Minimizing down-side risk probability and risk-sensitive asset allocation for linear Gaussian models

(Joint work with H. Hata and S.J. Sheu)

Hideo Nagai

Division of Mathematical Science Science for Social Systems,
Graduate School of Engineering Science,
Osaka University, Toyonaka, 560-8531, Japan,

E-mail: nagai@sigmath.es.osaka-u.ac.jp

- Risk-sensitive asset allocation for linear Gaussain models
  Bielecki-Pliska '99, Fleming-Sheu '00, '00, N. '00,
  Kuroda-N. '02, N.-Peng '02, . . .

- Up-side chance probability (1-dim.)
  Pham '03, Hata-Sekine '05, Hata-Iida '06, . . .

- Risk-sensitive asset allocation: bench marked case
  Davis-Lleo '06

Linear Gaussaian model

Consider a market with $m+1 \geq 2$ securities and $n \geq 1$ factors.
The bond price:

(2.1) $$dS^0(t) = rS^0(t)dt, \quad S^0(0) = s^0,$$

where r is a positive constant.
The other secutity prices and factors:

(2.2) $$dS^i(t) = S^i(t)\{\alpha^i(X_t)dt + \textstyle\sum_{k=1}^{n+m} \sigma^i_k dW^k_t\},$$
$$S^i(0) = s^i, \ \ i = 1, ..., m$$

and

(2.3) $$dX_t = \beta(X_t)dt + \lambda dW_t,$$
$$X(0) = x \in R^n,$$

$W_t = (W_t^k)_{k=1,..,(n+m)}$: an $m+n-$ dimensional standard B. M.

Assume that

(2.4)

$$\alpha(x) = Ax + a, \;\; \beta(x) = Bx + b, \quad x \in R^n, \;\; a \in R^m, \;\; b \in R^n$$

$$\sigma\sigma^* > 0$$

where σ^* stands for the transposed matrix of σ.

Let us denote investment strategy to i-th security $S^i(t)$ by $h^i(t)$, $i = 0, 1, ..., m$ and set

$$S(t) = (S^1(t), S^2(t), ..., S^m(t))^*,$$

$$h(t) = (h^1(t), h^2(t), ..., , h^m(t))^*$$

and

$$\mathcal{F}_t = \sigma(S(u), X(u); u \le t).$$

Definition 1 $(h^0(t), h(t)^*)_{0\le t\le T}$ *is said an invetment strategy if the following conditions are satisfied*

i) $h(t)$ *is a* R^m *valued* $\mathcal{F}_t$ *progressively measurable stochastic process such that*

$$\sum_{i=1}^{m} h^i(t) + h^0(t) = 1 \tag{2.5}$$

ii)

$$P\left(\int_0^T |h(s)|^2 ds < \infty\right) = 1, \quad \forall T > 0.$$

The set of all investment strategies will be denoted by $\mathcal{H}(T)$. When $(h^0(t), h(t)^*)_{0\le t\le T} \in \mathcal{H}(T)$ we will often write $h \in \mathcal{H}(T)$ for simplicity since h^0 is determined by (2.5).

For given $h \in \mathcal{H}(T)$ the process $V_t = V_t(h)$ representing the investor's capital at time t is determined by the stochastic differential equation:

$$
\begin{aligned}
\frac{dV_t}{V_t} &= \sum_{i=0}^{m} h^i(t)\frac{dS^i(t)}{S^i(t)} \\
&= h^0(t)rdt + \sum_{i=1}^{m} h^i(t)\{\alpha^i(X_t)dt + \sum_{k=1}^{m+n} \sigma_k^i dW_t^k\} \\
&= rdt + h(t)^*(\alpha(X_t) - r\mathbf{1})dt + h(t)^*\sigma dW_t, \\
V_0 &= v,
\end{aligned}
\tag{2.6}
$$

where $\mathbf{1} = (1, 1, ..., 1)^*$.

Down-side risk minimization

$$\hat{J}(k) := \liminf_{T\to\infty} \frac{1}{T} \inf_{h\in\mathcal{H}_T} \log P(\frac{\log V_T(h)}{T} \le k)$$

Risk-sensitive asset allocation

For a given constant $\theta > 0$ consider the following

(2.7) $$\chi(\theta) = \limsup_{T\to\infty} \frac{1}{T} \sup_{h\in\mathcal{A}_T} J(v, x; h; T)$$

where

(2.8) $$J(v, x; h; T) = -2 \log E[V_T(h)^{-\frac{\theta}{2}}] = -2 \log E[e^{-\frac{\theta}{2} \log V_T(h)}],$$

and h ranges over the set $\mathcal{A}(T)$ of all addmissible investment strategies defined later.

