AN ABC OF PORTFOLIO CHOICE: ASSET ALLOCATION WITH BANKRUPTCY AND CONTAGION

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The Problem

- Investors like to invest in all asset clases including corporate bonds.
- Especially in times of low interest rates corporate bonds are sometimes considered as a good investment alternative.
- However, there has been not much work on portfolio problems with default risk.
- We try to fill this gap by considering such a problem with contagion and bankruptcy.

Example for Contagion Effects

- On October 8, 2005 the auto parts maker Delphi Corp. filed for Chapter 11 bankruptcy protection.
- On December 12, 2005 the rating agency S&P cut GM's corporate credit rating to B, five steps below investment grade.
- On March 3, 2006 the auto parts maker Dana Corp. filed for bankruptcy protection defaulting on $2.5 billion of debt.

Related Work

Portfolio Optimization in Firm Value Models

- Korn, K. (2003)
- K., Steffensen (2004)

Portfolio Optimization in a Standard Reduced-from Model

- K., Steffensen (2005)

Portfolio Optimization using the JLY-Assumption

- Jarrow, Lando, Yu (2005): Conditionally Diversifiable Default Risk
- Hou (2003)

Pricing Defaultable Claims using Markov Chains

- K., Steffensen (2006)
- Schonbucher (2006)

JLY-Assumption: Conditionally Diversifiable Default Risk

Roughly speaking, JLY prove the following: If there are

- **infinitely** many firms and
- individual default-timing risk can be **diversified**

then default-timing risk premiums are zero.

Imposing this assumption in a portfolio problem means that one gets rid off all jump processes.

$\longrightarrow$ Such a problem is similar to a problem with stochastic interest rates.

(See Sørensen (1998), Korn, K. (2001), Munk, Sørensen (2004))

Is the JLY-assumption Supported by Empirical Evidence?

- After accounting for tax and liquidity, Driessen (2003) reports an average default-timing risk premium across his data of 1.89.

- This result is roughly in line with the results of Berndt, Douglas, Duffie, Ferguson, Schranz (2004).

- Yu (2002) demonstrates how a positive default-timing risk premium can be used to resolve some empirical puzzles in the credit risk literature.

These studies do not support the JLY-assumption.

$\Longrightarrow$ Default-timing risk should not be ignored in portfolio problems.

Agenda

1. **Portfolio Problem**
   - Merton's Results
   - Markov Chain Framework
2. **Solution**
   - First-order Conditions
   - Representation of the Investor's Indirect Utility
3. **Applications**
   - Reorganization vs. Liquidation
   - Contagion

Merton's Results

An Investor allocates his wealth between a stock and a risky bond:

$$\begin{aligned} dS(t) &= S(t)[\alpha dt + \sigma dW(t)], \quad S(0) = s_0, \\ dB(t) &= B(t)[r dt - dN(t)], \quad B(0) = b_0. \end{aligned}$$

r is the short rate and N is a Poisson process with intensity $\lambda > 0$.

Implication: Recovery rate is zero for the risky bond.

Merton's Results

Merton's HJB for the indirect utility G (π: stock proportion, x: wealth):

$$
\begin{aligned}
0 &= \sup_{\pi} \{G_t(t,x) + (r + (\alpha - r)\pi)G_x(t,x) + 0.5x^2\pi^2x^2G_{xx}(t,x) \\
&\quad + \lambda[G(t,\pi x) - G(t,x)]\}, \qquad G(T,x) = \tfrac{1}{\gamma}x^\gamma.
\end{aligned}
$$

Problem: In case of a default, the **bond is wiped out**, but Merton implicitely assumes that the investor can still invest in bonds.

