some notations concerning nuclear and Hilbert-Schmidt operators on H.

- $L(H, H)$ is the space of all bounded operators on H into H with the uniform norm $\| \,.\, \|$.
- $L^1(H, H)$ is the space of nuclear operators on H into H with the norm $\| \,.\, \|_1$.
- $L^2(H, H)$ is the space of Hilbert-Schmidt operators on H into H with the Hilbert Schmidt norm $\| \,.\, \|_2$.
- $H \widehat{\bigotimes}_1 H$ (resp. $H \widehat{\bigotimes}_2 H$) is the projective(resp. Hilbertian) tensor products of H with H.

For notational convenience, $H \widehat{\bigotimes}_1 H$ (resp. $H \widehat{\bigotimes}_2 H$) are identified with $L^1(H, H)$ (resp.$L^2(H, H)$ under the isometry which puts $h \bigotimes k$ into a one-to-one correspondence with $(., h)_H$. Here and in what follows $(\,.\, , \,.\,)_H$ denotes the scalar product on H. H-valued martingales are always taken with their cadlag versions.We denote by $\mathfrak{M}^2(H)$, the space of H(separable)-valued square-integrable martingales.

Definition 2.4. Given two martingales $M, N \in \mathfrak{M}^2(H)$, the space of square integrable H-valued martingale, there is a unique $H \widehat{\bigotimes}_1 H$-valued cadlag predictable process with integrable variation, denoted by $\langle M, N \rangle$ and called the "oblique" bracket of (M, N), such that $M \bigotimes N - \langle M, N \rangle$ is a $H \widehat{\bigotimes}_1 H$-valued martingale vanishing at $t = 0$. The bracket process $\langle M, M \rangle$ that we denote by $\langle M \rangle$ is called the increasing process of M. We put $\beta_t := \|\langle M \rangle_t\|_1$. This process is the unique predictable increasing process with integrable variation for which $\|M\|^2 - \beta$ is a martingale vanishing at $t = 0$.

From now on M will represent a given martingale in $\mathfrak{M}^2(H)$ and λ will denote the measure $d\mathbb{P}d\beta$. All the operations that we carry out here on stochastic processes and operators are only valued $\lambda - a.e.$ and in order to simplify the notations, we will not always mention it. There exists a predictable process Q with values in the cone of symmetric and non-negative elements of $L^1(H,H)$, unique up to a λ-equivalence, such that $\|Q\|_1 = 1$, $\lambda - a.e.$ and $\langle M\rangle_t = \int_{0-}^t Q_s d\beta_s$.

We consider the following factorization of $Q(t,\omega)$; for $\lambda - a.e.$ there is a predictable operator $D(t,w) \in L^2(H)$ such that $Q = DD^*$, where D^* denotes the transpose of D.

We define a new scalar product on H by

$$\forall f, g \in H \qquad (f,g)_{\widetilde{H}(t,w)} = (D^*(t,\omega)f, D^*(t,\omega)g)_H \tag{1}$$

We complete H with respect to this scalar product and get a Hilbert space that we denote by $\widetilde{H}(t,\omega)$. Obviously, $f \mapsto D^*(t,\omega)f$ is extended to an isometry from $\widetilde{H}(t,\omega)$ into H.

We can construct an orthonormal basis $\{\tilde{e}_n(t,\omega), n \in \mathbb{N}\}$ of $\widetilde{H}(t,\omega)$ of predictable processes such that $\{\tilde{e}_n(t,\omega)\}$ is also an element of H [13]. Let X be a predictable process such that $X(t,w) \in \widetilde{H}(t,\omega)$ and

$$\int_{0-}^{\infty} \|X(t,\omega)\|^2_{\widetilde{H}(t,\omega)} d\lambda(t,\omega) < \infty \tag{2}$$

Such a process can be written as follows

$$X(t,\omega) = \sum_{n=0}^{\infty} a_n(t,\omega)\tilde{e}_n(t,\omega) \tag{3}$$

where a_n is a predictable real process such that

$$\sum_{n=0}^{\infty} \int_{0-}^{\infty} a_n^2(s,\omega) d\lambda(s,w) < \infty. \tag{4}$$

