An ABC of Portfolio Choice: Asset Allocation with Bankruptcy and Contagion

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ABSTRACT: In this paper, we consider the asset allocation problem of an investor allocating his funds between several corporate bonds and a money market account. In particular, we provide a realistic model of financial distress: Firstly, we model Chapter 7 and Chapter 11 bankruptcies as different possible outcomes of financial distress. Secondly, we take into consideration that, in practice, "default" is not the end, but the beginning of financial distress, eventually leading to a reorganization or a liquidation of a distressed firm. Thirdly and most importantly, we are able to analyze the impact of contagion on an investor's demand for corporate bonds. Contagion is an important phenomenon, as it reduces the investor's ability to diversify his portfolio. Although widely recognized in the literature about the pricing of defaultable securities, to our knowledge, little work has been done quantifying the impact of contagion on security demands in a continuous-time framework.

KEYWORDS: portfolio optimization, liquidation, reorganization, default, finite state Markov chain

JEL-CLASSIFICATION: G11, G33

1 Introduction

Over the last decade, financial researchers have analyzed the pricing of defaultable securities in great detail. Motivated by economy-wide crises in Latin America or Asia, it was more recently recognized that an isolated view of a single firm is not appropriate, since the defaults of a small number of firms can have an economy-wide impact, triggering subsequent defaults of other firms.[1] Similar phenomena have been observed in certain sectors. A recent example is that of the carmaker industry, where the default of Delphi Corp. triggered a downgrade of GM's corporate credit rating. These so-called contagion effects have significant effects on the pricing of defaultable bonds and it is thus of great importance to analyze their impact on the portfolio choice of an investor, since the threat of contagion effects may obviously reduce an investor's ability to diversify his portfolio.[2] As far as we are aware, this has not been fully analyzed for defaultable claims yet, and our paper intends to fill this gap. Specifically, we are able to quantify the impact on the investor's demand for corporate bonds.

To reach this goal, we assume that the market for corporate bonds is driven by macroeconomic variables, modeled by a diffusion process, and additionally by a finite state Markov chain, characterizing the solvencies of the firms that have issued bonds. This is a flexible approach that contains, for instance, the model by Jarrow and Yu (2001) as special case. Besides, our Markov chain model enables us to analyze the impact of a bankruptcy code on the investor's portfolio decision. In particular, we demonstrate how the demand for corporate bonds changes, if the issuing firm is liquidated upon default instead of being reorganized and continuing to operate (Chapter 7 vs. Chapter 11).

Furthermore, a recent paper by Guo et al. (2005) emphasizes that, once a firm has defaulted on its debt, the company usually continues to operate (at least for a while) before it is liquidated or reorganized, i.e. the state of being financially distressed consists of two sub-states: Firstly, the firm defaults and then the financial problems are resolved by entering a formal bankruptcy (or a private workout). This distinction provides a more sophisticated model of recovery rates and allows for the pricing of defaulted debt, which is not possible in the classical models such as those by Jarrow and Turnbull (1995) or Duffie and Singleton (1999). Our framework is sufficiently general to analyze the impact of this distinction on portfolio choices as well.

[1]See, e.g., Jarrow and Turnbull (1995), Lando (1998), and Duffie and Singleton (1999) as well as Jarrow and Yu (2001) and Yu (2005).

[2]See also Kyle and Xiong (2001) for a similar conclusion in a different setting.

In contrast to portfolio problems with default-free bonds,[3] to the best of our knowledge, little work has been carried out on portfolio problems with defaultable bonds. Interestingly, Merton (1971) already considered such a portfolio problem in the special case of a reduced-form model with constant interest rates and constant default intensity. We shall discuss his approach in more detail in the following section, since it allows us to clarify some important points. Furthermore, Hou (2003) uses a diversification argument presented in an insightful paper by Jarrow, Lando, and Yu (2005) implying that event risk factors (counting processes) do not show up in the dynamics of well-diversified portfolios of corporate bonds. Consequently, the solution of these simplified portfolio problems can be found analogously to a problem with stochastic interest rates, but without default risk. This approach, however, disregards the impact of contagion effects on portfolio choices. To incorporate these effects, there are papers modeling contagion effects by so-called joint default events, i.e. several defaults are triggered at the same time by one Poisson process.[4] In practice, however, default events never occur simultaneously (except for parent and subsidiary companies) and thus this approach is not able to capture the rationale behind default clustering. Actually, contagion effects have a time-dimension, i.e. they spread like a disease in a population of firms and one firm after the other might get affected. Another drawback of these papers lies in the fact that, by definition, Poisson processes are memoryless and thus the probabilities of future (single as well as joint) defaults remain unchanged after a default has happened. In contrast to this, our paper can handle all these issues in a unified framework and the models with joint default events are nested as special cases.[5]

The contribution of our paper is manifold: Firstly, we suggest a general Markov chain setting in which several relevant portfolio problems with defaultable bonds can be analyzed. Secondly, we derive the corresponding system of Hamilton-Jacobi-Bellman equations that corresponds to a system of second-order partial differential equations and show how this system can be solved.[6] Thirdly, we provide closed-form solutions for several relevant portfolio problems. To make the intuition as clear as possible, we consider applications with one or two defaultable bonds only. This is however without loss of generality and our results generalize to problems with multiple

[3] See, e.g., Brennan, Schwartz, and Lagnado (1997), Brennan and Xia (2000), Munk and Sorensen (2005), and Sorensen (1998).

[4] See, e.g., Kraft and Steffensen (2006).

[5] To clarify this point consider two firms. If the default intensity of one firm jumps upwards upon the default of the second firm, then we have a contagion effect. In the special case where the default intensity jumps to infinity, a joint default event occurs.

[6] A suitable verification theorem is available from the authors upon request.

bonds. Our paper is structured as follows: In Section 2, we discuss the results by Merton (1971). Section 3 introduces our Markov chain framework. In Section 4, the investor's portfolio choice problem is formulated and optimality conditions are derived. Sections 5 demonstrates how the general results simplify if the market is complete. In Section 6, we discuss several important applications such as portfolio problems with contagion. All proofs are presented in the Appendix.

2 Merton's Results

Merton (1971) analyzes an interesting portfolio problem that allows us to clarify some important points. He considers an investor allocating his wealth between a stock and a so-called risky bond with dynamics:

$$\begin{aligned} dS(t) &= S(t)[\alpha dt + \sigma dW(t)], \quad S(0) = s_0 \\ dB(t) &= B(t)[rdt - dN(t)], \quad B(0) = b_0, \end{aligned}$$

where r is the constant short rate and N is a Poisson process with constant intensity $\lambda > 0$. As pointed out by Merton, the dynamics of the risky bond imply that its recovery rate is zero. Consider now an investor maximizing expected utility from terminal wealth at time T with respect to his utility function U.[7] Denoting the proportion invested in stock by π, Merton's Bellman equation for this problem reads

$$0 = \sup_{\pi} \Big\{ G_t(t,x) + x(r + (\alpha - r)\pi)G_x(t,x) + 0.5x^2\pi^2x^2G_{xx}(t,x) + \lambda\big[G(t,\pi x) - G(t,x)\big] \Big\} \quad (1)$$

with $G(T,x) = U(x)$, where G denotes the investor's indirect utility and x denotes his wealth. Since the indirect utility function does not qualitatively change upon default, – in fact, the investor solely loses $(1-\pi^*)x$, which corresponds to the amount of money invested in bonds – this formulation implicitly assumes that after a default has happened the investor can still invest in a risky bond with identical features (especially with the same default intensity). This assumption is a bit puzzling, as the recovery rate is zero. It is hard to believe that the issuing firm continues to exist if the bond is completely wiped out. Actually, one would guess that the firm is liquidated. In this case, the investor cannot invest in the risky bond anymore implying that his indirect utility corresponds to the indirect utility of a portfolio problem where the

[7]In contrast to Merton, we disregard consumption in this example. This is without loss of generality.

investor can put his funds in stock only.[8] For an investor with power utility $U(x) = x^\gamma/\gamma$, the indirect utility after liquidation reads

$$\tilde{G}(t,x) = \mathrm{E}^{t,x}\left[\frac{1}{\gamma}\left(\frac{x}{S(t)}\right)^\gamma S(T)^\gamma\right] = \frac{1}{\gamma}x^\gamma \exp\left(\gamma(\alpha - 0.5(1-\gamma)\sigma^2)(T-t)\right).$$

In contrast to (1), the Bellman equation now becomes

$$0 = \sup_\pi \left\{ G_t(t,x) + x(r + (\alpha - r)\pi)G_x(t,x) + 0.5x^2\pi^2x^2G_{xx}(t,x) + \lambda\big[\tilde{G}(t,\pi x) - G(t,x)\big]\right\}. \quad (2)$$

In general, there is no reason to believe that $G = \tilde{G}$ and thus the portfolio choice of an investor depends on whether he has the opportunity to invest in corporate bonds of the defaulted firm or not. This is one of the aspects that we will analyze in the sequel.

3 Model

This section generalizes ideas in Kraft and Steffensen (2007).[9] We consider an economy where uncertainty is described by the complete filtered probability space $(\Omega, \mathcal{P}, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T^*]})$ and $\mathcal{F} = \mathcal{F}_{T^*}$. To rule out arbitrage, we assume that an equivalent martingale measure Q exists under which discounted asset prices are (local) martingales.[10]

The economy is driven by an L-dimensional process $Y = (Y_1, \ldots, Y_L)$ consisting of macroeconomic state variables (e.g. interest rates) that possess the dynamics

$$dY_\iota(t) = \alpha_{Q,\iota}(t)dt + \beta_\iota(t)dW_Q(t),$$

where α_Q is an L-vector valued function of t and $Y(t)$ and β is an $L \times L$-matrix valued function of t and $Y(t)$ with β_ι being its ι-th row. The process W_Q is an L-dimensional standard Brownian motion under Q with uncorrelated coordinates. Here and in the following we assume that the coefficients of all stochastic differential equations (SDEs) are predictable processes which are sufficiently integrable such that the SDEs possess unique solutions.[11] In this economy, the short rate is a function of t and $Y(t)$, i.e. $r(t) = r(t, Y(t))$. Of course, the short rate itself can be one

[8] To make the portfolio problem a bit more interesting, one can assume that the investor can invest in more than one stock so that his portfolio problem is not trivial after the default of the risky bond.

[9] Especially, we also allow for dependencies of the parameters on the counting processes N and also derive the dynamics of the relevant processes under the physical measure.

[10] See Harrison and Kreps (1979) and Delbaen and Schachermayer (1999) for the essential equivalence of the existence of such a measure and the absence of arbitrage.

[11] See, e.g., Protter (2004), pp. 249ff.

of the components of Y. Investors can borrow and lend using a money market account with dynamics

$$dM(t) = M(t)r(t)dt, \quad M(0) = 1,$$

and there exists a market for zero-coupon bonds with risk-neutral dynamics

$$dP_f(t,T) = P_f(t,T_f)[r(t)dt + \sigma_f(t)dW_Q(t)], \quad P(T_f,T_f) = 1,$$

where T_f denotes the maturity of the bond. We assume that the price of a zero-coupon bond is a smooth function of time t and macroeconomic state variables $Y(t)$. Therefore, by Ito's lemma, the L-dimensional vector σ is a function of t and $Y(t)$.

Moreover, there exists a market for corporate zero-coupon bonds (defaultable bonds). This market is characterized by a finite set of states,[12] $\mathcal{J} = \{0,\dots,J\}$, where, by convention, 0 is the initial state at time 0. For instance, if the market consists of two corporate bonds and each bond can only be in default or in non-default, the state space corresponds to $\mathcal{J} = \{0,1,2,3\}$. The model of Jarrow and Yu (2001) is thus a special case of our framework. Alternatively, if the states are identified with rating classes, this framework nests the models by Jarrow, Lando, and Turnbull (1997), which is generalized by Lando (1998). Let $Z(t)$ denote the state at time $t \in [0,T^*]$ and let Z be a right-continuous process with left limits (RCLL). Then the associated $(J+1)$-dimensional counting process $N = (N^k)_{k\in\mathcal{J}}$ is an RCLL process, where N^k counts the number of transitions into state k, i.e.

$$N^k(t) = \#\{s | s \in (0,t], Z(s-) \neq k, Z(s) = k\}.$$

We assume that there exist sufficiently regular functions $\lambda_Q^{jk,n}$ depending on t and $Y(t)$ such that N^k admits the risk-neutral stochastic intensity process $\{\lambda_Q^{Zk,N}\}$, i.e.

$$M_Q^k(t) = N^k(t) - \int_0^t \lambda_Q^{Z(s)k,N(s)}(s)\,ds$$

is a Q-martingale. Let $\mathcal{F}_t^Y = \sigma\{Y(s), s \leq t\}$ and set $\lambda_Q^{j,n*} = \sum_{k\neq j} \lambda_Q^{jk,n}$. By construction, (Z,Y,N) is a Markov process.