We shall show that $\chi(\theta)$ is differentiable and concave and that

$$\hat{J}(k) = \frac{1}{2}\inf_{\theta>0}\{\theta k - \chi(\theta)\}$$

for $\chi'(0+) > k > \lim_{\theta\to\infty}\chi'(\theta)$ under suitable conditions through seeing that $\chi(\theta)$ satisfies the ergodic type H-J-B equation corresponding to the risk-sensitive asset allocation on infinite time horizon :

$$\chi(\theta) = \tfrac{1}{2}\mathrm{tr}(\lambda\lambda^* D^2 w) + (\beta - \tfrac{\theta}{\theta+2}\lambda\sigma^*(\sigma\sigma^*)^{-1}(\alpha - r\mathbf{1}))^* Dw + \theta r$$
$$-\tfrac{1}{4}(Dw)^*\lambda(I - \tfrac{\theta}{\theta+2}\sigma^*(\sigma\sigma^*)^{-1}\sigma)\lambda^* Dw + \tfrac{\theta}{\theta+2}(\alpha - r\mathbf{1})^*(\sigma\sigma^*)^{-1}(\alpha - r\mathbf{1})$$

H-J-B equation: finite time horizon case

Risk-sensitive asset allocation on a finite time horizon:

(3.1) $$I(v,x;T) := \sup_{h\in\mathcal{A}_T} J(v,x;h;T),$$

$$J(v,x;h;T) = -2\log E[V_T(h)^{-\frac{\theta}{2}}] = -2\log E[e^{-\frac{\theta}{2}\log V_T(h)}]$$

H-J-B equation:

(3.2) $$\frac{\partial u}{\partial t} + \sup_{h\in R^m} L^h u = 0,$$
$$u(T,x) = \theta \log v,$$

where L^h is defined by

$$L^h u(t,x) \;=\; \tfrac{1}{2}\mathrm{tr}(\lambda\lambda^* D^2 u) + (\beta(x) - \tfrac{\theta}{2}\lambda\sigma^* h)^* Du$$
$$-\tfrac{1}{4}(Du)^*\lambda\lambda^* Du - \theta\eta(x,h),$$

$$\eta(x,h) = \frac{\theta+2}{4}h^*\sigma\sigma^* h - r - h^*(\alpha(x) - r\mathbf{1}).$$

Note that $\sup_{h\in R^m} L^h u$ can be written as

$$\sup_{h\in R^m} L^h u(t,x) = \tfrac{1}{2}\mathrm{tr}(\lambda\lambda^* D^2 u) + (\beta - \tfrac{\theta}{\theta+2}\lambda\sigma^*(\sigma\sigma^*)^{-1}(\beta - r\mathbf{1}))^* Du$$

$$-\tfrac{1}{4}(Du)^*\lambda(I - \tfrac{\theta}{\theta+2}\sigma^*(\sigma\sigma^*)^{-1}\sigma)\lambda^* Du + \tfrac{\theta}{\theta+2}(\alpha - r\mathbf{1})^*(\sigma\sigma^*)^{-1}(\alpha - r\mathbf{1}) + \theta r.$$

(3.2) is written as:

(3.3)

$$\frac{\partial u}{\partial t} + \frac{1}{2}\mathrm{tr}(\lambda\lambda^* D^2 u) + F(x)^* Du - \frac{1}{2}(Du)^*\lambda N^{-1}\lambda^* Du + U(x) = 0,$$

$$u(T,x) = \theta \log v,$$

where

(3.4)

$$F(x) = \beta(x) - \frac{\theta}{\theta+2}\lambda\sigma^*(\sigma\sigma^*)^{-1}(\alpha(x) - r\mathbf{1})$$

$$N^{-1}(x) = \frac{1}{2}(I - \frac{\theta}{\theta+2}\sigma^*(\sigma\sigma^*)^{-1}\sigma)$$

$$U(x) = \frac{\theta}{\theta+2}(\alpha - r\mathbf{1})^*(\sigma\sigma^*)^{-1}(\alpha - r\mathbf{1}) + \theta r.$$