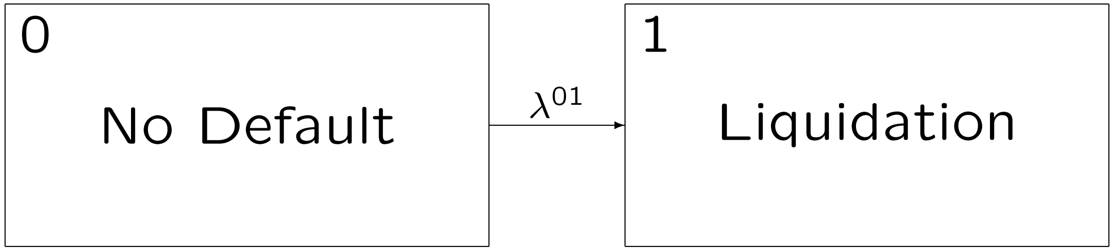
$$
\begin{aligned}
0 &= \sup_{\pi} \{G_t(t,x) + (r + (\alpha - r)\pi)G_x(t,x) + 0.5x^2\pi^2x^2G_{xx}(t,x) \\
&\quad + \lambda[\tilde{G}(t,\pi x) - G(t,x)]\}.
\end{aligned}
$$

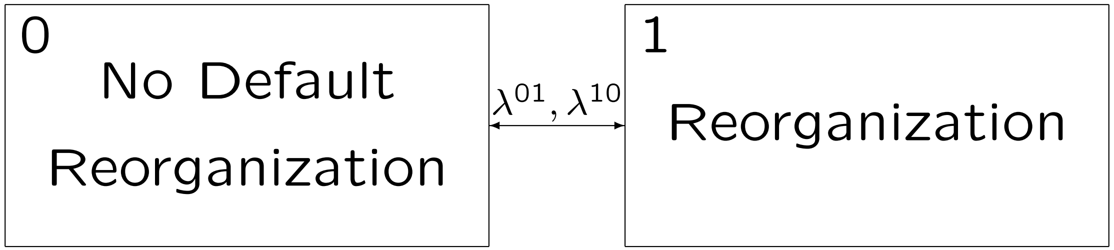
$$
\tilde{G}(t,x) = \mathsf{E}^{t,x}\left[\frac{1}{\gamma}\left(\frac{x_t}{s_t}\right)^\gamma S(T)^\gamma\right] = \frac{1}{\gamma}x^\gamma \exp(\gamma(\alpha - 0.5(1-\gamma)\sigma^2)(T-t)),
$$

In general, $G \neq \tilde{G}$

Why are Markov Chains Useful?

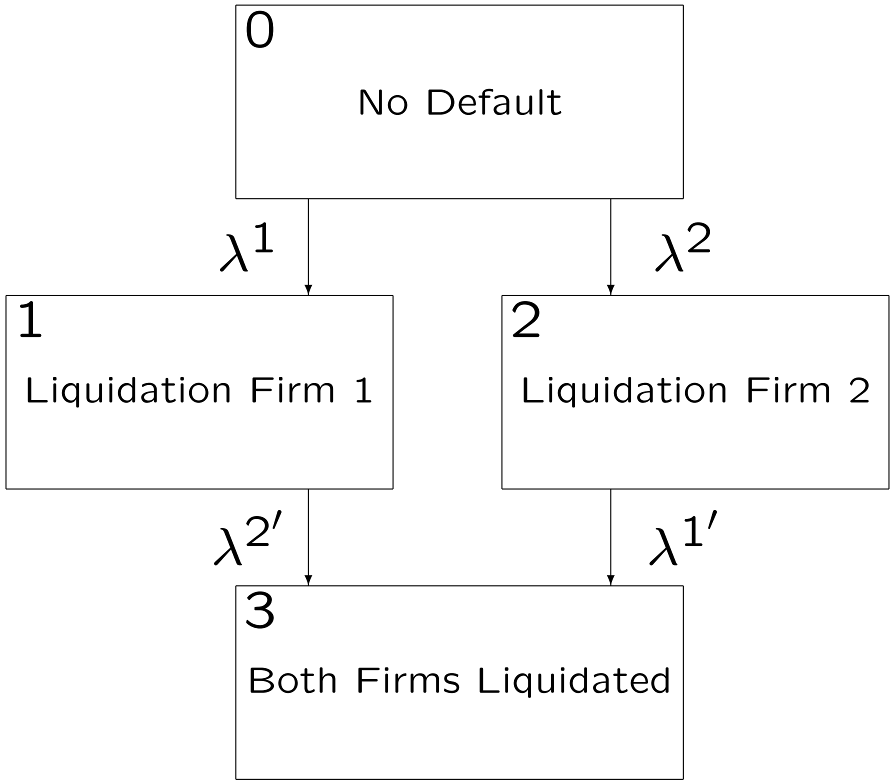
Models for a Single Firm: Liquidation vs. Reorganization





Classical Models like Jarrow/Turnbull (1995) or Duffie/Singleton (1999) can be embedded in a Markov Chain Framework.