The vector space Λ^2 of all predictable $\widetilde{H}$-valued processes X satisfying (2) or equivalently (4) is a Hilbert Space, denoted as $\Lambda^2(D,H)$ with the scalar product,

$$(X,Y)_{\Lambda^2} = \int_{0-}^{\infty} (X(s,\omega), Y(s,\omega))_{\widetilde{H}(s,\omega)} d\lambda(s,\omega) \tag{5}$$

or equivalently

$$(X,Y)_{\Lambda^2} = \sum_{n=0}^{\infty} \int_{0-}^{\infty} a_n(s,\omega) b(s,\omega) d\lambda(s,\omega) \tag{6}$$

with

$$Y(t,\omega) = \sum_{n=0}^{\infty} b_n(t,\omega)\tilde{e}_n(t,\omega) \tag{7}$$

We refer to [14], [17] and [13] for the definitions and properties of a stochastic integral on Hilbert spaces. Let $X.M$ be the square integrable martingale defined by

$$(X.M)(t,\omega) = \int_{0-}^{t} X(s,\omega)dM(s,\omega) \tag{8}$$

We have

$$\langle X.M, Y.M\rangle(t,\omega) = \int_{0-}^{t} (X(s,\omega), Y(s,\omega))_{\widetilde{H}(s,\omega)} d\beta(s,\omega) \tag{9}$$

Let us put

$$M_n(t,\omega) = \int_{0-}^{t} \tilde{e}_n(s,\omega)dM(s,\omega) \tag{10}$$

Finally, using (9) we get

$$\begin{aligned} \langle M_n, M_m\rangle(t,\omega) &= \int_{0-}^{t} (\tilde{e}_n(s,\omega), \tilde{e}_m(s,\omega))_{\widetilde{H}(s,\omega)} d\beta(s,\omega) \\ &= \beta(t,\omega), \qquad if \quad n = m \\ &= 0, \qquad\qquad if \quad n \neq m \end{aligned} \tag{11}$$

$\{M_n, n \in \mathbb{N}\}$ is thus a sequence of mutually orthogonal square-integrable real martingales, with a common increasing process $\beta(t,\omega)$. If X is represented by (3) the stochastic integral (8) is then given by

$$(X.M)(t,\omega) = \sum_{n=0}^{\infty} \int_{0-}^{t} a_n(s,\omega)dM_n(s,\omega) \tag{12}$$

with

$$\langle X.M\rangle(t,\omega) = \sum_{n=0}^{\infty} \int_{0-}^{t} a_n^2(s,\omega)d\beta(s,\omega) \tag{13}$$

and

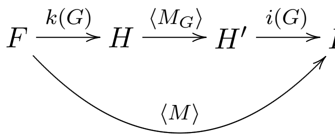
$$\mathbb{E}[(X.M)_t]^2 = \sum_{n=0}^{\infty} \int_{0-}^{t} a_n^2(s,\omega d\lambda(s,\omega) \tag{14}$$

In conclusion, the stochastic integral (with respect to M) of any process X in $\Lambda^2(D,H)$ provides a real square integrable martingale such that $\|X\|_{\Lambda^2}^2$ is equal to the left hand side of (2). Such a process X has the representation (3), with $\|X\|_{\Lambda^2}^2$ given by the left hand side of (4). Each element of Λ^2 has the representation (3) with the stochastic integral given by (12) and having the properties (13) and (14).

Now, we go back to our martingale $M \in \mathfrak{M}^2(F, F')$ considered as a $H(\cong F(G) \cong F'[G^0])$-valued squared martingale. We know from [13] that there is a predictable process $\langle M \rangle$, unique up to an evanescent set (with respect to λ), with values in the set of symmetric, nonnegative nuclear operators $L(F, F')$ such that, $\forall f, g \in F$

$$(\langle M \rangle f, g)_{F',F} = \langle M(f), M(g) \rangle \tag{15}$$

with $M(f)(t, \omega) = (M(t, \omega), f)_{F',F}$.We call $\langle M \rangle$ the increasing process of M. Let $\langle M_G \rangle$ be the increasing process of M considered as a H-valued martingale. We then have the following diagram.