The solution of (3.2) is expressed as

$$u(t,x) = \frac{1}{2}x^* P(t)x + q(t)^* x + k(t). \tag{3.5}$$

Here $P(t)$ is the symmetric nonnegative definite solution of the Riccati differential equation:

(3.6)

$$\dot{P}(t) - P(t)K_0P(t) + K_1^*P(t) + P(t)K_1 + \tfrac{2\theta}{\theta+2}A^*(\Sigma\Sigma^*)^{-1}A = 0,$$

$$P(T) = 0,$$

where

$$K_0 = \frac{1}{2}\lambda(I - \frac{\theta}{\theta+2}\sigma^*(\sigma\sigma^*)^{-1}\sigma)\lambda^* \equiv \lambda N^{-1}\lambda^*$$

$$K_1 = B - \frac{\theta}{\theta+2}\lambda\sigma^*(\sigma\sigma^*)^{-1}A.$$

The term $q(t)$ in (3.5) is a solution of the following linear ordinary differential equation

$$\begin{aligned}(3.7)\qquad & \dot{q}(t) + (K_1^* - P(t)K_0)q(t) + P(t)b \\ & \quad + (\tfrac{2\theta}{\theta+2}A^* - \tfrac{\theta}{\theta+2}P(t)\lambda\sigma^*)(\sigma\sigma^*)^{-1}(a - r)\mathbf{1}) = 0, \\ & q(T) = 0,\end{aligned}$$

and $k(t)$ is a solution of

(3.8)

$$\begin{aligned}& \dot{k}(t) + \tfrac{1}{2}\mathrm{tr}(\lambda\lambda^* P(t)) - \tfrac{1}{4}q(t)^*\lambda\lambda^* q(t) + b^* q(t) + \theta r \\ & \quad + \tfrac{\theta}{\theta+2}(a - r\mathbf{1})^*(\sigma\sigma^*)^{-1}(a - r\mathbf{1}) + \tfrac{\theta}{4(\theta+2)}q(t)^*\lambda\sigma^*(\sigma\sigma^*)^{-1}\sigma\lambda^* q(t) \\ & \quad - \tfrac{\theta}{\theta+2}(a - r\mathbf{1})^*(\sigma\sigma^*)^{-1}\sigma\lambda^* q(t) = 0, \\ & k(T) = \theta \log v.\end{aligned}$$

Optimal strategy (finite time horizon case)

Lemma 1 (cf. Kuroda-N.'02) *We assume the assumptions in the above theorem and let u be a solution of (2.14). Define*

$$\begin{aligned}\widehat{h}_t &= \widehat{h}(t, X_t)\\ \widehat{h}(t,x) &= \tfrac{2}{\theta+2}(\sigma\sigma^*)^{-1}(\alpha - r\mathbf{1} - \tfrac{1}{2}\sigma\lambda^* Du)(t,x),\\ &= \tfrac{2}{\theta+2}(\sigma\sigma^*)^{-1}[a - r\mathbf{1} - \tfrac{1}{2}\sigma\lambda^* q(t) + (A - \tfrac{1}{2}\sigma\lambda^* P(t))x]\end{aligned}$$

where X_t is the solution of (2.3). Then, $\widehat{h}_t$ is an optimal strategy for problem (3.1) and

$$J(x, v, \widehat{h}; T) = \frac{1}{2}x^* P(0)x + q(0)^* x + k(0) = u(0, x).$$

$\mathcal{A}_T$ is the set of investment strategies $h \in \mathcal{H}_T$ such that

$$E[e^{-\frac{1}{2}\int_0^T [(P(s)+q_s)^*\lambda+\theta h_s^*\sigma]dW_s - \frac{1}{8}\int_0^T [(P(s)+q_s)^*\lambda+\theta h_s^*\sigma][(P(s)+q_s)^*\lambda+\theta h_s^*\sigma]^* ds}] = 1$$