Why are Markov Chains Useful?



The Contagion Model by Jarrow/Yu (2001) can be embedded in a Markov Chain Framework.

Standing Assumption

To rule out arbitrage, we assume that an **equivalent martingale measure Q exists** under which discounted asset prices are (local) martingales.

See Harrison and Kreps (1979) and Delbaen and Schachermayer (1999) for the essential equivalence of the existence of such a measure and the absence of arbitrage.

Markov Chain Framework

$\mathcal{J} = \{0, \ldots, J\}$: **finite set of states** characterizing the market for corporate claims

$Z(t)$: **state at time $t \in [0, T^*]$** (RCLL process)

$N = (N^k)_{k \in \mathcal{J}}$: J-dimensional **counting process** counting the number of transitions into the states, i.e.

$$N^k(t) = \#\{s | s \in (0, t], Z(s-) \neq k, Z(s) = k\}.$$

$Y = (Y_1, \ldots, Y_L)$: **macroeconomic variables** with dynamics

$$dY_\iota(t) = \alpha_{Q,\iota}(t)dt + \beta_\iota(t)dW_Q(t),$$

where $W_Q(t) = W(t) + \int_0^t \eta_y(s)\, ds$ is an L-dimensional standard Brownian increment under Q.

Markov Chain Framework

1st Assumption: The processes N^K have **Q-intensities** $\lambda_Q^{jk,n}(t, Y(t))$ such that

$$M_Q^k(t) = N^k(t) - \int_0^t \lambda_Q^{Z(s)k,N(s)}(s)\, ds$$

is a Q-martingale.

2nd Assumption: There exist strictly positive processes $\eta^{jk,n}(t, Y(t))$ such that $\lambda_Q^{jk,n} = \eta^{jk,n}\lambda^{jk,n}$, where $\lambda^{jk,n}$ denote the **physical intensities**, and

$$M^k(t) = N^k(t) - \int_0^t \lambda^{Z(s)k,N(s)}(s)\, ds$$

is a P-martingale.

Markov Chain Framework

Consider a **corporate contingent claim** characterized by a payment stream

$$dA(t) = c^{Z(t)}dt + \sum_{k:k\neq Z(t-)} a^{Z(t-)k,N(t-)}(t)dN^k(t) + a^{Z(t),N(t)}d\varepsilon_T(t),$$

c^j:	contingent coupon payment (continuously paid)
$a^{jk,n}$:	payment upon transition from state j into state k
$a^{j,n}$:	final payment given state j at time T
ε_T:	Dirac mass at T

Example: Zero-coupon bond

state 0:	firm solvent
state 1:	firm in default
a^{01}:	recovery payment upon default
$a^0 = 1$, $a^1 = 0$:	state dependent final payments

Markov Chain Framework

Under a risk-neutral measure, the **time-t value** reads

$$\begin{aligned} B(t) &= \mathsf{E}_Q[\int_{(t,T]} e^{-\int_t^s r(u)\,du} dA(s)|\mathcal{F}_t] \\ &= \mathsf{E}_Q[\int_{(t,T]} e^{-\int_t^s r(u)\,du} dA(s)|Z(t), Y(t), N(t)], \end{aligned}$$

Theorem (System of Pricing Equations) There exist functions $\mathcal{B}^{j,n}$, $j \in \mathcal{J}$ and $n \in I\!N^{J+1}$, such that $B(t) = \mathcal{B}^{Z(t),N(t)}(t, Y(t))$ and the functions $\mathcal{B}^{j,n}$ satisfy the system of PDEs

$$\mathcal{B}_t^{j,n} = r\mathcal{B}^{j,n} - \alpha_Q \mathcal{B}_y^{j,n} - 0.5\,\mathrm{tr}[\beta^T \mathcal{B}_{yy}^{j,n} \beta] - \sum_{k:k\neq j} \lambda_Q^{jk,n} \left(a^{jk,n} + \mathcal{B}^{k,n+1^k} - \mathcal{B}^{j,n} \right)$$

with terminal conditions $\mathcal{B}^{j,n}(T, y) = a^{j,n}(y)$, $j \in \mathcal{J}$ and $n \in I\!N^{J+1}$.