We consider a corporate contingent claim that is characterized by a payment stream given by

$$dA(t) = \sum_{k:k\neq Z(t-)} a^{Z(t-)k,N(t-)}(t)dN^k(t) + a^{Z(t),N(t)}d\varepsilon_T(t),$$

[12]These states should not be mixed up with the macroeconomic state Y.

where for fixed states j, k, and $n \in I\!N^{J+1}$ the variables $a^{j,n}$ and $a^{jk,n}$ are functions depending on t and $Y(t)$. Moreover, ε_T is the Dirac mass at T. The function a^{jk} is a payment upon transition from state j into state k and a^j is the final payment given the chain is in state j at time T. In our applications, the process A usually models the payment stream of a corporate zero-coupon bond and that is why we refer to A as a corporate bond. To clarify the meaning of $a^{jk,n}$ and $a^{j,n}$, let us consider a simple example, where we can disregard the N-dependence of $a^{jk,n}$ and $a^{j,n}$: If we wish to analyze a single zero-coupon bond where the issuing firm can be in non-default (state 0) or in default (state 1), then a^{01} models the recovery payment upon default. The final payment is then either 0 or 1, i.e. $a^1 = 0$ and $a^0 = 1$.

Our framework allows all coefficients to depend on the $J + 1$-dimensional counting process $N = (N^0, N^1, \ldots, N^J)$ counting the number of transitions into each state. One of the reasons for this formulation is that we also wish to analyze situations where the (final) payments depend on an *unbounded* number of default events.

By assumption, a risk-neutral measure exists implying that the time-t value of the payment stream A is given by

$$\begin{aligned} B(t) &= \mathrm{E}_Q\Big[\int_{(t,T]} e^{-\int_t^s r(u)\,du} dA(s)\Big|\mathcal{F}_t\Big] \qquad (3)\\ &= \mathrm{E}_Q\Big[\int_{(t,T]} e^{-\int_t^s r(u)\,du} dA(s)\Big|Y(t), Z(t), N(t)\Big], \end{aligned}$$

where the second equality sign follows from the fact that $(Z(t), Y(t), N(t))$ is a Markov process. It is important to realize that $B(t)$ equals the value of the contingent claim's future payments ex a (possible) payment upon transition at time t. However, similarly to dividend payments, holding the contingent claim also includes all payments upon transition such as recovery payments. An application of Ito's lemma leads to the price dynamics of the bond under the risk-neutral measure.

Proposition 3.1 (Risk-neutral Bond Dynamics) *There exist functions $\mathcal{B}^{j,n}$, $j \in \mathcal{J}$ and $n \in I\!N^{J+1}$, such that $B(t) = \mathcal{B}^{Z(t),N(t)}(t, Y(t))$. If each $\mathcal{B}^{j,n}$ is sufficiently smooth (more precisely, $\mathcal{B}^{j,n} \in C^{1,2}$), the risk-neutral price dynamics are given by*[13]

$$\begin{aligned} dB(t) &= \big(\mathcal{B}_t^{Z(t),N(t)}(t) + \alpha_Q(t)\mathcal{B}_y^{Z(t),N(t)}(t) + 0.5\,\mathrm{tr}\big[\beta(t)^T\mathcal{B}_{yy}^{Z(t),N(t)}(t)\beta(t)\big]\big)dt \\ &\quad + \mathcal{B}_y^{Z(t),N(t)}(t)\beta(t)dW_Q(t) + \sum_{k:k\neq Z(t-)} (\mathcal{B}^{k,N(t-)+1^k}(t) - \mathcal{B}^{Z(t-),N(t-)}(t))dN^k(t) \end{aligned}$$

[13]tr stands for trace and $\beta(t)^T$ denotes the transpose of the matrix $\beta(t)$.

with $1^k = (0^{k\times 1}, 1, 0^{(J-k)\times 1})^T$. *To shorten notation, we have omitted the dependence of* $\mathcal{B}^{j,n}$ *on* Y.

Remark. a) If $\mathcal{B}^{j,n}$ is in $C^{1,2}$, then $\mathcal{B}^{j,n}$ and its derivatives are continuous in t and y implying that, for instance, $\mathcal{B}^{j,n}(t-, Y(t-)) = \mathcal{B}^{j,n}(t, Y(t))$. Besides, Z jumps at only countably many points, i.e. in Ito and Lebesgue integrals we can replace all $Z(t-)$ by $Z(t)$.

b) Note that the drift of B under a risk-neutral measure is not equal to r, since, by definition, $B(t)$ does not include payments at time t.

The system (4) of partial differential equations (PDEs) in the following theorem is a generalization of the Black-Scholes PDE that is satisfied by contingent claims depending on the macroeconomic state variables Y only. If the state process Z cannot jump, this system reduces to the Black-Scholes PDE.

Theorem 3.1 (System of Pricing Equations) *The functions* $\mathcal{B}^{j,n}$, $j \in \mathcal{J}$ *and* $n \in I\!N^{J+1}$, *satisfy the system of PDEs*

$$\mathcal{B}^{j,n}_t = r\mathcal{B}^{j,n} - \alpha_Q \mathcal{B}^{j,n}_y - 0.5\,\mathrm{tr}\big[\beta^T \mathcal{B}^{j,n}_{yy}\beta\big] - \sum_{k:k\neq j} \lambda^{jk,n}_Q \left(a^{jk,n} + \mathcal{B}^{k,n+1^k} - \mathcal{B}^{j,n}\right) \tag{4}$$

with terminal conditions $\mathcal{B}^{j,n}(T, y) = a^{j,n}(y)$, $j \in \mathcal{J}$ *and* $n \in I\!N^{J+1}$.

Remark. If $a^{jk,n} = 0$ for all $j \neq k$ and $n \in I\!N_0^{J+1}$, then the drift of B simplifies into r. This can be verified by substituting (4) into the risk-neutral dynamics for B.

Let $\Gamma = B + A$ denote the gains process of the bond. The drift of the total return on a bond investment equals r and its dynamics are given in the following corollary.

Corollary 3.1 *The risk-neutral dynamics of the total return of a bond investment are given by*

$$\begin{aligned} d\Gamma(t) &= r(t)\mathcal{B}^{Z(t),N(t)}(t)dt + \mathcal{B}^{Z(t),N(t)}_y(t)\beta(t)dW_Q(t) \\ &\quad + \sum_{k:k\neq Z(t-)} \left(a^{Z(t-)k,N(t-)}(t) + \mathcal{B}^{k,N(t-)+1^k}(t) - \mathcal{B}^{Z(t-),N(t-)}(t)\right) dM^{k,N(t)}_Q(t), \end{aligned}$$

where $dM^{k,N(t)}_Q(t) = dN^k(t) - \lambda^{Z(t)k,N(t)}_Q(t)dt$.

To link the risk-neutral intensities $\lambda^{jk,n}_Q$ to their counterparts under the physical measure, $\lambda^{jk,n}$, we introduce strictly positive processes $\eta^{jk,n}$ being functions of t and $Y(t)$ such that $\lambda^{jk,n}_Q = \eta^{jk,n}\lambda^{jk,n}$ for fixed j, $k \in \mathcal{J}$, and $n \in I\!N_0^{J+1}$. The market price of risk for a transition of state

j into state k is thus given by $\eta^{jk,n} - 1$, i.e. there is a positive risk premium if $\eta^{jk,n} > 1$ and a negative risk premium if $\eta^{jk,n} < 1$. Specifically, if k is the default state, then $\eta^{jk,n} - 1$ equals the risk premium for default event risk given that the current state is j. From a probabilistic point of view, $\eta^{jk,n} - 1$ is a Girsanov kernel analogous to the corresponding kernel η_y for diffusive risk. Note that under the physical measure $dW(t) = dW_Q(t) - \eta_y(t)dt$ is a Brownian increment.

Proposition 3.2 (Physical Return Dynamics) *Under the physical measure, the dynamics of the total return on a bond investment are given by*

$$\begin{aligned} \frac{d\Gamma(t)}{B(t-)} &= (r(t) + \chi^{Z(t),N(t)}(t))dt - D^{Z(t),N(t)}(t)\beta(t)dW(t) \\ &\quad + \sum_{k:k\neq Z(t-)} \Big(R^{Z(t-)k,N(t-)}(t) + L^{Z(t-)k,N(t-)}(t)\Big)dN^k(t)\Big] \end{aligned}$$

with

$$\chi^{j,n}(t) = -\sum_{k:k\neq j} (R^{jk,n}(t) + L^{jk,n}(t))\eta^{jk,n}(t)\lambda^{jk,n}(t) - \beta(t)D^{j,n}(t)\eta_y(t),$$

where $D^{j,n} = -\mathcal{B}_y^{j,n}/\mathcal{B}^{j,n}$ denotes the state-dependent duration of the corporate bond, $R^{jk,n} = a^{jk,n}/\mathcal{B}^{j,n}$ the relative payment upon transition, and $L^{jk,n} = (\mathcal{B}^{k,n+1^k} - \mathcal{B}^{j,n})/\mathcal{B}^{j,n}$ the relative price jump.

Remark. In the sequel, with a slight abuse of notation, we will not distinguish between the process B and the function $\mathcal{B}$ to simplify notations. For instance, we write B_y instead of $\mathcal{B}_y$.

We wish to remark that $\chi^{j,n}$ is not the excess return of the corporate bond because N^k is not a (local) martingale. Rewritting the dynamics in terms of the compensated process $M^{k,N(t)}(t) = N^k(t) - \int_0^t \lambda^{Z(t)k,N(t)}(s)\, ds$, it follows that the excess return equals

$$\chi^{j,n} + \sum_{k:k\neq j} (R^{jk,n} + L^{jk,n})\lambda^{jk,n} = -\sum_{k:k\neq j} (R^{jk,n} + L^{jk,n})\lambda^{jk,n}(\eta^{jk,n} - 1) - D^{j,n}\beta\eta_y.$$

From this representation, it can be seen that $\eta^{jk,n} - 1$ plays the same role as η_y, i.e. both are market prices of risk for jump and diffusive risks, respectively, scaled by corresponding elasticity measures. To simplify our exposition, we make the following

Assumption. The state process is identical to the short rate, i.e. $Y = r$, and W is a one-dimensional Brownian motion such that

$$dr(t) = \alpha(t)dt + \beta(t)dW(t),$$

where α and β are functions of t and r.

We emphasize that the more general situation can easily be handled following the methodology presented below.

4 Portfolio Problem

Our goal is to analyze portfolio problems where an investor can allocate his funds between a (locally risk-free) money market account M, a Treasury bond (syn. default-free bond), and several defaultable bonds.[14] We assume that the dynamics of I corporate bonds are given. At time t not all of these bonds need to be available for trading. For some of them the issuing firm may have already been liquidated so that the bond is not traded any more.[15] To model this fact, we introduce the set

$$\mathcal{M}_A^{j,n}(t) = \{i \in \{1, \dots, I\} : \text{bond } i \text{ is traded at time } t\},$$

where $Z(t) = j$ and $N(t) = n$. To simplify our exposition, we assume that all maturities of the bonds, T_i, are greater than the investor's horizon T, but it would be straightforward to allow for situations with $T_i < T$.

The investor maximizes expected utility from intermediate consumption and terminal wealth at final time $T \leq \min\{T_i, i = 1, \dots, I\}$ with respect to the following utility function

$$U(t,x) = \frac{1}{\gamma}\psi(t)^{1-\gamma}x^{\gamma}, \qquad \gamma < 1,$$

where $\psi(t) = \psi(t, r(t))$ is a non-negative discount process reflecting the investor's time preferences. For instance, one can choose $\psi(t) = e^{-\frac{1}{1-\gamma}\rho t}$ with a constant $\rho > 0$. Disregarding consumption for the moment, the investor's time-t wealth is given by

$$X(t) = \varphi_f(t)P_f(t,T_f) + \sum_{i=1}^{I} \varphi_i(t)B_i(t,T_i) + \varphi_M(t)M(t),$$

where $B_i(t,T_i)$ denotes the time-t price of the i-th corporate bond with maturity T_i. The product $\varphi_M(t)M(t)$ stands for the money invested in the money market account at time t. The processes $\varphi_f(t)$ and $\varphi_i(t)$ denote the number of shares of default-free zero-coupon bonds and the i-th corporate bond held by the investor at time t. If $i \notin \mathcal{M}_A^{j,n}(t)$, by convention, we set $\varphi_i(t) = 0$. Restricting our considerations to self-financing strategies $(\varphi_f, \varphi, \varphi_M)$ and applying

[14] To make the intuition as clear as possible, we do not consider stocks here, which is without loss of generality.

[15] In principle, our framework is also able to cover situations where corporate bonds are exogenously issued.