Consider the following representation of $\langle M_G \rangle$:

$$\langle M_G \rangle(t, \omega) = \int_{0-}^{t} Q_G(s, \omega) d\beta_G(s, \omega) \tag{16}$$

We use again the factorization $Q_G = D_G \circ D_G^*$ where D_G is a predictable process with values in the Hilbert-Schmidt operators on H. Let us put $D = i(G) \circ D_G$. Then $D^* = D_G^* \circ k(G)$. For $f, g \in F$ we introduce a scalar product

$$(f, g)_{\widetilde{H}(t,\omega)} = (D^* f, D^* g)_H \tag{17}$$

We complete F with respect to this scalar product. As in the Hilbertian case we can construct a Hilbert space of $\widetilde{H}(t, \omega)$-valued predictable process X such that

$$\int_{0-}^{\infty} \|X(t, \omega\|^2_{\widetilde{H}(t,\omega)} d\lambda(s, \omega) < \infty. \tag{18}$$

Remark 2.2. Remember that this space is constructed over the equivalence classes with equivalence relation $f_1 \sim f_2$ if and only if $p_G(f_1 - f_2) = 0$. We can also identify this space with space Λ^2 generated by $H(\cong F(G) \cong F'[G^0])$.

Here $\Lambda^2(D, F, F')$ will denote the space of $\widetilde{H}(t, \omega)$-valued predictable processes satisfying the above diagram. We can repeat here verbatim what we have said for $\Lambda^2(D, F)$. We are interested in the representation (3) of elements of $\Lambda(D, F, F')$ with the series representation (10) of the elements of $\mathfrak{M}^2(F, F')$ and the definition of the stochastic integral by (12), followed by (13) and (14).

Remark 2.3. The construction of the stochastic integral we developed here entirely depends on the factorization of $Q = DD^*$. One can prove that $\Lambda^2(D, H)$ does not depend on this factorization. If $Q = BB^*$ is another factorization, then $\Lambda^2(D, H) = \Lambda^2(B, H)$. Moreover, each element of $\Lambda^2(B, H)$ is an isometric image of an element of $\Lambda^2(D, H)$ [13]. One can prove a better statement. $X.M$ can be defined independently on the factorization of Q. The proof of this fact will be given in a forthcoming paper.

Next we will give the modeling approach of bond prices by using the above construction.

3 Application to Bond Prices

In this section we consider zero-coupon bond prices where forward interest rates are expressed in terms of semi-martingales. More precisely, if $P(t, T)$ represents the zero-coupon bond price at t with maturity T, then

$$P(t,T) = exp\left[-\int_t^T f(t,s)ds\right] \tag{19}$$

where the forward interest rate $f(t, s)$ is

$$f(t,s) = f(0,s) + \int_0^t \mu(u, s-u)du + \int_0^t \sigma_c(u, s-u)dM_c + \int_0^t \sigma_d(u, s-u)dM_d \tag{20}$$

with M_c and M_d being square integrable continuous and discontinuous martingales, respectively. This type of forward interest rate formula was already considered by [8] under the name of mixed HJM-Musiela type model.

Note 3.1. *We use here the notation of the preceding section by putting an index c or d to the element, corresponding to M_c and M_d, respectively. For β_i, λ_i, Q_i, D_i, $H_i(t, \omega)$, $e_{i,n}(t, \omega)$, Λ_i^2 with $i = c$ or d, respectively.*

In what follows, M_d can be seen as the compensated process of X_d, a pure jump process which may have the semimartingale representation

$$X_d(t) = M_d(t) + K(t)$$

where $K(t)$ is the compensator of the process $X_d(t)$. We can define the random measure of the jumps of the process X_d as

$$\delta((0,t] \times B) = \sum_{u \le t} \mathbb{I}_B(\Delta X_d)$$

for a Borel set of H. Additionally, the compensator of δ^d, ν^d can be defined such that

$$M_d = \int_0^t \int_H m(\delta^d - \nu^d)(du, dm).$$

We suppose that the probability measure $\mathbb{P}$ we have been considering here is a risk neutral probability measure, implying that the discounted bond prices $P(t,T)$ are $\mathbb{P}$ martingales. We have the following assumptions for forward rate dynamics.

Assumption 3.1. *For each (u,s), such that $u \leq t \leq s \leq T$, $\sigma_i(u, s-u)$ is supposed to belong to Λ_i^2. Moreover it is assumed that*

$$\int_0^T \int_0^T |\mu(u,s)| du ds < \infty \tag{21}$$

We define

$$\mu^*(u,T) = \int_0^{T-u} \mu(u,s) ds \tag{22}$$

and

$$\sigma_i^*(u,T) = \int_0^{T-u} \sigma_i(u,s) ds \tag{23}$$

It can be seen that $\sigma_i^*(.,T)$ is also in Λ_i^2. Notice that the short term interest rate is $f(t,t)$.