Markov Chain Framework

Proposition (Physical Return Dynamics) Under the physical measure, the dynamics of the gains process $\Gamma = B + A$ are given by

$$\frac{d\Gamma(t)}{B(t-)} = (r(t) + \widehat{\chi}^{Z(t),N(t)}(t))dt - D^{Z(t),N(t)}(t)\beta(t)dW(t) + \sum_{k:k\neq Z(t-)} \tilde{L}^{Z(t-)k,N(t-)} dM^k(t)]$$

with

$$\widehat{\chi}^{j,n} = -\sum_{k:k\neq j} \tilde{L}^{jk,n}(\eta^{jk,n} - 1)\lambda^{jk,n} - \beta D^{j,n}\eta_y,$$

$D^{j,n} = -\mathcal{B}^{j,n}_y/\mathcal{B}^{j,n}$:	state dependent duration,
$\tilde{L}^{jk,n} = R^{jk,n} + L^{jk,n}$:	actual loss
$R^{jk,n} = a^{jk,n}/\mathcal{B}^{j,n}$:	relative payment upon transition
$L^{jk,n} = (\mathcal{B}^{k,n+1^k} - \mathcal{B}^{j,n})/\mathcal{B}^{j,n}$:	relative price jump.

Portfolio Problem

For simplicity: $Y = r$, i.e. short rate is the only macro variable

Money Market Account

$$dM(t) = M(t)r(t)dt, \quad M(0) = 1,$$

Default-free zero-coupon bond

$$dP_f(t,T) = P_f(t,T)[(r(t)+\sigma_f(t)\eta_y(t))dt+\sigma_f(t)dW(t)], \quad P(T,T) = 1.$$

Corporate zero-coupon bonds

$$\begin{aligned} \frac{d\Gamma(t)}{B(t-)} &= (r(t) + \chi^{Z(t),N(t)}(t))dt - D^{Z(t),N(t)}(t)\beta(t)dW(t) \\ &\quad + \sum_{k:k\neq Z(t-)} \tilde{L}^{Z(t-)k,N(t-)}dN^k(t)] \end{aligned}$$

Investor's time-t wealth is given by

$$X(t) = \varphi_f(t)P_f(t,T_f) + \sum_{i=1}^{I} \varphi_i(t)B_i(t,T_i) + \varphi_M(t)M(t),$$

For a self-financing strategy $(\varphi_f, \varphi, \varphi_M)$,

$$dX(t) = \varphi_f(t)dP_f(t) + \sum_{i=1}^{I} \varphi_i(t-)d\Gamma_i(t,T_i) + \varphi_M(t)dM(t),$$

Seting $\pi_0 = \varphi_f P_f/X$ and $\pi_i = \varphi_i B_i/X$, the **wealth equation** reads:

$$dX = X^-[(r + \sum_{i=0}^{I} \pi_i \chi_i^{Z,N})dt - \beta D dW + \sum_{i=1}^{I} \sum_{k \neq j} \pi_i^- \tilde{L}_i^{Z^- k, N^-} dN^k],$$

where $X^- := X(t-)$, $Z^- = Z(t-)$, $\pi_i^- := \pi_i(t-)$, $\chi_0 = \chi_0^Z := \eta_r \sigma_f$, $D := \sum_{i=0}^{I} \pi_i D_i^{Z,N}$ (portfolio duration).

0-th bond is identified with the default-free zero-coupon bond

Portfolio Problem

The investor **maximizes expected utility** from intermediate consumption and terminal wealth at final time T w.r.t.

$$U(t,x) = \frac{1}{\gamma}\psi(t)^{1-\gamma}x^{\gamma}, \qquad \gamma < 1,$$

where $\psi(t) = \psi(t, r(t))$ models the investor's time-preferences.