Ito's lemma yields the dynamics

$$dX(t) = \varphi_f(t)dP_f(t,T_f) + \sum_{i=1}^{I} \varphi_i(t-)d\Gamma_i(t,T_i) + \varphi_M(t)dM(t),$$

where Γ_i denotes the gain process of the i-th corporate bond. The proportions invested in the default-free bond and the i-th bond are given by $\pi_0 = \varphi_f P_f/X$ and $\pi_i = \varphi_i B_i/X$. Therefore, we obtain the so-called wealth equation:

$$dX = X^-\big[(r + \sum_{i=0}^{I} \pi_i \chi_i^{Z,N})dt - \beta D dW + \sum_{i=1}^{I} \sum_{k \neq Z^-} \pi_i^-(R_i^{Z^-k,N^-} + L_i^{Z^-k,N^-})dN^k\big],$$

where $X^- \equiv X(t-)$, $Z^- \equiv Z(t-)$, $\pi_i^- \equiv \pi_i(t-)$, and $\chi_0 = \chi_0^Z \equiv \eta_r \sigma_f$. The process $D \equiv \sum_{i=0}^{I} \pi_i D_i^{Z,N}$ denotes the portfolio duration and the 0-th bond is identified with the default-free zero-coupon bond, i.e. $D_0 = D_0^Z = D_f = -(P_f)_r/P_f$. Unless necessary for clarity, here and in the following we suppress functional dependencies. Besides, $\sum_{k \neq j}$ is short hand for $\sum_{k:k \neq j}$. Since

$$\sum_{i=0}^{I} \pi_i \chi_i^{Z,N} = -\sum_{i=1}^{I} \pi_i \sum_{k \neq Z} (R_i^{Zk,N} + L_i^{Zk,N}) \eta^{Zk,N} \lambda^{Zk,N} - \beta \eta_r D,$$

rewriting the wealth equation gives

$$\begin{aligned} dX &= X^-\big[(r - \sum_{k \neq j} \eta^{Zk,N} \lambda^{Zk,N} \sum_{i=1}^{I} \pi_i (R_i^{Zk,N} + L_i^{Zk,N}) - \beta \eta_r D)dt \\ &\quad - \beta D dW + \sum_{k \neq Z^-} \sum_{i=1}^{I} \pi_i^-(R_i^{Z^-k,N^-} + L_i^{Z^-k,N^-})dN^k\big]. \end{aligned} \tag{5}$$

Since the investor can invest in a default-free bond, the portfolio duration can be treated as a control variable. For $Z(t) = j$ and $N(t) = n$, the investor's indirect utility (syn. value function) is defined by

$$G^{j,n}(t,x,r) = \sup_{\pi,D,c} \mathrm{E}_{j,n}^{t,x,r}\left[\int_t^T \tfrac{1}{\gamma}\psi(s)^{1-\gamma} c(s)^\gamma ds + \tfrac{1}{\gamma}\psi(T)^{1-\gamma}(X(T))^\gamma\right],$$

where $\mathrm{E}_{j,n}^{t,x,r}[\cdot] = \mathrm{E}[\cdot|X(t) = x, r(t) = r, Z(t) = j, N(t) = n]$. To shorten notations, we set $\tilde{L}_i^{jk,n} = R_i^{jk,n} + L_i^{jk,n}$. The following theorem summarizes the first-order conditions of the portfolio problem.

Theorem 4.1 (Optimality Conditions) *The first-order conditions (FOCs) for D, π, and c read*

$$D^{j,n*} = -\frac{1}{1-\gamma}\frac{\eta_r}{\beta} - \frac{f_r^{j,n}}{f^{j,n}}, \tag{6}$$

$$\frac{c^{j,n*}}{x} = \frac{\psi}{f^{j,n}} \tag{7}$$

$$0 = -\sum_{k\neq j}\tilde{L}_{\nu}^{jk,n}\eta^{jk,n}\lambda^{jk,n}, +\sum_{k\neq j}\lambda^{jk,n}(1+\sum_i \pi_i^{j,n*}\tilde{L}_i^{jk,n})^{\gamma-1}\left(\frac{f^{k,n+1^k}}{f^{j,n}}\right)^{1-\gamma}\tilde{L}_{\nu}^{jk,n}, \tag{8}$$

$\nu = 1,\ldots,I$, where the functions $f^{j,n}$ satisfy the system of PDEs

$$\begin{aligned} 0 &= \psi + f_t^{j,n} + \tfrac{\gamma}{1-\gamma}\left[r + 0.5\tfrac{1}{1-\gamma}\eta_r^2 - \sum_{k\neq j}\eta^{jk,n}\lambda^{jk,n}\sum_{i=1}^{I}\pi_i^{j,n*}\tilde{L}_i^{jk,n}\right]f^{j,n} \\ &\quad +(\alpha + \tfrac{\gamma}{1-\gamma}\beta\eta_r)f_r^{j,n} + 0.5\beta^2 f_{rr}^{j,n} \\ &\quad +\tfrac{1}{1-\gamma}f^{j,n}\sum_{k\neq j}\lambda^{jk,n}\left[(1+\sum_{i=1}^{I}\pi_i^{j,n*}\tilde{L}_i^{jk,n})^{\gamma}\left(\frac{f^{k,n+1^k}}{f^{j,n}}\right)^{1-\gamma} - 1\right] \end{aligned} \tag{9}$$

with $f^{j,n}(T,r) = \psi(T)$, $j \in J$ and $n \in I\!N_0^{J+1}$. The investor's indirect utility is given by $G^{j,n}(t,x,r) = \frac{1}{\gamma}x^{\gamma}f^{j,n}(t,r)^{1-\gamma}$.

Interest rate dynamics play a major role in explaining the prices of corporate bonds. However, in this paper we are mainly interested in analyzing the effect of credit risk on portfolio decisions. In order to distinguish between effects stemming from interest rate risk and credit risk, it is thus reasonable to start with the assumption that interest rates are constant. Specifically, equation (9) simplifies significantly:

$$\begin{aligned} 0 &= \psi + f_t^{j,n} + \tfrac{\gamma}{1-\gamma}\left[r - \sum_{k\neq j}\eta^{jk,n}\lambda^{jk,n}\sum_{i=1}^{I}\pi_i^{j,n*}\tilde{L}_i^{jk,n}\right]f^{j,n} \\ &\quad +\tfrac{1}{1-\gamma}f^{j,n}\sum_{k\neq j}\lambda^{jk,n}\left[(1+\sum_{i=1}^{I}\pi_i^{j,n*}\tilde{L}_i^{jk,n})^{\gamma}\left(\frac{f^{k,n+1^k}}{f^{j,n}}\right)^{1-\gamma} - 1\right], \end{aligned} \tag{10}$$

where the optimal portfolio strategy is still determined by (8). These equations form a system of first-order ODEs (compared to a system of second-order PDEs). We study this case in order to obtain analytic solutions of the optimal portfolio strategy. Analytic solutions facilitate intuition, comparative statics and empirical estimation. As discussed below, this structure is readily generalized, but at the expense of losing the analytic solutions. In any case, it is crucial to get a tractable representation for $f^{k,n+1^k}/f^{j,n}$.

In the following section, we analyze the problem under the assumption that the market is complete. Without going into details, we remark that the first-order conditions also simplify if the interest rate model is affine.

5 Complete Markets

Our financial market is (dynamically) complete if the number of tradable corporate bonds at any time $t \in [0,T]$ is (at least) equal to the number of states into which the Markov chain Z

can jump at that time. To formalize the notion of completeness, we define the set

$$\mathcal{M}_S^{j,n}(t) = \{k \in \{0, \ldots, J\} : \lambda^{jk,n}(t) > 0\}$$

and recall the definition

$$\mathcal{M}_A^{j,n}(t) = \{i \in \{1, \ldots, I\} : \text{bond } i \text{ is traded at time } t\}$$

from Section 4. The set $\mathcal{M}_S^{j,n}(t)$ consists of all states into which the chain can jump at time t given that the current state is $Z(t) = j$ and that $N(t) = n$. Using these notations, completeness formally means essentially that $|\mathcal{M}_S^{j,n}(t)| \leq |\mathcal{M}_A^{j,n}(t)|$ for all $t \in [0, T]$, $j \in \{0, \ldots, J\}$, and $n \in I\!N_0^{J+1}$.[16] In this section, we assume without loss of generality that $|\mathcal{M}_S^{j,n}(t)| = |\mathcal{M}_A^{j,n}(t)|$. To exploit the completeness assumption, we rewrite the FOC (8) as follows:

$$\textstyle\sum_{k \neq j} \lambda^{jk,n}(R_\nu^{jk,n} + L_\nu^{jk,n})\delta^{jk,n} = \sum_{k \neq j} \lambda^{jk,n}(R_\nu^{jk,n} + L_\nu^{jk,n})\eta^{jk,n}, \quad \nu \in \mathcal{M}_A^j, \tag{11}$$

where $\delta^{jk,n} \equiv (1 + \sum_i \pi_i^{j,n*}(R_i^{jk,n} + L_i^{jk,n}))^{\gamma-1}(f^{k,n+1^k}/f^{j,n})^{1-\gamma}$. Relation (11) defines a system of linear equation for $\delta^{jk,n}$. Rewriting (11) leads to an homogeneous system of linear equations for $\delta^{jk,n} - \eta^{jk,n}$:

$$\textstyle\sum_{k \neq j} \lambda^{jk,n}(R_\nu^{jk,n} + L_\nu^{jk,n})(\delta^{jk,n} - \eta^{jk,n}) = 0, \quad \nu \in \mathcal{M}_A^{j,n}.$$

Therefore, we conclude $\delta^{jk,n} = \eta^{jk,n}$. This solution is unique if the matrix $(\tilde{L}_\nu^{jk,n})_{\nu k}$ has full rank, i.e. the actual losses of all firms are linearly independent. By definition of $\delta^{jk,n}$, the optimal proportions $\pi_i^{j,n*}$ satisfy

$$1 + \textstyle\sum_i \pi_i^{j,n*}(R_i^{jk,n} + L_i^{jk,n}) = (\eta^{jk,n})^{\frac{1}{\gamma-1}} f^{k,n+1^k}/f^{j,n}, \quad k \in \mathcal{M}_S^{j,n} \tag{12}$$

and we can rewrite the PDEs (9) as follows:

$$0 = \psi + f_t^{j,n} - \tilde{r}^{j,n} f^{j,n} + \tilde{\alpha} f_r^{j,n} + 0.5\beta^2 f_{rr}^{j,n} + \textstyle\sum_{k \neq j} \tilde{\lambda}^{jk,n}(f^{k,n+1^k} - f^{j,n}), \tag{13}$$

where

$$\begin{aligned}
\tilde{r}^{j,n} &= -\tfrac{\gamma}{1-\gamma}\Big(r + 0.5\tfrac{1}{1-\gamma}\eta_r^2 + \sum_{k \neq j}(\eta^{jk,n} - 1)\lambda^{jk,n}\Big) + \sum_{k \neq j}\lambda^{jk,n} - \sum_{k \neq j}\tilde{\lambda}^{jk,n}, \\
\tilde{\alpha} &= \alpha + \tfrac{\gamma}{1-\gamma}\beta\eta_r, \\
\tilde{\lambda}^{jk,n} &= \lambda^{jk,n}(\eta^{jk,n})^{\frac{\gamma}{\gamma-1}}.
\end{aligned}$$

Note that due to the completeness assumption, the system of linear equations (12) can be solved for the optimal proportions. Using a similar argument as in the proof of Proposition 3.1, we obtain the following result.

[16] See Norberg (2003).

Proposition 5.1 (Stochastic Representation of Indirect Utility) *The system of PDEs for $f^{j,n}$, $j = 0, \ldots, J$ and $n \in \mathbb{N}_0^{J+1}$, has the following Feynman-Kac representation*

$$f^{j,n}(t,r) = \tilde{\mathrm{E}}_{j,n}^{t,r}\Big[\int_t^T e^{-\int_t^s \tilde{r}^{Z(u),N(u)}(u)\,du}\psi(s)\,ds + e^{-\int_t^T \tilde{r}^{Z(s),N(s)}(s)\,ds}\psi(T)\Big],$$

where expectations are taken under an utility-dependent measure $\tilde{P}$ under which the short rate has the drift $\tilde{\alpha}$ and the Markov chain admits the intensity process $\tilde{\lambda}^{jk,n}$.

Substituting the solutions for $f^{j,n}$ back into (7) and (12), one can finally compute the consumption rate $c^{j,n*}$ and the proportions $\pi_i^{j,n*}$. Note that for fixed $r(t) = r$, $Z(t) = j$, and $N(t) = n$ the equations (12) form a system of linear equations for $\pi_i^{j,n*}$. To conclude this section, we remark that, if interest rates are assumed to be constant, equation (13) becomes

$$0 = \psi + f_t^{j,n} - \tilde{r}^{j,n} f^{j,n} + \textstyle\sum_{k \neq j} \tilde{\lambda}^{jk,n}(f^{k,n+1^k} - f^{j,n}), \tag{14}$$

where $\tilde{r}^{j,n} = -\frac{\gamma}{1-\gamma}\big(r + \sum_{k\neq j}(\eta^{jk,n} - 1)\lambda^{jk,n}\big) + \sum_{k\neq j}\lambda^{jk,n} - \sum_{k\neq j}\tilde{\lambda}^{jk,n}$.