Passage to the limit as $T \to \infty$

Lemma 2 (cf. ibid.) *Under the assumption that*

$$G \equiv B - \lambda\sigma^*(\sigma\sigma^*)^{-1}A \quad \text{is stable} \tag{4.1}$$

i) $P(0) = P(0;T)$ *converges as* $T \to \infty$ *to a nonnegative definite matrix* $\bar{P}$*, which is a solution of algebraic Riccati equation:*

$$-\bar{P}K_0\bar{P} + K_1^*\bar{P} + \bar{P}K_1 + \tfrac{2\theta}{\theta+2}A^*(\Sigma\Sigma^*)^{-1}A = 0, \tag{4.2}$$

and satisfies the estimate

$$0 \le \bar{P} \le 2\int_0^\infty e^{sG^*}A^*(\sigma\sigma^*)^{-1}Ae^{sG}ds. \tag{4.3}$$

Moreover $K_1 - \lambda N^{-1}\lambda^*\bar{P}$ *is stable.*

ii) $q(0) = q(0;T)$ *converges as* $T \to \infty$ *to a constant vector* $\bar{q}$*, which satisfies*

(4.4)

$$(K_1^* - \bar{P}K_0)\bar{q} + \bar{P}b + (\frac{2\theta}{\theta+2}A^* - \frac{\theta}{\theta+2}\bar{P}\lambda\sigma^*)(\sigma\sigma^*)^{-1}(a - r\mathbf{1}) = 0,$$

iii) $\frac{k(0;T)}{T}$ *converges to a constant* $\chi(\theta)$ *defined by*

(4.5)

$$\chi(\theta) = \frac{1}{2}tr(\lambda\lambda^*\bar{P}) - \frac{1}{4}\bar{q}^*\lambda\lambda^*\bar{q} + b^*\bar{q} + \theta r$$
$$+ \frac{\theta}{\theta+2}(a - r\mathbf{1})^*(\sigma\sigma^*)^{-1}(a - r\mathbf{1}) + \frac{\theta}{4(\theta+2)}\bar{q}^*\lambda\sigma^*(\sigma\sigma^*)^{-1}\sigma\lambda^*\bar{q}$$
$$- \frac{\theta}{\theta+2}(a - r\mathbf{1})^*(\sigma\sigma^*)^{-1}\sigma\lambda^*\bar{q} = 0,$$

Ergodic type H-J-B eqution

Let us set

$$G(x) = \beta(x) - \lambda\sigma^*(\sigma\sigma^*)^{-1}(\alpha(x) - r\mathbf{1})$$
$$H(x) = \frac{2}{\theta+2}\sigma^*(\sigma\sigma^*)^{-1}(\alpha(x) - r\mathbf{1})$$

and rewrite equation (3.2) as

(4.7)

$$\frac{\partial u}{\partial t} + \frac{1}{2}\mathrm{tr}(\lambda\lambda^* D^2 u) + G(x)^* Du$$
$$-\frac{1}{2}(-\lambda^* Du + NH)^* N^{-1}(-\lambda^* Du + NH) + \frac{\theta+2}{4}H^* NH + \theta r = 0,$$
$$u(T,x) = \theta \log v,$$

Ergodic type H-J-B equation:

(4.8)

$$\chi(\theta) = \frac{1}{2}\text{tr}(\lambda\lambda^* D^2 w) + G(x)^* Dw$$
$$-\frac{1}{2}(-\lambda^* Dw + NH)^* N^{-1}(-\lambda^* Dw + NH) + \frac{\theta+2}{4} H^* NH + \theta r,$$

Corollary 1 *Assume (4.1). Then, as* $T \to \infty$

$$u(0, x; T) - u(0, 0; T) \to w(x),$$
$$\frac{1}{T} u(0, x; T) \to \chi(\theta),$$

uniformly on each compact set, where

(4.9) $$w(x) = \frac{1}{2} x^* \bar{P} x + \bar{q}^* x.$$

and $(w, \chi(\theta))$ *is the solution of (4.8).*

Concavity of $\chi(\theta)$

Lemma 3 *Assume (4.1) and that*

(5.1) $$(B, \lambda) \text{ is controllable.}$$

Then, $\chi(\theta)$ is twice differentiable and concave. Furthermore

$$\frac{\partial \chi}{\partial \theta} = L(\theta)\xi$$

$$+\tfrac{1}{2(\theta+2)^2}(-\lambda^* Dw + NH)^* \sigma^* (\sigma\sigma^*)^{-1} \sigma(-\lambda^* DW + NH) + r,$$

$$\frac{\partial^2 \chi}{\partial \theta^2} = L(\theta)\zeta - \frac{1}{2}(D\xi)^* \lambda (I - \sigma^*(\sigma\sigma^*)^{-1}\sigma)\lambda^* D\xi$$