Under the above assumptions, we can prove the following theorem.

Theorem 3.1. *Let $\widetilde{P}(t,T)$ be the discounted bond price,that is,*

$$\widetilde{P}(t,T) = \left[exp(-\int_0^t f(u,u)du) \right] P(t,T).$$

Then it has the following expression as a semi-martingale.

$$\begin{aligned} \widetilde{P}(t,T) = \widetilde{P}(0,T) &- \int_0^t \widetilde{P}(u-,T)\mu^*(u,T)du - \int_0^t \widetilde{P}(u-,T)\sigma_c^*(u,T)dM_c(u) \\ &- \int_0^t \widetilde{P}(u-,T)\sigma_d^*(u,T)dM_d(u) + \frac{1}{2}\int_0^t \sum_{n=0}^{\infty} \widetilde{P}(u-,T)[\sigma_{c,n}^*(u,T)]^2 d\beta_{c,n} \\ &+ \int_{0-}^t \int_H \widetilde{P}(u-,T) \left\{ exp\left[\sigma_d^*(u-,T)m\right] - 1 - \sigma^*(u-,T)m \right\} \delta^d(du,dm) \end{aligned} \tag{24}$$

Remark 3.1. Let us consider the series expansion

$$\sigma_c^*(t,T) = \sum_{n=0}^{\infty} \int_0^{\infty} \sigma_{c,n}^*(u,T) e_{c,n} \tag{25}$$

and

$$\int_0^t \sigma_c^*(u,T)dM_c(u) = \sum_{n=0}^{\infty}\int_0^t \sigma_{c,n}^*(u,T)dM_{c,n}(u) \tag{26}$$

where

$$\sigma_{c,n}^*(u,T) = \int_0^{T-u} \sigma_{c,n}(u,s)ds$$

We could then write

$$\int_0^t \widetilde{P}(u-,T)\sigma_c^*(u,T)dM_c(u) = \int_0^t \sum_{n=0}^{\infty} \widetilde{P}(u-,T)\sigma_{c,n}^*(u,T)dM_{c,n}(u) \tag{27}$$

and

$$\int_0^t \widetilde{P}(u-,T)d\langle\sigma_c^*(.,T).M\rangle_u = \int_0^t \sum_{n=0}^{\infty} \widetilde{P}(u-,T)[\sigma_{c,n}^*(u,T)]^2 d\beta_{c,n} \tag{28}$$

Proof. The proof of the theorem is an immediate consequence of the Ito formula applied to $\widetilde{P}(t,T)$. Let us first give its expression. Put

$$Y(t) = \int_0^T f(t,s)ds$$

Differentiating $Y(t)$ and using the Fubini Theorem we find,

$$\begin{aligned} Y(t) =& -\int_0^t f(u,u)du + \int_0^t \left(\int_t^T \mu(u,s-u)ds\right)du \\ &+ \int_0^t \left(\int_t^T \sigma_c(u,s-u)ds\right)dM_c(u) + \int_0^t \left(\int_t^T \sigma_d(u,s-u)ds\right)dM_d(u) \\ =& -\int_0^t f(u,u)du + \int_0^t \mu^*(u,T)du \\ &+ \int_0^t \sigma_c^*(u,T)dM_c(u) + \int_0^t \sigma_d^*(u,T)dM_d(u) \end{aligned} \tag{29}$$

According to the above assumptions, the last two stochastic integrals are well defined and give square-integrable martingales. The Ito formula applied to

$$\widetilde{P}(t,T) = exp\left[\left(-\int_0^t f(u,u)du - Y(t)\right)\right]$$

gives

$$\begin{aligned} \widetilde{P}(t,T) =& \widetilde{P}(0,T) - \int_0^t \widetilde{P}(u-,T)\mu^*(u,T)du - \int_0^t \widetilde{P}(u-,T)\sigma_c^*(u,T)dM_c(u) \\ &- \int_0^t \widetilde{P}(u-,T)\sigma_d^*(u,T)dM_d(u) + \frac{1}{2}\int_0^t \widetilde{P}(u-,T)d\langle\sigma_c^*(.,T).M\rangle_u \\ &+ \sum_{u\le t}\Big[\widetilde{P}(u,t) - \widetilde{P}(u-,T) - \widetilde{P}(u-,T)\Delta Y(u)\Big] \end{aligned} \tag{30}$$