6 Applications

6.1 Reorganization vs. Liquidation

If a firm is in financial distress, then it either continues to exist after being reorganized or the business is closed down and the firm is liquidated. The main factor driving the final result is the strategic position of the firm. Another crucial aspect is the philosophy behind the bankruptcy code of a country. For instance, in the United States a firm filing for bankruptcy usually continues to operate (Chapter 11), whereas in some European countries (e.g. Germany) liquidation and sale of the assets for the benefit of creditors is more likely the result.[17] In the following, we will demonstrate that this has an effect on investors' portfolio choices.

To make the rational behind our analysis as clear as possible, we look at a portfolio problem where, apart from the Treasury bond, the investor can invest in a single corporate bond and maximizes utility from terminal wealth only. Upon default the firm is either reorganized or liquidated. This formulation, though simple, is capable of capturing the impact of the bankruptcy code on the investor's decision. Later on we generalize the results of these extreme cases to

[17]For Germany, this was at least true until 1999. See Davydenko and Franks (2004) for an insightful survey over the bankruptcy codes in UK, France, and Germany and their implications for distressed reorganizations and lending practices.

a multiple bond problem and to a situation where, with certain probabilities, the firm is reorganized or liquidated. Furthermore, we abstract from the fact that usually some time elapses between the default of a firm and its liquidation or its reorganization, respectively. Our framework is general enough to deal with this more realistic situation as well and we will address this point in the following subsection. In the sequel, we use the following

Notation. If coefficients are assumed to be independent of N, then we drop the superindex n (e.g. λ^{01} instead of $\lambda^{01,n}$).

We assume that in the problem with reorganization neither the default intensity nor the corresponding risk premium depends on the default history. Therefore, the Markov chain starts in a non-default state and upon default jumps into a second state. Since the firm is reorganized and continues to operate, this state is not absorbing and upon the second default the chain jumps back into the initial state and so on. This situation is depicted in Figure 1. In particular, the investor's indirect utility functions are identical in the two states and thus the states are indistinguishable. Since the market is complete, this follows directly from (13) and the fact that, by assumption, $\eta = \eta^{01} = \eta^{10}$ and $\lambda = \lambda^{01} = \lambda^{10}$.

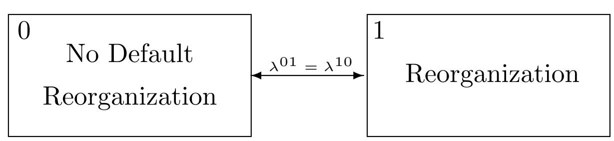

Figure 1: Default with Reorganization

If the firm is liquidated, jumping back and forth between the two states is not possible. This is reflected by the fact that upon default the chain jumps into an absorbing state. The investor is then forced to invest all his funds in the money market account and in the Treasury bond. For this reason, the indirect utility functions of the investor differ in the two states. This situation is detailed in Figure 2.

Firstly, we analyze the case where the firm continues to operate after default. Since there is no payment upon default, but only a downward jump of the corporate bond price, we set $R^{01} = R^{10} = 0$. Hence, the FOC (12) for the optimal proportion invested in the corporate bond

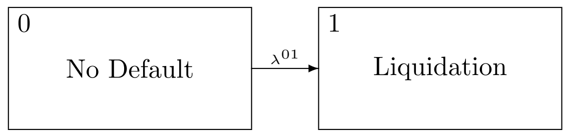

Figure 2: Default with Liquidation

can be rewritten as follows:[18]

$$\pi^{j,n*} = \frac{1 - \eta^{\frac{1}{\gamma - 1}}}{-L}, \tag{15}$$

where $j \in \{0, 1\}$ and $\eta = \eta^{01} = \eta^{10}$. For simplicity, it is assumed that $L = L^{01} = L^{10}$ is a constant being independent of N.[19] Note that $-L > 0$, since L denotes the downward price jump upon default. The optimal demand for corporate bonds, $\pi^{j,n*}$, has reasonable properties: It is decreasing in the investor's risk aversion γ as well as the fractional loss L and increasing in the risk premium η. A similar result was derived by Liu, Longstaff, and Pan (2003) who consider portfolio problems with event risk.

If the firm is liquidated upon default, we are in a different situation: Upon default a fraction of the pre-default value of the bond is paid out to the investor, the bond price jumps to zero, and the bond cannot be traded any more. Denoting the initial state by 0 and the liquidation state by 1, we thus have $L^{01} = -1$, where in the liquidation case we suppress the N-dependency everywhere (even in the value function), as only one default can occur. The optimal proportion invested in corporate bonds reads

$$\pi^{0*} = \frac{1 - (\eta^{01})^{\frac{1}{\gamma - 1}} \frac{f^1}{f^0}}{1 - R^{01}}. \tag{16}$$

The functions f^0 and f^1 stand for the corresponding functions in the state-dependent indirect utilities G^0 and G^1 of the investor. Note that the liquidation state is absorbing implying that it is not necessary to specify L^{10}, R^{10}, and η^{10}. It follows from (16) and $f^0(T) = f^1(T) = 1$, that (16) collapses into (15) if time t approaches the horizon T and $-L = 1 - R^{01}$.

To reach a closed-form solution for the ratio f^1/f^0, we firstly make the simplifying assumption that interest rates are constant. This assumption will be relaxed later on.

[18] We always drop the i-dependencies of processes like L if we are considering a portfolio problem with one bond only.

[19] See Appendix A.2 for a discussion of this assumption.

Proposition 6.1 *If interest rates are constant, we obtain*

$$f^0(t) = f^1(t)\Big[1 + (1 - e^{-a(T-t)})(\tfrac{\tilde{\lambda}^{01}}{a} - 1)\Big], \tag{17}$$

where $a = -\frac{\gamma}{1-\gamma}(\eta^{01} - 1)\lambda^{01} + \lambda^{01}$. *Therefore, for positive risk premiums, i.e.* $\eta^{01} \geq 1$, *we get* $f^0 \leq f^1$ *if* $\gamma < 0$ *and* $f^0 \geq f^1$ *if* $\gamma > 0$.

Remark. Note that $\tilde{\lambda}^{01}/a = (\eta^{01})^{\frac{\gamma}{\gamma-1}}/(1 - \frac{\gamma}{1-\gamma}(\eta^{01} - 1))$, i.e. this ratio depends on the risk aversion and the market price of risk only.

If the actual losses are assumed to be equal in both cases, i.e. $-L = 1 - R^{01}$, Proposition 6.1 allows us to compare the corporate bond demands (15) and (16): An investor who is less risk averse than logarithmic ($\gamma > 0$) will increase his exposure in corporate bonds, whereas an investor who is more risk averse than logarithmic will reduce his exposure. This is because the investor's utility function is bounded from below and unbounded from above for $\gamma > 0$, while the opposite is true for $\gamma < 0$.[20] Our situation is similar to the one in Longstaff (2001) because liquidation of a firm causes an extreme form of illiquidity, since after a liquidation has taken place investors cannot trade in any firm securities.

We now consider a combination of both effects, i.e. upon default the firm is either reorganized or liquidated with a certain probability. Putting legal details aside, we can think of the going concern state as formal bankruptcy under Chapter 11 and the liquidation state as formal bankruptcy under Chapter 7. Private workouts can either be subsumed under the continuation state or be modeled as a separate state. This depends on whether the recovery rates in private workouts are significantly different from the recovery rates in formal bankruptcy cases under Chapter 11.[21] For the moment, we restrict our considerations to the model depicted in Figure 3. The Markov chain modeling the default mechanism behind our portfolio problem starts in state 0. Then the chain can either jump into state 1, i.e. the firm defaults, but is reorganized and continues to operate, or the chain can jump into state 2 meaning that the firm is liquidated and disappears from the market. In contrast to the default regimes detailed in Figures 1 and 2, this leads to an incomplete market if the investor can only invest in one corporate bond. This is so because the Markov chain can jump into two states. Note that the same is true if the chain is currently in state 1. We assume $\lambda^{01} = \lambda^{10}$, $\lambda^{02} = \lambda^{12}$, $\eta^{01} = \eta^{10}$, and $\eta^{02} = \eta^{12}$. As will

[20] Samuelson (1991) shows that investors with $\gamma < 0$ prefer "smooth" distributions of their portfolio returns, whereas less risk averse investor dislike smoothing of their returns. This phenomenon is also documented in portfolio problems with event risk as analyzed by Liu, Longstaff, and Pan (2003).

[21] See, e.g., Bris, Welch, Zhang (2005) for a discussion of this point.

be shown in the following propositions, jumping back and forth between states 0 and 1 does not change the form of the indirect utility function. Nevertheless, although the indirect utility functions are the same, the investor's utility changes because he loses money upon default.

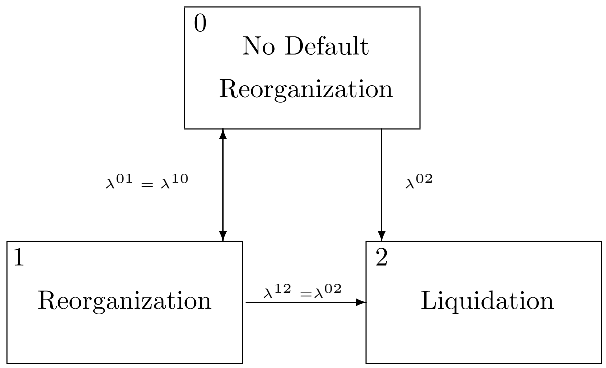

Figure 3: Default with Reorganization or Liquidation

We firstly assume that the investor can put his funds into two corporate zero-coupon bonds issued by the firm. Then the market is complete and the solution is given in the following proposition where, without loss of generality, it is assumed that the chain is currently in state 0. Note that $R_1^{01} = R_1^{10} = 0$ and $L_1^{02} = L_1^{12} = L_2^{02} = L_2^{12} = -1$. To shorten notation, $\tilde{L}_i^{02} = R_i^{02} + L_i^{02}$ denotes the actual loss upon a default with liquidation.

Proposition 6.2 (Complete Market) *The optimal portfolio proportions invested in the corporate bonds are given by*

$$\begin{aligned} \pi_1^{0,n*} &= \frac{\frac{L_1^{01}}{L_2^{01}} \frac{1-(\eta^{01})^{\frac{1}{\gamma-1}}}{-L_1^{01}} - \frac{\tilde{L}_1^{02}}{\tilde{L}_2^{02}} \frac{1-(\eta^{02})^{\frac{1}{\gamma-1}} f^2/f^0}{-\tilde{L}_1^{02}}}{\frac{L_1^{01}}{L_2^{01}} - \frac{\tilde{L}_1^{02}}{\tilde{L}_2^{02}}}, \\ \pi_2^{0,n*} &= \frac{1-(\eta^{01})^{\frac{1}{\gamma-1}}}{-L_2^{01}} - \frac{L_1^{01}}{L_2^{01}} \pi_1^{0,n*}, \end{aligned}$$

where f^j, $j = 0, 1, 2$, are independent of N. For constant interest rates, we obtain

$$f^0(t) = f^2(t)\Big[1 + (1 - e^{-a(T-t)})(\tfrac{\tilde{\lambda}^{02}}{a} - 1)\Big],$$

with $a = -\frac{\gamma}{1-\gamma}[(\eta^{01} - 1)\lambda^{01} + (\eta^{02} - 1)\lambda^{02}] + \lambda^{01} + \lambda^{02} - \tilde{\lambda}^{01}$.

Remarks. a) According to Appendix A.2, we can choose L_i^{01} and $\tilde{L}_i^{02}$ to be constants.

b) Obviously we need to require that $L_1^{01}/L_2^{01} \neq \tilde{L}_1^{02}/\tilde{L}_2^{02}$ ensuring that π_1^{0*} is real-valued. This condition implies that the actual losses of the bonds are linearly independent. If this requirement is violated, the market is incomplete because one cannot separately hedge against the different types of defaults.

Proposition 6.2 shows that the optimal demand for the first corporate bond equals a weighted average of the two extreme cases discussed previously. The corresponding weights are the actual loss ratios of the possible states, L_1^{01}/L_2^{01} and $-\tilde{L}_1^{02}/\tilde{L}_2^{02}$.

If the investor can invest in a single corporate bond only, then he faces an incomplete market. In general, an explicit solution is not available. However, under the simplifying assumption that intensities and actual losses are the same in both states, we are able to achieve an explicit solution for the optimal bond demand, which provides further insights. For notational simplicity, let $\bar{\eta} = \eta^{01} + \eta^{02}$. Note that $R^{01} = 0$ and $L^{02} = -1$.