$$-\tfrac{1}{\theta+2}\{\lambda^* D\xi - \tfrac{1}{\theta+2}(\lambda^* Dw - NH)\}^* \sigma^* (\sigma\sigma^*)^{-1} \sigma \{\lambda^* D\xi - \tfrac{1}{\theta+2}(\lambda^* Dw - NH)\}$$

where $\xi = \frac{\partial w}{\partial \theta}$, $\zeta = \frac{\partial \xi}{\partial \theta}$ *and*

$$L(\theta)\psi := \frac{1}{2}tr[\lambda\lambda^* D^2\psi] + G(x)^* D\psi + (-\lambda^* Dw + NH)^* N^{-1}\lambda^* D\psi$$

- $L(\theta)$ is the generator of the ergodic diffusion process Y_t:

$$dY_t = \{(K_1 - \lambda N^{-1}\lambda^* \bar{P})Y_t + f\}dt + \lambda dW_t$$

$$f := b - \frac{\theta}{\theta+2}\lambda\sigma^*(\sigma\sigma^*)^{-1}(a - r\mathbf{1}) - \lambda N^{-1}\lambda^* \bar{q}$$

Definition 2 *The pair (K, L) of an $n \times n$ matrix K and an $n \times l$ matrix L is said to be controllable if an $n \times nl$ matrx $(L, KL, K^2L, \dots, K^{n-1}L)$ has rank n*

Asymptotics as $\theta \to \infty$

Proposition 1 *Assume (4.1). Then* $\chi'(\infty) \equiv \lim_{\theta\to\infty} \chi'(\theta) = r$

Main theorem

Theorem 1 *Under the assumptions of Lemma 3*

$$\hat{J}(k) = \begin{cases} \frac{1}{2}\inf_{\theta>0}\{\theta k - \chi(\theta)\}, & \chi'(0+) > k > r \\ 0, & k \geq \chi'(0+) \end{cases}$$

where

$$\hat{J}(k) = \liminf_{T\to\infty} \frac{1}{T} \inf_{h\in\mathcal{H}_T} \log P\left(\frac{\log V_T(h)}{T} \leq k\right)$$

$$P(\log V_T(h) \leq kT) \leq E[V_T(h)^{-\frac{\theta}{2}}]e^{\frac{\theta}{2}kT}$$

$$-2 \log P(\frac{\log V_T(h)}{T} \leq k) \geq -2 \log E[V_T(h)^{-\frac{\theta}{2}}] - \theta kT$$

$$\limsup_{T\to\infty} -\frac{2}{T} \inf_h \log P(\frac{\log V_t(h)}{T} \leq k) \geq \limsup_{T\to\infty} \frac{1}{T} \sup_h \{-2 \log E[V_T(h)^{-\frac{\theta}{2}}]\} - \theta k$$

$$\liminf_{T\to\infty} \frac{2}{T} \inf_h \log P(\frac{\log V_T(h)}{T} \leq k) \leq \theta k - \chi(\theta)$$

Converse inequality:

$\forall \epsilon > 0$ such that $k - \epsilon > r$ and take $\bar{\theta}$ such that

$$\hat{\chi}(k-\epsilon) := \inf_{\theta>0}\{\theta(k-\epsilon) - \chi(\theta)\} = \bar{\theta}\chi'(\bar{\theta}) - \chi(\bar{\theta})$$

Then we can show that there exists T_0 such that $\forall T \geq T_0$

$$P(\frac{V_T(h)}{T} \leq k) \geq \exp\{(\frac{\bar{\theta}}{2}\chi'(\bar{\theta}) - \frac{1}{2}\chi(\bar{\theta}) - \epsilon)T\}(1-4\epsilon)$$

Therefore

$$\frac{1}{T}\inf_h \log P(\frac{V_T(h)}{T} \leq k) \geq \frac{\bar{\theta}}{2}\chi'(\bar{\theta}) - \frac{1}{2}\chi(\bar{\theta}) - \epsilon + \frac{1}{T}(1-4\epsilon)$$

Hence

$$\liminf_{T\to\infty} \frac{1}{T}\inf_h \log P(\frac{V_T(h)}{T} \leq k) \geq \frac{1}{2}\hat{\chi}(k-\epsilon)$$