We need to express the last sum in terms of ΔM_d. We have

$$\frac{\widetilde{P}(u,T)}{\widetilde{P}(u-,T)} = exp\left[\sigma_d^*(u,T)\Delta M_d(u)\right] \tag{31}$$

and

$$\Delta\widetilde{P}(u,T) = \widetilde{P}(u-,T)\left[\frac{\widetilde{P}(u,T)}{\widetilde{P}(u-,T)} - 1\right] \tag{32}$$

Therefore,

$$\begin{aligned} &\sum_{u\le t}\left[\widetilde{P}(u,t) - \widetilde{P}(u-,T) - \widetilde{P}(u-,T)\Delta Y(u)\right] \\ =&\sum_{u\le t}\widetilde{P}(u-,T)\left\{exp[\sigma_d^*(u,T)\Delta M_d(u)] - 1 - \sigma_d^*(u,T)\Delta M_d(u)\right\} \end{aligned} \tag{33}$$

The last term can also be written as

$$\int_{0-}^t\int_H \widetilde{P}(u-,T)\left\{exp\left[\sigma_d^*(u-,T)m\right] - 1 - \sigma^*(u-,T)m\right\}\delta^d(du,dm)$$

Then by using remark (3.1), we find the bond price formula. □

From this we deduce the following proposition.

Proposition 3.1 (Extended Heath-Jarrow-Morton drift condition)**.** *Since* $\widetilde{P}(t,T)$ *should be a martingale, we have*

$$\begin{aligned} \int_0^t \widetilde{P}(u-,T)\mu^*(u,T)du = &\frac{1}{2}\int_0^t\sum_{n=0}^{\infty}\widetilde{P}(u-,T)[\sigma_{c,n}^*(u,T)]^2 d\beta_{c,n} \\ &+ \int_{0-}^t\int_H \widetilde{P}(u-,T)\rho_m(u,T)\nu^d(du,dm) \end{aligned} \tag{34}$$

where

$$\rho_m(u,T) = exp\left[\sigma_d^*(u-,T).m\right] - 1 - \sigma^*(u-,T).m$$

Proof. In order for $\widetilde{P}(t,T)$ to be a martingale we need to cancel all the terms of (24) which are not stochastic integrals. By rearranging the equation (24) as

$$\begin{aligned} \widetilde{P}(t,T) = \widetilde{P}(0,T) &- \int_0^t \widetilde{P}(u-,T)\mu^*(u,T)du - \int_0^t \widetilde{P}(u-,T)\sigma_c^*(u,T)dM_c(u) \\ &- \int_0^t \widetilde{P}(u-,T)\sigma_d^*(u,T)dM_d(u) + \frac{1}{2}\int_0^t\sum_{n=0}^{\infty}\widetilde{P}(u-,T)[\sigma_{c,n}^*(u,T)]^2 d\beta_{c,n} \\ &+ \int_{0-}^t\int_H \widetilde{P}(u-,T)\rho_m(u,T)(\delta^d(du,dm) - \nu^d(du,dm)) \\ &+ \int_{0-}^t\int_H \widetilde{P}(u-,T)\rho_m(u,T)\nu^d(du,dm) \end{aligned} \tag{35}$$

the martingale condition implies

$$\begin{aligned}\widetilde{P}(t,T) =&\widetilde{P}(0,T) - \int_0^t \widetilde{P}(u-,T)\sigma_c^*(u,T)dM_c(u) - \int_0^t \widetilde{P}(u-,T)\sigma_d^*(u,T)dM_d(u)\\ &+ \int_{0-}^t \int_H \widetilde{P}(u-,T)\rho_m(u,T)(\delta^d(du,dm) - \nu^d(du,dm))\end{aligned} \tag{36}$$

Then condition (34) is automatically satisfied. □

Corollary 3.2. *If the measures β and ν are absolutely continuous with respect to the Lebesque measure, then the drift condition becomes,*

$$\mu^*(t,T) = \frac{1}{2}\sum_{n=0}^{\infty}[\sigma_{c,n}^*(u,T)]^2\frac{d\beta_c(t)}{dt} + \frac{d\chi(t)}{dt} \tag{37}$$

where

$$\chi(t) = \int_0^t \int_H \{exp\,[\sigma_d^*(u-,T).m] - 1 - \sigma_d^*(u-,T).m\}\,\nu(du,dm)$$

We will see various cases where the above absolute continuity conditions are satisfied. As the first example, the absolute continuity of ν is obtained in the case where M_d is derived from a Markov Jump Process.