Proposition 6.3 (Incomplete Market) *Assume $\lambda^0 \equiv \lambda^{01} = \lambda^{02}$ and $\tilde{L}^0 \equiv \tilde{L}^{01} = \tilde{L}^{02} =$ const. If only one corporate bond is traded, then the optimal proportion is given by*

$$\pi^{0,n*} = \frac{1 - \bar{\eta}^{\frac{1}{\gamma-1}} \left(1 + \left(f^2/f^0\right)^{1-\gamma}\right)^{\frac{1}{1-\gamma}}}{-\tilde{L}^0}, \tag{18}$$

where f^j, $j = 0, 1, 2$, is independent of N. Furthermore, setting $e^{-k} \equiv f^2/f^0$ the function k with $k(T) = 0$ satisfies the ODE

$$k_t = -\lambda^0 \bar{\eta}^{\frac{\gamma}{\gamma-1}} (1 + e^{-k})^{\frac{1}{1-\gamma}} - \lambda^0 \Big(\tfrac{\gamma}{1-\gamma} \bar{\eta} - \tfrac{2}{1-\gamma} \Big).$$

To understand the intuition behind the results of Proposition 6.3, firstly note that f^j captures the part of the indirect utility which is not wealth-dependent. Therefore, $(f^j/f^0)^{1-\gamma}$ is a measure for the relative loss of the non-wealth-dependent part of the indirect utility. We now consider the second terms in the numerator of the optimal portfolio demands (15), (16), and (18), which can be rewritten as follows:[22]

$$\left(\frac{1}{\eta^{01}}\right)^{\frac{1}{1-\gamma}}, \quad \left(\frac{(f^2/f^0)^{1-\gamma}}{\eta^{02}}\right)^{\frac{1}{1-\gamma}}, \quad \left(\frac{1 + (f^2/f^0)^{1-\gamma}}{\eta^{01} + \eta^{02}}\right)^{\frac{1}{1-\gamma}}.$$

The greater these terms are, the smaller are the corresponding corporate bond demands. Each of these terms reflects the trade-off between the investor's utility loss upon default and the

[22] For expositional clarity, we denote the default state in the second case by 2. However, we emphasize that the functions f^0 and f^2 in the second case usually differ from the corresponding functions in the third case. Therefore, these cases cannot be compared easily.

default risk premium. If the utility loss is relatively large compared to the risk premium, the demand is low. The incomplete market situation is obviously a mixture of the first two extreme cases. In general it is not obvious whether the third term is greater or smaller than the first one. However, in the special case $\eta = \eta^{01} = \eta^{02}$, the third term is smaller than the first one if $f^2 < f^0$.

Finally, we briefly discuss the impact of a stochastic short rate on our previous results. In the reorganization case, the demand for corporate bonds (15) may depend on the short rate, since the risk premium η^{01} and the loss rate L can be short rate-dependent. However, (15) does not explicitly depend on f^j. This is in contrast to the liquidation case where (16) involves the functions f^0 and f^1, each of which has a more complicated representation if interest rates are stochastic. However, if we keep the assumption that the risk premium and the intensity are deterministic, our above results still hold.

Proposition 6.4 (Stochastic Short Rate) *Given a stochastic short rate the results of Proposition 6.1 are still valid if the intensity λ^{01} and the risk premium η^{01} are deterministic.*

We remark that this result holds for all possible specifications of the short rate. Because of this result and to keep our exposition as simple as possible, in the following it is assumed that interest rates are deterministic.

6.2 Default vs. Insolvency

In the previous subsection, we have not distinguished between default and insolvency of a firm. Guo, Jarrow, and Zeng (2005) emphasize that this distinction is crucial for the pricing of defaultable assets, since it allows for a more realistic modeling of recovery rates. In our framework, one can incorporate this idea by adding an additional state. This is detailed in Figure 4. Here, we have modeled the insolvency state as absorbing meaning that the firm is liquidated upon transition into this state. Since only finitely many credit events can occur, the N-dependence of the parameters is redundant and we drop all N-dependencies in this subsection. Guo et al. (2005) model the recovery rates via the final payments a^1 and a^2, where $1 = a^0 > a^1 > a^2$. Alternatively, one can assume that the recovery payment in case of bankruptcy is paid upon transition, i.e. $a^{12} > 0$, but $a^2 = 0$.

If the firm is already in default, i.e. that the chain is currently in state 1, then the situation is identical to the one with liquidation of the previous section (see Figure 2). Therefore, the

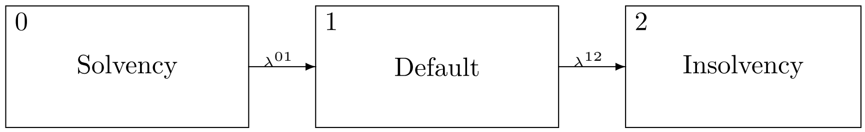

Figure 4: Guo-Jarrow-Zeng Recovery Model

optimal proportion invested in a corporate bond issued by the firm is given by (16):

$$\pi^{1*} = \frac{1 - (\eta^{12})^{\frac{1}{\gamma-1}} \frac{f^2}{f^1}}{1 - R^{12}} \tag{19}$$

with $L^{12} = -1$, since the firm is liquidated upon transition. The ratio f^2/f^1 equals the ratio f^1/f^0 of Proposition 6.1 with all indices shifted by one. If the firm is still solvent, i.e. the chain is currently in state 0, then the situation is different. The optimal proportion is a mixture of (15) and (16):

$$\pi^{0*} = \frac{1 - (\eta^{01})^{\frac{1}{\gamma-1}} \frac{f^1}{f^0}}{-L^{01}}.$$

The numerator corresponds to the numerator of (16), i.e. it contains a ratio of value functions, f^1/f^0, because in the current setting there is no way back in the solvency state, once a firm is in default. The denominator corresponds to the denominator of (15), since the firm is not liquidated upon default, i.e. $R^{01} = 0$ and $-L^{01} > 0$. It is worth noting that L^{01} is endogenously given, i.e. in general one cannot choose the value of L^{01} exogenously. The only exception is a situation where $-L^{01} = 1 - R^{12} = const$ (see Appendix A.2). To solve the problem completely, we finally need to determine the ratio f^1/f^0.

Proposition 6.5 (Guo-Jarrow-Zeng Recovery Model) *If interest rates are deterministic, we obtain*

$$\frac{f^1(t)}{f^0(t)} = \frac{\psi(t)}{\tilde{\lambda}^{01} \int_t^T e^{-a_0(s-t)} \psi(s)\, ds + e^{-a_0(T-t)}},$$

where $\psi(t) = f^1(t)/f^2(t) = 1 + (1 - e^{-a_1(T-t)})(\frac{\tilde{\lambda}^{12}}{a_1} - 1)$ *and* $a_j = -\frac{\gamma}{1-\gamma}(\eta^{jj+1} - 1)\lambda^{jj+1} + \lambda^{jj+1}$, $j = 0, 1$.

Remark. The integral can be calculated explicitly:

$$\int_t^T e^{-a_0(s-t)} \psi(s)\, ds = \frac{\tilde{\lambda}^{12}}{a_0 a_1}(1 - e^{-a_0(T-t)}) + \frac{1}{a_1 - a_0}(1 - \frac{\tilde{\lambda}^{12}}{a_1})(e^{-a_0(T-t)} - e^{-a_1(T-t)}),$$

where for $a_0 = a_1$ the second term on the right-hand side becomes $(1 - \frac{\tilde{\lambda}^{12}}{a_1}) e^{-a_1(T-t)}(T - t)$.

In contrast to our previous results, depending on the parameter choice the ratio f^1/f^0 can now be greater or smaller than one implying that the optimal bond demand can be greater or smaller than the demand (15) in the benchmark case. Hence, a more realistic model of the recovery rates changes the bond demands not only quantitatively, but also qualitatively. The reason for this result is that insolvency is not only "one credit event away" from solvency, but there is the default state as a "buffer state" between these two states. Consequently, if the default probability (or equivalently λ^{01}) is small and the time to maturity of the bond is short, then the proportion invested in corporate bonds may be greater than in the benchmark case even for investors that are more risk averse than logarithmic. For the same reason, the demand is not necessarily monotonous in the time to the investment horizon of the portfolio problem, which is in contrast to the optimal demand (19) when the firm has already defaulted on its debt. In this case, f^2/f^1 is monotonous in the time to the investment horizon of the problem. However, when the investment horizon approaches infinity both ratios behave similarly, and we obtain

$$\begin{aligned}\lim_{T\to\infty} f^1(t)/f^0(t) &= a_0/\tilde{\lambda}^{01} = (1 - \tfrac{\gamma}{1-\gamma}(\eta^{01}-1))(\eta^{01})^{\frac{\gamma}{1-\gamma}},\\ \lim_{T\to\infty} f^2(t)/f^1(t) &= a_1/\tilde{\lambda}^{12} = (1 - \tfrac{\gamma}{1-\gamma}(\eta^{12}-1))(\eta^{12})^{\frac{\gamma}{1-\gamma}}.\end{aligned}$$

Therefore, if both risk premiums are equal, the limits are equal. Additionally, when time approaches the investment horizon, both ratios converge to one.

6.3 Contagion and Competition Effects

In the previous subsections, we have analyzed several institutional factors affecting the portfolio decision of an investor who puts his funds into corporate bond exposed to various types of default risks. All these factors, however, were firm specific. On a portfolio level there is one big issue which is at the center of current research in the area of credit risk and these are so-called contagion effects: Firms' defaults are correlated due to industry or economy-wide factors like business relations or liquidity breakdowns. As a consequence, a firm's default probability shall depend on default events of other firms.[23] In general, two effects can occur: Either a firm's default probability goes up if another firm defaults on its debt or it goes down indicating positive or negative default correlation. The former phenomena is usually called contagion, whereas the latter is referred to as competition effect. In a recent paper, Jorrion and Zhang (2005) analyze Credit Default Swap data and find strong evidence of dominant contagion effects for Chapter 11 bankruptcies as well as competition effects for Chapter 7 bankruptcies.

[23] See, e.g., Jarrow and Yu (2001) or Yu (2005).

We consider a portfolio problem where the investor can allocate his funds between two corporate bonds of different firms. As in Subsection 6.1, one can then assume that upon default the firms are either reorganized or liquidated or that both reorganization and liquidation can occur with certain probabilities. We firstly concentrate our analysis on the case where a firm is liquidated upon default. Since only finitely many defaults can occur, we drop all N-dependencies.

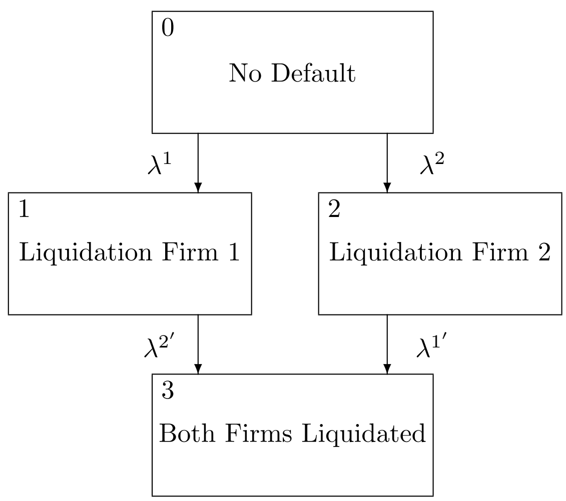

Figure 5: Contagion Effects with Liquidation

In Figure 5, the variables $\lambda^1 = \lambda^{01}$ and $\lambda^2 = \lambda^{02}$ denote the default intensities of firm 1 and 2 given that no default has occurred yet. Besides, $\lambda^{1'} = \lambda^{23}$ and $\lambda^{2'} = \lambda^{13}$ denote the default intensities given that the other firm has already defaulted on its debt. Under the physical measure contagion effects occur if $\lambda^{1'} > \lambda^1$ and $\lambda^{2'} > \lambda^2$, i.e. the default intensity of a firm increases if the other firm defaults implying positive default correlation, whereas competition effects occur if $\lambda^{1'} < \lambda^1$ and $\lambda^{2'} < \lambda^2$. Since competition effects can be interpreted as negative contagion effects, in the sequel we refer to both effects as contagion. To solve the portfolio problem for the model depicted in Figure 5, we firstly remark that, given one default has already occurred, i.e. the chain is currently in state 1 or 2, the problem reduces to the portfolio problem of Section 6.1 that is detailed in Figure 2. Therefore, the corresponding bond demands and indirect utility functions of the investor are given by (16) and (17). If the chain is currently in state 0, then, by (12), the optimal demands are given by

$$\pi_1^{0*} \tilde{L}_1^{01} + \pi_2^{0*} \tilde{L}_2^{01} = (\eta^{01})^{\frac{1}{\gamma-1}} f^1 / f^0 - 1, \tag{20}$$

$$\pi_1^{0*}\tilde{L}_1^{02} + \pi_2^{0*}\tilde{L}_2^{02} \quad = \quad (\eta^{02})^{\frac{1}{\gamma-1}} f^2/f^0 - 1.$$

The variables $\tilde{L}_1^{01}$ and $\tilde{L}_2^{02}$ model the actual losses if firm 1 or firm 2 default. Since it is assumed that the firm is liquidated upon default, both variables can be chosen exogenously. On the other hand, $\tilde{L}_1^{02}$ ($\tilde{L}_2^{01}$) models the price jump of the first (second) bond if firm 2 (firm 1) defaults. These jumps are triggered by the contagion effect and are *endogenously* determined, i.e. one cannot simply fix the jump sizes exogenously. In the special case when there are no contagion effects, i.e. $\lambda^1 = \lambda^{1'}$ and $\lambda^2 = \lambda^{2'}$, the price of one bond is not affected by a default of the other bond and thus $\tilde{L}_1^{02} = \tilde{L}_2^{01} = 0$. Appendix A.2 discusses this point in more detail. The following proposition summarizes the solution to the portfolio problem.

Proposition 6.6 (Contagion and Liquidation) *If interest rates are deterministic, we obtain*

$$\frac{f^0(t)}{f^3(t)} = \tilde{\lambda}^1 \int_t^T e^{-b_0(s-t)}\psi^2(s)\,ds + \tilde{\lambda}^2 \int_t^T e^{-b_0(s-t)}\psi^1(s)\,ds + e^{-b_0(T-t)}.$$

where $b_0 = -\frac{\gamma}{1-\gamma}[(\eta^1-1)\lambda^1 + (\eta^2-1)\lambda^2] + \lambda^1 + \lambda^2$ *and* $\psi^k(t) = 1 + (1 - e^{-b_k(T-t)})(\frac{\tilde{\lambda}^{k'}}{b_k} - 1)$ *with* $b_k = -\frac{\gamma}{1-\gamma}(\eta^{k'}-1)\lambda^{k'} + \lambda^{k'}$, $k = 1, 2$. *Furthermore,* $f^1/f^3 = \psi^2$ *and* $f^2/f^3 = \psi^1$. *The optimal demands are given as the solution to (20) if*

$$\tilde{L}_1^{01}\tilde{L}_2^{02} - \tilde{L}_2^{01}\tilde{L}_1^{02} \neq 0. \tag{21}$$

Remarks. a) Note that (21) is satisfied for a reasonable parametrization of the model. Even if it is violated, by the monotonicity of the bond price for a fixed state, it is violated only for a single $t \in [0, T]$, which is a null-set.

b) The integrals can be calculated explicitly:

$$\int_t^T e^{-b_0(s-t)}\psi^k(s)\,ds = \frac{\tilde{\lambda}^{k'}}{b_0 b_k}(1 - e^{-b_0(T-t)}) + \frac{1}{b_k - b_0}(1 - \frac{\tilde{\lambda}^{k'}}{b_k})(e^{-b_0(T-t)} - e^{-b_k(T-t)}).$$

In the previous model, by assumption, a firm is liquidated upon default. If the defaulted firm is reorganized and continues to operate, then we need to consider a model as depicted in Figure 6. In this model, contagion effects occur after the first default of one of the firms in which case the default intensities become $\lambda^{1'}$ and $\lambda^{2'}$. For simplicity, it is assumed that the intensities remain constant after the first default, but this assumption can be relaxed. Besides, the model can be generalized by introducing intensities depending on the number of defaults.

Since infinitely many defaults of both firms are possible, we cannot directly drop the N-dependence of the value functions. We solve the problem recursively and assume that one

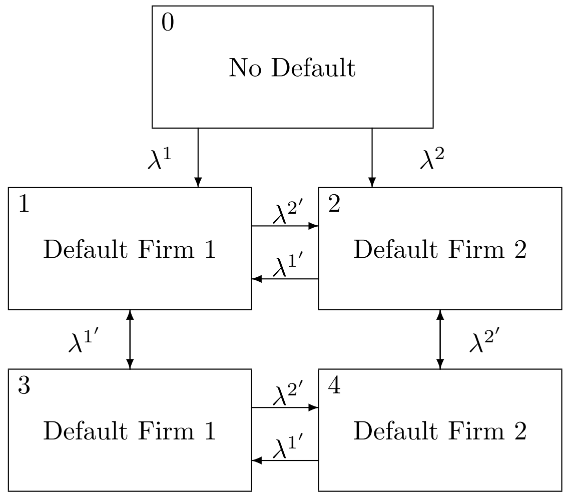

Figure 6: Contagion Effects with Reorganization

default has already occurred, i.e. the chain is currently in state 1, 2, 3, or 4. Besides, we assume that interest rates are constant. If the current state is 1, then (14) becomes

$$0 = f_t^{1,n} - \tilde{r}^{1,n} f^{1,n} + \tilde{\lambda}^{1'}(f^{3,n} - f^{1,n}) + \tilde{\lambda}^{2'}(f^{2,n} - f^{1,n}), \tag{22}$$

where $\tilde{r}^{1,n} = -\frac{\gamma}{1-\gamma}(r + (\eta^{13,n} - 1)\lambda^{1'} + (\eta^{12,n} - 1)\lambda^{2'}) + \lambda^{1'} + \lambda^{2'} - \tilde{\lambda}^{1'} - \tilde{\lambda}^{2'}$. Under the assumption that $\eta^{1'} \equiv \eta^{13,n} = \eta^{31,n} = \eta^{21,n} = \eta^{43,n}$ and $\eta^{2'} \equiv \eta^{12,n} = \eta^{24,n} = \eta^{42,n} = \eta^{34,n}$, i.e. the risk premiums are independent of the number of subsequent defaults if already one default has occurred, $\tilde{r}^{j,n}$, $j = 1, 2, 3, 4$, are identical and independent of n. We thus drop the N-dependencies and set $\tilde{r}' \equiv \tilde{r}^j$, $j = 1, 2, 3, 4$. As a consequence, the equations for $f^{j,n}$, $j = 2, 3, 4$, are identical to (22) and also independent of N. Therefore, we obtain $f^j(t) = e^{-\tilde{r}'(T-t)}$ for $j = 1, 2, 3, 4$. Finally, we consider the problem if the chain is currently in state 0. Since the chain cannot revisit this state and the value functions in all other states are independent of N, we can drop the N dependencies. Equation (14) becomes

$$\begin{aligned} 0 &= f_t^0 - \tilde{r}^0 f^0 + \tilde{\lambda}^1(f^1 - f^0) + \tilde{\lambda}^2(f^2 - f^0) \\ &= f_t^0 - \tilde{r}^0 f^0 + (\tilde{\lambda}^1 + \tilde{\lambda}^2)(f^1 - f^0) \end{aligned} \tag{23}$$

with $\tilde{r}^0 = -\frac{\gamma}{1-\gamma}(r + (\eta^1 - 1)\lambda^1 + (\eta^2 - 1)\lambda^2) + \lambda^1 + \lambda^2 - \tilde{\lambda}^1 - \tilde{\lambda}^2$, where $\eta^1 = \eta^{01}$ and $\eta^2 = \eta^{02}$. Note that we allow the risk premiums to change after the first default. The following proposition summarizes our results.

Proposition 6.7 (Contagion and Reorganization) *We assume that the bond of firm 1 (firm 2) loses a constant fraction $l^{1'}$ ($l^{2'}$) of the pre-default market value if a default occurs in one of the states 1-4. If interest rates are deterministic, then we obtain*

$$f^0(t) = \Big(1 + \frac{(\tilde{\lambda}^1 + \tilde{\lambda}^2)\big(e^{\tilde{r}^0+\tilde{\lambda}^1+\tilde{\lambda}^2-\tilde{r}')(T-t)} - 1\big)}{\tilde{r}^0 + \tilde{\lambda}^1 + \tilde{\lambda}^2 - \tilde{r}'}\Big) e^{-(\tilde{r}^0+\tilde{\lambda}^1+\tilde{\lambda}^2)(T-t)}$$

and $f^j(t) = e^{-\tilde{r}'(T-t)}$, $j = 1, 2, 3, 4$. In state 0, the optimal demands are given by (20) if (21) is satisfied. In all other states, the optimal demands are given by

$$\pi_1^{j,n*} = \frac{1 - (\eta^{1'})^{\frac{1}{\gamma-1}}}{l^{1'}}, \qquad \pi_2^{j,n*} = \frac{1 - (\eta^{2'})^{\frac{1}{\gamma-1}}}{l^{2'}}. \tag{24}$$

Remarks. a) In state 0, the loss rates are endogenously determined and time-dependent. Actually, the loss rates are also endogenously given in the other states, but it can be shown that the loss rates can be assumed to be constant.

b) Note that $\tilde{L}_i^{0k} = L_i^{0k}$ for the model with reorganization.

c) By assumption, after the first default there are no further contagion effects and thus the optimal demands (24) have the same structure as (15).

In both models (with liquidation or with reorganization), one can gain further insights about the initial optimal bond demands (20) if it is assumed that both bonds are characterized by the same set of parameters, i.e. $\lambda \equiv \lambda^1 = \lambda^2$, $\lambda' \equiv \lambda^{1'} = \lambda^{2'}$, $l^1 = l^2$, $l^{1'} = l^{2'}$, $\eta \equiv \eta^1 = \eta^2$, and $\eta' \equiv \eta^{1'} = \eta^{2'}$. This implies $f^1 = f^2$, $L_1^{01} = L_2^{02}$, and $L_1^{02} = L_2^{01}$, which leads to the following optimal demands

$$\pi_1^{0*} = \pi_2^{0*} = \frac{1 - \eta^{\frac{1}{\gamma-1}} \frac{f^1}{f^0}}{-(\tilde{L}_1^{01} + \tilde{L}_1^{02})}. \tag{25}$$

In general, the effect of f^1/f^0 on the optimal demands is of minor relevance compared to the effect of $\tilde{L}_1^{02}$. Specifically, this ratio goes to one if the time to maturity goes to zero. Therefore, (25) has a nice interpretation: If we have a positive contagion effect under the risk-neutral measure, i.e. $\eta\lambda = \lambda_Q < \lambda'_Q = \eta'\lambda'$, then $\tilde{L}_1^{02} < 0$ and the demand decreases, whereas the opposite is true for a negative contagion effect (competition effect). Figure 7 depicts the effect of contagion on the optimal portfolio demands (25). Considering an investor with an investment horizon of 5 years and $\gamma = -5$, we set $\lambda = 0.025$, $l^1 = l^2 = l^{1'} = l^{2'} = 0.5$, $r = 0.04$ and vary λ'. The maturities of the bonds are assumed to be 10 years. The risk premiums are $\eta = \eta' = 1.5$.[24] It can be seen that positive contagion effects ($\lambda' > 0.025$) reduce the bond demands, whereas the opposite is true for negative contagion effects ($\lambda' < 0.025$).

[24] This value is in line with the findings of Berndt et al. (2004).

[INSERT FIGURE 7 ABOUT HERE]

Comparing these results with our findings for a single bond (see Proposition 6.1), we conclude the following: If λ' is small, then the demand is still bigger for a model with reorganization. On the other hand, if λ' increases, then there is an interval where the result changes. If λ' is large enough, then the demand in the model with reorganization is again greater than in the model with liquidation. The reason is that high contagion causes a high frequency of defaults in such a model and thus such a situation is disadvantageous for an investor compared to a model with liquidation where only one default can occur. Consequently, in a model with reorganization the initial demand for corporate bonds increases eventually if the contagion effect is large enough.

7 Conclusion

This paper analyzes portfolio choice problems with defaultable bonds in great detail. Instead of interpreting a default as a "0-1-event", it develops a sophisticated framework covering various realistic features of defaultable bonds. We distinguish between Chapter 7 and Chapter 11 bankruptcies as well as between default and bankruptcy. Additionally, we explicitly model contagion effects, which are of particular interest, since they reduce the investor's ability to diversify his portfolio. Specifically, we are able to quantify the impact of contagion on portfolio choices. All these aspects can significantly change the demand for corporate securities. It is worth noting that due to its generality our approach can be applied to other situations as well. For instance, one can analyze how portfolio decisions are affected if we distinguish between several rating classes for bonds. Besides, one can solve a portfolio problem with exogenous (re)issuing of bonds.