3.1 Bond Prices with Markov Jump Process

Let $\{X(t),\, t \in [0,T]\}$ be a Markov Jump Process with values $\{f_n,\, n \in \mathbb{N}\} \subset F'$.In order to guarantee the right continuity we suppose that X takes its values at the jump points. The sojourn time at the value $X(t)$ is an exponential random variable with parameter $\lambda(X(t)) \in (0,+\infty)$. We suppose that

$$sup_n\, \lambda(f_n) < \infty$$

This ensures the regularity, i.e., the process does not explode. Let $p(f_j, f_i)$ be the transition probability of X from state f_j to state f_k. If $X(t)$ is the state just before the jump to f_j, the jump size is represented by

$$\xi(X(t)) = f_j - X(t)$$

Suppose that

$$\mathbb{E}[|\xi(X(t))|] = \sum_{j\in\mathbb{N}} p(X(t), f_j)|f_j - X(t)| < \infty. \tag{38}$$

we then put

$$\mathbb{E}[\xi(X(t))] = \sum_{j\in\mathbb{N}} p(X(t), f_j)(f_j - X(t)) \tag{39}$$

This is the conditional expected size of jumps. We define

$$K(t) = \int_0^t \lambda(X(s))\mathbb{E}[\xi(X(s))]ds \tag{40}$$

We claim that this is the compensator of X. In fact, the probability that a jump occurs in the elementary interval $(t, t+dt]$ when X is in state $X(t)$ is $\lambda(X(t))dt$ and the conditional expected jump size is $\mathbb{E}[\xi(X(t))]$. Therefore $dX(t) - dK(t)$ is the increment of a martingale.

In view of Definiton 2.3, we would like to look for conditions under which

$$N(t) = X(t) - K(t)$$

is a square integrable martingale. Let us choose $\phi \in F$ and define,

$$N_\phi(t) := (N(t), \phi)_{F,F'}$$

$$X_\phi(t) := (X(t), \phi)_{F,F'}$$

$$K_\phi(t) := (K(t), \phi)_{F,F'}$$

Consider first the real valued Markov Jump Process $X_\phi(t)$. It has a cadlag version because of the right continuity and regularity of X. Let us put

$$\begin{aligned} \mathbb{E}[\xi_\phi(X_\phi(t))] &= \sum_{j=0}^{\infty} p\left(X(t), f(j)\right)[(f_j, \phi)_{F,F'} - X_\phi(t)] \\ &= \sum_{j=0}^{\infty} p\left(X(t), f(j)\right)(f_j - X(t), \phi)_{F,F'} \\ &= (\mathbb{E}[\xi(X(t))], \phi)_{F'F}. \end{aligned} \tag{41}$$

It is also seen that

$$K_\phi(t) = (K(t), \phi)_{F,F'} = \int_0^t \lambda(X(s))(\mathbb{E}[\xi(X(s))]ds \tag{42}$$

is the compensator of X_ϕ. Here again $\lambda(X(t))(\mathbb{E}[\xi(X(t))]dt$ represents the conditional expectation of the jump size of X_ϕ given that that there is a jump of $X(t)$ in the interval $(t, t+dt]$. Now, we need to give conditions under which N_ϕ is a square integrable martingale.

Proposition 3.2. *For $k = 1\, and\, 2$, define*

$$\mathbb{E}[|\xi_\phi(X_\phi(t))|^k] = \sum_{j=0}^{\infty} p\,(X(t), f(j))|(f_j, \phi)_{F,F'} - X_\phi(t)|^k \tag{43}$$

If there is a positive constant C such that

$$\mathbb{E}[|\xi_\phi(X_\phi(t))|^k] \leq C(1 + |X_\phi(t)|^k) \tag{44}$$

then N_ϕ is a square integrable martingale.

Proof. If (44) holds we can write

$$\lambda(X(t))\mathbb{E}[|\xi_\phi(X(t))|^k] \leq sup_n\, \lambda(f_n) C(1 + |X_\phi(t)|^k)$$

Therefore, according to Klebaner[19] and Hamza and Klebaner[20], N_ϕ is a square-integrable martingale. □

As a consequence, according to Definition 2.3 we see that the compensated Markov Jump Process $N(t)$ is a F'-valued square-integrable martingale. This will represent our M^d in the general setting of the previous section.