A Appendix

A.1 Proofs

Proof of Proposition 3.1. By definition of a conditional expected value, we obtain

$$B(t) = \mathrm{E}_Q\Big[\int_{(t,T]} e^{-\int_t^s r(u)\,du} dA(s)\Big| Y(t), Z(t), N(t)\Big] = \sum_j B^{j,n}(t)\mathbf{1}_{\{Z(t)=j,N(t)=n\}},$$

where $B^{j,n}(t) \equiv \mathrm{E}_Q[\int_{(t,T]} e^{-\int_t^s r(u)\,du} dA(s)|Y(t), Z(t) = j, N(t) = n]$. Due to the Markovian property of (Y, Z, N), there exist functions $\mathcal{B}^{j,n}$, $j \in \mathcal{J}$ and $n \in I\!N_0^{J+1}$, such that $B^{j,n}(t) = \mathcal{B}^{j,n}(t, Y(t))$ implying $B(t) = \mathcal{B}^{Z(t),N(t)}(t, Y(t))$. If the functions $\mathcal{B}^{j,n}$, $j \in \mathcal{J}$ and $n \in I\!N_0^{J+1}$, are in $C^{1,2}$, we can apply Ito's product rule:[25]

$$\begin{aligned}
dB(t) &= \sum_j \sum_n \mathbf{1}_{\{Z(t)=j,N(t)=n\}} d\mathcal{B}^{j,n}(t, Y(t)) + \sum_j \sum_n \mathcal{B}^{j,n}(t, Y(t)) d\mathbf{1}_{\{Z(t)=j,N(t)=n\}} \\
&= \sum_j \sum_n \mathbf{1}_{\{Z(t)=j,N(t)=n\}} \Big\{ \big(\mathcal{B}_t^{j,n}(t, Y(t)) + \alpha_Q(t)\mathcal{B}_y^{j,n}(t, Y(t)) \\
&\qquad +0.5\mathrm{tr}[\beta(t)^T \mathcal{B}_{yy}^{j,n}(t, Y(t))\beta(t)]\big)dt + \mathcal{B}_y^{j,n}(t, Y(t))\beta(t)dW_Q(t)\Big\} \\
&\quad + \sum_j \sum_n \mathcal{B}^{j,n}(t, Y(t)) d\mathbf{1}_{(Z(t)=j,N(t)=n)}
\end{aligned}$$

Here,

$$\begin{aligned}
&\sum_j \sum_n \mathcal{B}^{j,n}(t, Y(t)) d\mathbf{1}_{\{Z(t)=j,N(t)=n\}} \\
=\;& \sum_j \sum_n \mathcal{B}^{j,n}(t, Y(t))(\mathbf{1}_{\{N(t-)=n-1^j\}} dN^j(t) - \sum_{k:k\neq j} \mathbf{1}_{(Z(t-)=j,N(t-)=n)} dN^k(t)) \\
=\;& \sum_j \sum_n \mathcal{B}^{j,n}(t, Y(t))\mathbf{1}_{(N(t-)=n-1^j)} dN^j(t) - \sum_j \sum_n \sum_{k:k\neq j} \mathcal{B}^{j,n}(t, Y(t))\mathbf{1}_{(Z(t-)=j,N(t-)=n)} dN^k(t) \\
=\;& \sum_k \sum_n \mathcal{B}^{k,n}(t, Y(t))\mathbf{1}_{(N(t-)=n-1^k)} dN^k(t) - \sum_k \sum_{j:j\neq k} \sum_n \mathcal{B}^{j,n}(t, Y(t))\mathbf{1}_{(Z(t-)=j,N(t-)=n)} dN^k(t) \\
=\;& \sum_k \mathcal{B}^{k,N(t-)+1^k}(t, Y(t)) dN^k(t) - \sum_k \mathcal{B}^{Z(t-),N(t-)}(t, Y(t)) dN^k(t) \\
=\;& \sum_k (\mathcal{B}^{k,N(t-)+1^k}(t, Y(t)) - \mathcal{B}^{Z(t-),N(t-)}(t, Y(t))) dN^k(t)
\end{aligned}$$

□

Proof of Proposition 3.1. By definition, $\mathcal{Y}(t) \equiv \mathrm{E}_Q\Big[\int_0^T e^{-\int_0^s r(u)\,du} dA(s)\Big|\mathcal{F}_t\Big]$ is a Q-

[25]Note that we can drop the minuses in $\mathbf{1}_{\{Z(t-)=j,N(t-)=n\}}$ because one can change the integrand of a Lebesgue and an Ito integral on a null set. Besides, we use the fact that $B^{j,n}$ is continuous in t and y.

martingale which has the representation

$$\mathcal{Y}(t) = \int_0^t e^{-\int_0^s r(u)\,du} dA(s) + e^{-\int_0^t r(u)\,du} B(t)$$

and is called the normalized gain process. To simplify notations, in the following we drop the dependencies. Due to Proposition 3.1, we obtain

$$\begin{aligned}
d\mathcal{Y} &= e^{-\int_0^{\cdot} r(u)\,du}\Big[dA + dB - Brdt\Big] \\
&= e^{-\int_0^{\cdot} r(u)\,du}\Big[\sum_{k\neq Z^-} a^{Z^-k,N^-} dN^k + a_T^{Z,N} d\varepsilon_T + \mathcal{B}_t^{Z,N} dt + \mathcal{B}_y^{Z,N}\alpha_Q dt \\
&\quad + \mathcal{B}_y^{Z,N}\beta dW_Q + 0.5\mathrm{tr}[\beta^T \mathcal{B}_{yy}^{Z,N}\beta]dt + \sum_{k\neq Z^-}(\mathcal{B}^{k,N^-+1^k} - \mathcal{B}^{Z^-,N^-})dN^k - \mathcal{B}^Z rdt\Big] \\
&= e^{-\int_0^{\cdot} r(u)\,du}\Big[a_T^{Z,N} d\varepsilon_T + \mathcal{B}_t^{Z,N} dt + \mathcal{B}_y^{Z,N}\alpha_Q dt + 0.5\mathrm{tr}[\beta^T \mathcal{B}_{yy}^{Z,N}\beta]dt - \mathcal{B}^{Z,N} rdt \\
&\quad + \sum_{k\neq Z^-}(a^{Z^-k,N^-} + \mathcal{B}^{k,N^-+1^k} - \mathcal{B}^{Z^-,N^-})\lambda_Q^{Z^-k,N^-} dt \\
&\quad + \sum_{k\neq Z^-}(a^{Z^-k,N^-} + \mathcal{B}^{k,N^-+1^k} - \mathcal{B}^{Z^-,N^-})dM_Q^{k,N} + \mathcal{B}_y^{Z,N}\beta dW_Q\Big],
\end{aligned}$$

where $dM_Q^{k,N} = dN^{kN} - \lambda_Q^{Z^-k,N^-} dt$. Since $\mathcal{Y}$ is a Q-martingale, the dt-terms need to vanish implying (4). □

Proof of Theorem 4.1. The Hamilton-Jacobi-Bellman equation (HJB) for the state-dependent candidate G^j of the investor's indirect utility is given by

$$0 = \sup_{\pi,D,c}\Big\{A^{\pi,D,c}G^{j,n} + \tfrac{1}{\gamma}\psi^{1-\gamma}c^\gamma\Big\} \tag{26}$$

with terminal condition $G^{j,n}(T,x,r) = \frac{1}{\gamma}\psi(T)^{1-\gamma}x^\gamma$ and

$$\begin{aligned}
A^{\pi,D,c}G^{j,n} &= G_t^{j,n} + \{x(r - \textstyle\sum_k \eta^{jk,n}\lambda^{jk,n}\sum_{i=1}^I \pi_i \tilde{L}_i^{jk,n} - \beta\eta_r D) - c\}G_x^{j,n} \\
&\quad + \alpha G_r^{j,n} + 0.5x^2\beta^2 D^2 G_{xx}^{j,n} - x\beta^2 D G_{xr}^{j,n} + 0.5\beta^2 G_{rr}^{j,n} \\
&\quad + \textstyle\sum_{k\neq j}\lambda^{jk,n}\Big\{G^{k,n+1^k}(t, x(1+\sum_{i=1}^I \pi_i\tilde{L}_i^{jk,n}), r) - G^{j,n}(t,x,r)\Big\}.
\end{aligned}$$

We conjecture $G^{j,n}(t,x,r) = \frac{1}{\gamma}x^\gamma f^{j,n}(t,r)^q$, where q is a constant and $f^{j,n}$ is a function with $f^{j,n}(T,r) = 1$. This leads to

$$\begin{aligned}
0 &= \sup_{\pi,D,c}\Big\{\tfrac{1}{\gamma}\psi^{1-\gamma}(c/x)^\gamma (f^{j,n})^{1-q} - (c/x)f^{j,n} + \tfrac{q}{\gamma}f_t^{j,n} \\
&\quad + (r - \textstyle\sum_k \eta^{jk,n}\lambda^{jk,n}\sum_{i=1}^I \pi_i\tilde{L}_i^{jk,n} - \beta\eta_r D)f^{j,n} + \tfrac{q}{\gamma}\alpha f_r^{j,n} \\
&\quad + 0.5(\gamma-1)\beta^2 D^2 f^{j,n} - q\beta^2 D f_r^{j,n} + 0.5\tfrac{q(q-1)}{\gamma}\beta^2\tfrac{(f_r^{j,n})^2}{f^{j,n}} + 0.5\tfrac{q}{\gamma}\beta^2 f_{rr}^{j,n} \\
&\quad + \tfrac{1}{\gamma}f^{j,n}\textstyle\sum_{k\neq j}\lambda^{jk,n}\Big[(1+\sum_{i=1}^I \pi_i\tilde{L}_i^{jk,n})^\gamma\Big(\tfrac{f^{k,n+1^k}}{f^{j,n}}\Big)^q - 1\Big]\Big\}.
\end{aligned} \tag{27}$$

Therefore, the first-order conditions are given by (6)-(8). Substituting these conditions into (27) and choosing $q = 1 - \gamma$ gives (9). $\square$

Lemma A.1 *The ODE given by* $0 = g_t + (\frac{\gamma}{1-\gamma} r - a) g + bh$ *with* $g(T) = 1$ *and constants* a, b *as well as* $h(t) = e^{\frac{\gamma}{1-\gamma}(T-t)r}$ *possesses the unique solution*

$$g(t) = h(t)\Big[1 + (1 - e^{-a(T-t)})(\tfrac{b}{a} - 1)\Big].$$

Proof of Proposition 6.1. Once a default has occurred, the investor can only put his funds into the default-free bond implying $G^1(t,x) = \frac{1}{\gamma} x^\gamma e^{\gamma(T-t)r}$, i.e. $f^1(t) = e^{\frac{\gamma}{1-\gamma}(T-t)r}$. According to (14), f^0 satisfies $0 = f^0_t - \tilde{r}^0 f^0 + \tilde{\lambda}^{01}(f^1 - f^0)$, with $\tilde{r}^0 = -\frac{\gamma}{1-\gamma}[r + (\eta^{01} - 1)\lambda^{01}] + \lambda^{01} - \tilde{\lambda}^{01}$ and $f^0(T) = 1$, which, by Lemma A.1, has the solution stated in the proposition.

To prove the second part, assume $a > 0$. This is the case for $\gamma < 0$ and some $\gamma > 0$. Since $\tilde{\lambda}^{01}/a = (\eta^{01})^{\frac{\gamma}{\gamma-1}}/(1 - \frac{\gamma}{1-\gamma}(\eta^{01} - 1))$, we show that the function $h(y) = y^b/(1 + b(y-1))$, $y \geq 1$ is smaller than one if $b \in (0,1)$ and greater than one if $b \in (-\infty, 0)$. The first (second) statement is equivalent to $y^b \leq (\geq) 1 + b(y-1)$, which holds because $\partial y^b/\partial y = by^{b-1} \leq (\geq) b$ for $b \in (0,1)$ (for $b \in (-\infty, 0)$). Finally, if $a < 0$ which is only possible if γ is close to one, then $\tilde{\lambda}^{01}/a < 0$ and the claim follows. $\square$

Proof of Proposition 6.2. An inspection of the equations (13) shows that f^0, f^1, and f^2 do not depend on N and that f^0 and f^1 are identical. By (12),

$$\begin{aligned} 1 + \pi_1^{0,n*} L_1^{01} + \pi_2^{0,n*} L_2^{01} &= (\eta^{01})^{\frac{1}{\gamma-1}}, \\ 1 + \pi_1^{0,n*} \tilde{L}_1^{02} + \pi_2^{0,n*} \tilde{L}_2^{02} &= (\eta^{02})^{\frac{1}{\gamma-1}} \frac{f^2}{f^0}, \end{aligned} \tag{28}$$

where $f^2(t) = e^{\frac{\gamma}{1-\gamma}(T-t)r}$ and f^0 satisfies $0 = f^0_t - \tilde{r}^0 f^0 + \tilde{\lambda}^{02}(f^2 - f^0)$ with $\tilde{r}^0 = -\frac{\gamma}{1-\gamma}[r + (\eta^{01} - 1)\lambda^{01} + (\eta^{02} - 1)\lambda^{02}] + \lambda^{01} + \lambda^{02} - \tilde{\lambda}^{01} - \tilde{\lambda}^{02}$ and $f^0(T) = 1$. Solving (28) for $\pi_1^{0,n*}$ and $\pi_2^{0,n*}$ gives the stated result. Furthermore, by Lemma A.1, the ODE for f^0 has the solution stated in the proposition. $\square$

Proof of Proposition 6.3. The representation of $\pi^{0,n*}$ follows from the first-order condition (8). Substituting $\pi^{0,n*} L^0 = \bar{\eta}^{\frac{1}{\gamma-1}} (1 + (\frac{f^{2,n+1^2}}{f^{0,n}})^{1-\gamma})^{\frac{1}{1-\gamma}}$ into (10) leads to the following ODE for f^0:

$$0 = f_t^{0,n} + \big[\tfrac{\gamma}{1-\gamma}(r + \lambda^0 \bar{\eta}) - \tfrac{2}{1-\gamma}\lambda^0\big] f^{0,n} + \lambda^0 \bar{\eta}^{\frac{\gamma}{\gamma-1}} (1 + (\tfrac{f^{2,n+1^2}}{f^{0,n}})^{1-\gamma})^{\frac{1}{1-\gamma}} f^{0,n}.$$

Consequently, $f^{0,n}$ is independent of N, since $f^{2,n}(t) = e^{\frac{\gamma}{1-\gamma} r(T-t)}$ is independent of N. Defining $h = f^2/f^0$, we obtain $0 = h_t + \left[\frac{\gamma}{1-\gamma}\lambda^0\bar{\eta} - \frac{2}{1-\gamma}\lambda^0\right]h + \lambda^0\bar{\eta}^{\frac{\gamma}{\gamma-1}}\left(1 + h^{\gamma-1}\right)^{\frac{1}{1-\gamma}}h$. Setting $h = e^k$ we arrive at the PDE for k. □

Proof of Proposition 6.4. For notational convenience, assume that the intensity and the risk premium are constant. As before, we drop the N-dependence in the notation, as only one default can occur. We need to solve the PDE

$$0 = f_t^0 - (\tilde{r}^0 + \tilde{\lambda}^{01})f^0 + \tilde{\alpha} f_r^0 + 0.5\beta^2 f_{rr}^0 + \tilde{\lambda}^{01} f^1, \tag{29}$$

where f^1 is given by $f^1(t,r) = \tilde{\mathrm{E}}_1^{t,r}\left[e^{-\int_t^T \tilde{r}^{Z(s)}(s)\,ds}\right] = \tilde{\mathrm{E}}^{t,r}\left[e^{-\int_t^T \tilde{r}^1(s)\,ds}\right]$, because state 1 is absorbing and $\tilde{r}^0 = \tilde{r}^1 + c$ with $c = -\frac{\gamma}{1-\gamma}(\eta^{01}-1)\lambda^{01} + \lambda^{01} - \tilde{\lambda}^{01}$. The solution to (29) has the following Feynman-Kac representation

$$f^0(t,r) = \int_t^T \tilde{\lambda}^{01}\tilde{\mathrm{E}}^{t,r}\left[f^1(s,r(s))e^{-\int_t^s \tilde{r}^0(u)+\tilde{\lambda}^{01}\,du}\right]ds + \tilde{\mathrm{E}}^{t,r}\left[e^{-\int_t^T \tilde{r}^0(u)+\tilde{\lambda}^{01}\,du}\right].$$

Since $\tilde{\mathrm{E}}^{t,r}\left[e^{-\int_t^T \tilde{r}^0(u)+\tilde{\lambda}^{01}\,du}\right] = f^1(t,r)\,e^{-a(T-t)}$ and

$$\begin{aligned}
\tilde{\mathrm{E}}^{t,r}\left[f^1(s,r(s))e^{-\int_t^s \tilde{r}^0(u)+\tilde{\lambda}^{01}\,du}\right] &= \tilde{\mathrm{E}}^{t,r}\left[\tilde{\mathrm{E}}\left[e^{-\int_s^T \tilde{r}^1(u)\,du}\middle| r(s)\right]e^{-\int_t^s \tilde{r}^1(u)+a\,du}\right] \\
&= \tilde{\mathrm{E}}^{t,r}\left[e^{-\int_t^T \tilde{r}^1(u)\,du}\right]e^{-a(s-t)} = f^1(t,r)\,e^{-a(s-t)},
\end{aligned}$$

we obtain $f^0(t,r) = f^1(t,r)\left\{\int_t^T \tilde{\lambda}^{01}e^{-a(s-t)}\,ds + e^{-a(T-t)}\right\}$, which gives (17). □

Proof of Proposition 6.5. According to (14), f^0 satisfies $0 = f_t^0 - \tilde{r}^0 f^0 + \tilde{\lambda}^{01}(f^1 - f^0)$, with $f^0(T) = 1$, $\tilde{r}^0 = -\frac{\gamma}{1-\gamma}[r + (\eta^{01}-1)\lambda^{01}] + \lambda^{01} - \tilde{\lambda}^{01}$, $f^1(t) = f^2(t)\psi(t)$, and $f^2(t) = e^{\frac{\gamma}{1-\gamma}(T-t)r}$. The solution to the ODE is given by

$$\begin{aligned}
f^0(t) &= \int_t^T \tilde{\lambda}^{01} f^1(s)e^{-(\tilde{r}^0+\tilde{\lambda}^{01})(s-t)}\,ds + e^{-(\tilde{r}^0+\tilde{\lambda}^{01})(T-t)} \\
&= \int_t^T \tilde{\lambda}^{01} f^2(s)\psi(s)e^{(\frac{\gamma}{1-\gamma}r - a_0)(s-t)}\,ds + f^2(t)e^{-a_0(T-t)} \\
&= f^2(t)\left\{\tilde{\lambda}^{01}\int_t^T \psi(s)e^{-a_0(s-t)}\,ds + e^{-a_0(T-t)}\right\}
\end{aligned}$$

which gives the desired result. □

Proof of Proposition 6.6. Due to (14), f^0 satisfies $0 = f_t^0 - \tilde{r}^0 f^0 + \tilde{\lambda}^1(f^1 - f^0) + \tilde{\lambda}^2(f^2 - f^0)$, where $f^1/f^3 = \psi^2$ and $f^2/f^3 = \psi^1$. Integrating this equation gives the representation for f^0. The representations of the investor's optimal bond demands follow from (12). □

Proof of Proposition 6.7. The solution to (23) has the representation

$$f^0(t) = \int_t^T (\tilde{\lambda}^1 + \tilde{\lambda}^2) f^1(s) e^{-(\tilde{r}^0+\tilde{\lambda}^1+\tilde{\lambda}^2)(s-t)}\, ds + e^{-(\tilde{r}^0+\tilde{\lambda}^1+\tilde{\lambda}^2)(T-t)}.$$

Solving gives the desired result. □

A.2 Bond Prices and Loss Rates

The relative payments upon transitions, $R^{jk,n}$ are *exogenously* determined, since the payment upon transition $a^{jk,n}$ is exogenously given. Choosing $a^{jk,n} = l^{jk,n} B^{j,n}$ with $l^{jk,n}$ depending on t and $Y(t)$ yields $R^{jk,n} = l^{jk,n}$. One can thus mimic the so-called recovery of market value assumption by Duffie and Singleton (1999). On the contrary, the relative price jumps upon a default event, $L^{jk,n}$, are *endogenously* determined by the pricing equations (4) and cannot be arbitrarily chosen. The goal of this part of the Appendix is to show that, under specific assumptions, $L^{jk,n}$ becomes a constant. By setting $l^{jk,n}$ equal to this constant, one thus obtains a model where the loss rates upon a liquidation and upon a reorganization are the same.

We consider the model depicted in Figure 1. The counting process N consists of two components, N^0 and N^1. Set $a^{j,n}(T_B, y) = (1-l)^{n_0+n_1}$ with a constant $l \in (0,1)$ and T_B denotes the maturity of the bond. Since $\lambda = \lambda^{01,n} = \lambda^{10,m}$ and $\eta = \eta^{01,n} = \eta^{10,m}$ are independent of Z and N, $\lambda_Q = \eta\lambda$ is independent of Z and N as well. The pricing equations (4) become

$$\mathcal{B}_t^{j,n} = r\mathcal{B}^{j,n} - \alpha_Q \mathcal{B}_y^{j,n} - 0.5\text{tr}\left[\beta^T \mathcal{B}_{yy}^{j,n} \beta\right] - \lambda_Q \left(\mathcal{B}^{k,n+1^k} - \mathcal{B}^{j,n}\right)$$

with $j,\, k \in \{0,1\}$, $j \neq k$, and $n = (n_0, n_1) \in I\!N_0^2$. The solution reads[26]

$$\mathcal{B}^{j,n}(t) = (1-l)^{n_0+n_1} \mathrm{E}_Q^t\left[e^{-\int_t^{T_B} r(s) + l\lambda_Q(s)\, ds}\right]$$

implying $\mathcal{B}^{k,n+1^k} = (1-l)\mathcal{B}^{j,n}$ and thus $L^{jk,n} = -l$. Although $L^{jk,n}$ is actually endogenously determined, in this setting fixing l is equivalent to fixing $-L^{jk,n}$. For the model depicted in Figure 3, the result is similar. The problem becomes more involved for the model depicted in Figure 4. The pricing equations are given by[27]

$$\begin{aligned}
\mathcal{B}_t^1 &= r\mathcal{B}^1 - \alpha_Q \mathcal{B}_y^1 - 0.5\text{tr}\left[\beta^T \mathcal{B}_{yy}^1 \beta\right] - \lambda_Q^{12}\left(a^{12} - \mathcal{B}^1\right), \\
\mathcal{B}_t^0 &= r\mathcal{B}^0 - \alpha_Q \mathcal{B}_y^0 - 0.5\text{tr}\left[\beta^T \mathcal{B}_{yy}^0 \beta\right] - \lambda_Q^{01}\left(\mathcal{B}^1 - \mathcal{B}^0\right).
\end{aligned}$$

We would like to be in a situation where there exist constants l^0 and $l^1 \in (0,1)$ such that $\mathcal{B}^1 = (1-l^0)\mathcal{B}^0$ and $\mathcal{B}^2 = (1-l^1)\mathcal{B}^1$. The second requirement is satisfied by setting $a^{12} =$

[26] We use the short-hand notation $\mathrm{E}_Q^t[\cdot]$ for $\mathrm{E}_Q[\cdot|\mathcal{F}_t]$.

[27] As in the core part of the text, we drop the N-dependencies.

$(1-l^1)\mathcal{B}^1$, whereas the relation between $\mathcal{B}^0$ and $\mathcal{B}^1$ is endogenously determined. By definition, the firm is not in financial distress in state 0 implying that $a^0(T_B)=1$. We now try to achieve $\mathcal{B}^1=(1-l^0)\mathcal{B}^0$. Since this condition also needs to hold when time approaches maturity of the bond, we set $a^1(T_B)=(1-l^0)$. Therefore,

$$\begin{aligned} B^1(t) &= (1-l^0)\mathrm{E}_Q^t\left[e^{-\int_t^{T_B} r(u)+l^1\lambda_Q^{12}(u)\,du}\right] \\ B^0(t) &= \mathrm{E}_Q^t\left[e^{-\int_t^{T_B} r(u)+\lambda_Q^{01}(u)\,du}\int_t^{T_B}\lambda_Q^{01}(s)(1-l^0)e^{-\int_s^{T}\lambda_Q^{01}(u)-l^1\lambda_Q^{12}(u)\,du}\,ds\right] \\ &\quad +\mathrm{E}_Q^t\left[e^{-\int_t^{T_B} r(u)+\lambda_Q^{01}(u)\,du}\right]. \end{aligned}$$

For the special case $l^0\lambda_Q^{01}=l^1\lambda_Q^{12}$, this expression simplifies into

$$B^0(t)=\mathrm{E}_Q^t\left[e^{-\int_t^{T_B} r(u)+\lambda_Q^{01}(u)\,du}\left(e^{\int_t^{T_B}\lambda_Q^{01}(u)-l^1\lambda_Q^{12}(u)\,du}-1\right)\right]+\mathrm{E}_Q^t\left[e^{-\int_t^{T_B} r(u)+\lambda_Q^{01}(u)\,du}\right]=\frac{B^1(t)}{1-l^0},$$

which gives the desired result. In general, however, this is not true.

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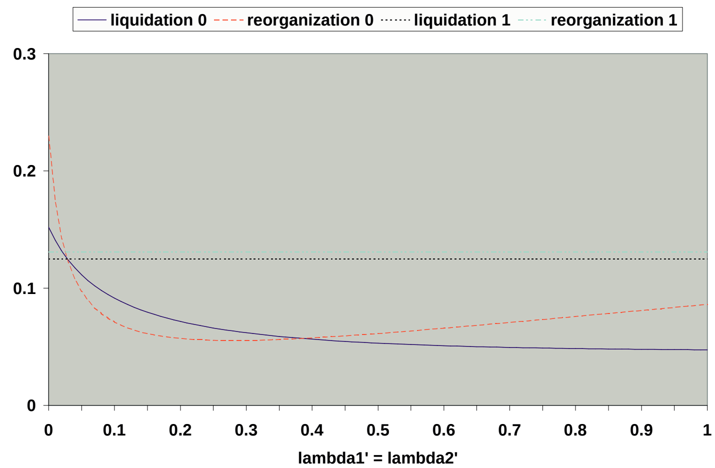

Figure 7: Optimal Demands in State 0 and State 1