Proposition 3.3. *If the dynamics of the forward interest rates are given by the following,*

$$f(t,s) = f(0,s) + \int_0^t \mu(u, s-u)du + \int_0^t \sigma(u, s-u)dX(u) \tag{45}$$

where X is a Markov Jump Process, then the drift condition becomes

$$\mu^*(t,T) = \int_H \{exp\,[\sigma^*(u-,T).m] - 1 - \sigma^*(u-,T).m\}\, \lambda(X(t))\mathbb{E}(\xi(X(t)) \tag{46}$$

Proof. By using the definition of the compensator of $X(t)$ and the Corollary (3.2) □

3.2 Bond Prices with Levy Processes

Here we consider a centered Levy process

$$L(t) = W(t) + M_d(t)$$

where $M_d(t)$ is the compensated jump part of L. In order to follow our approach, we suppose that L is a square integrable martingale. We use again the Hilbertian space

$H(\cong F(G) \cong F'[G])$ where all trajectories of L are concentrated. It is known [21] that the Hilbert space valued Levy Process has the increasing process

$$\langle L \rangle_t = \int_0^t Q dt = \int_0^t (Q/TrQ)(TrQ)dt.$$

With our preceding notations, the covariance operator $Q_t = Q/TrQ$ and $\beta_t^L = tTrQ$. Instead of factorizing Q_t we construct $\widetilde{H}(t,\omega)$ in such a way that the measure $\lambda(dt, d\omega)$ is replaced by $dt\mathbb{P}(d\omega)$. We consider the factorization $D \circ D^*$ of Q with $D \in L^2(H, H)$. Let us define a scalar product $(f,g)_{\widetilde{H}} = (D^* f, D^* g)_H$. The completion of H under this scalar product $(f,g)_{\widetilde{H}}$ is denoted by $\widetilde{H}$. If $\{\tilde{e}_n, \, n \in \mathbb{N}\}$ is an orthogonal basis in $\widetilde{H}$, then the space Λ^2 is the space of predictable processes X such that

$$X(t,\omega) = \sum_{n=0}^{\infty} a_n(t,\omega)\tilde{e}_n(t,\omega) \tag{47}$$

where a_n is a predictable real process such that

$$\sum_{n=0}^{\infty} \int_{0-}^{\infty} a_n^2(s,\omega) ds\mathbb{P}(d\omega) < \infty. \tag{48}$$

The vector space Λ^2 of all predictable $\widetilde{H}$-valued processes X satisfying (2) or equivalently (4) is a Hilbert Space, denoted as $\Lambda^2(D, H)$ with the scalar product,

$$(X,Y)_{\Lambda^2} = \int_{0-}^{\infty} (X(s,\omega), Y(s,\omega))_{\widetilde{H}(s,\omega)} ds\mathbb{P}(d\omega) \tag{49}$$

or equivalently

$$(X,Y)_{\Lambda^2} = \sum_{n=0}^{\infty} \int_{0-}^{\infty} a_n(s,\omega) b(s,\omega) ds\mathbb{P}(d\omega) \tag{50}$$

with

$$Y(t,\omega) = \sum_{n=0}^{\infty} b_n(t,\omega)\tilde{e}_n(t,\omega) \tag{51}$$

With this setting, we have the following HJM condition.

Proposition 3.4. *If the dynamics of the forward interest rates are given by the following,*

$$f(t,s) = f(0,s) + \int_0^t \mu(u, s-u)du + \int_0^t \sigma(u, s-u)dL(u) \tag{52}$$

where

$$L(t) = W(t) + M_d(t),$$

then the drift condition becomes

$$\mu^*(t,T) = \frac{1}{2}\sum_{n=0}^{\infty}[\sigma^*(u,T)]^2 + \int_H \{exp\,[\sigma^*(u-,T).x] - 1 - \sigma^*(u-,T).x\}\, F(dx) \quad (53)$$

where $F(dx)$ denotes the Levy measure of the jumps.

Proof. By taking $M_c(t) = W(t)$ and using corollary 3.2 with $\nu(dt,dx) = dtF(dx)$. □

Remark 3.2. The above condition is similar to the one found in [11] for default free case where the Levy random field drives the forward interest rates. The condition in Proposition (3.4) differs from the one in [11] that the eigenvalues of the covariance operator do not appear in (53). This is due to the construction of the stochastic integral described in section 2.

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