Investor's indirect utility (syn. value function) is defined by

$$G^{j,n}(t,x,r) = \sup_{\pi,D,c} \mathsf{E}_{j,n}^{t,x,r}\left[\int_t^T \tfrac{1}{\gamma}\psi(s)^{1-\gamma}c(s)^{\gamma}ds + \tfrac{1}{\gamma}\psi(T)^{1-\gamma}\left(X(T)\right)^{\gamma}\right]$$

Hamiltion-Jacobi-Bellman Equation

$$0 = \sup_{\pi,D,c} \{A^{\pi,D,c}G^{j,n} + \tfrac{1}{\gamma}\psi^{1-\gamma}c^{\gamma}\}, \qquad j \in \{0,\dots,J\},$$

with terminal condition $G^{j,n}(T,x,r) = \frac{1}{\gamma}\psi(T)^{1-\gamma}x^{\gamma}$ and

$$\begin{aligned} A^{\pi,D,c}G^{j,n} \;=\; & G^{j,n}_t + \{x(r - \textstyle\sum_k \eta^{jk,n}\lambda^{jk,n}\sum_{i=1}^{I}\pi_i \tilde{L}_i^{jk,n} - \beta\eta_r D) - c\}G^{j,n}_x \\ & + \alpha G^{j,n}_r + 0.5x^2\beta^2 D^2 G^{j,n}_{xx} - x\beta^2 D G^{j,n}_{xr} + 0.5\beta^2 G^{j,n}_{rr} \\ & + \textstyle\sum_{k\neq j}\lambda^{jk,n}\left\{G^{k,n+1^k}(t, x(1+\sum_{i=1}^{I}\pi_i\tilde{L}_i^{jk,n}), r) - G^{j,n}(t,x,r)\right\}. \end{aligned}$$

We conjecture $G^{j,n}(t,x,r) = \frac{1}{\gamma}x^{\gamma}f^{j,n}(t,r)^{1-\gamma}$ with $f^{j,n}(T,r) = \psi(T)$

...

Theorem (FOC) First-order conditions for D, π, and c:

$$D^{j,n*} = -\frac{1}{1-\gamma}\frac{\eta_r}{\beta} - \frac{f_r^{j,n}}{f^{j,n}}, \qquad \frac{c^{j,n*}}{x} = \frac{\psi}{f^{j,n}}$$

$$0 = -\sum_{k\neq j}\tilde{L}_\nu^{jk,n}\eta^{jk,n}\lambda^{jk,n} + \sum_{k\neq j}\lambda^{jk,n}(1+\sum_i \pi_i^{j,n*}\tilde{L}_i^{jk,n})^{\gamma-1}\left(\frac{f^{k,n+1^k}}{f^{j,n}}\right)^{1-\gamma}\tilde{L}_\nu^{jk,n},$$

where the functions $f^{j,n}$ satisfy the system of PDEs

$$0 = \psi + f_t^{j,n} + \frac{\gamma}{1-\gamma}[r + 0.5\frac{1}{1-\gamma}\eta_r^2 - \sum_{k\neq j}\eta^{jk,n}\lambda^{jk,n}\sum_{i=1}^I \pi_i^{j,n*}\tilde{L}_i^{jk,n}]f^{j,n} + (\alpha + \frac{\gamma}{1-\gamma}\beta\eta_r)f_r^{j,n} + 0.5\beta^2 f_{rr}^{j,n} + \frac{1}{1-\gamma}f^{j,n}\sum_{k\neq j}\lambda^{jk,n}\left[(1+\sum_{i=1}^I \pi_i^{j,n*}\tilde{L}_i^{jk,n})^\gamma\left(\frac{f^{k,n+1^k}}{f^{j,n}}\right)^{1-\gamma} - 1\right]$$

with $f^{j,n}(t,r) = 1$, $j \in J$ and $n \in IN_0^{J+1}$. The **investor's indirect utility** is given by $G^{j,n}(t,x,r) = \frac{1}{\gamma}x^\gamma f^{j,n}(t,r)^{1-\gamma}$.

Applications: Reorganization vs. Liquidation

Simple Problem: Money market account and one corporate bond

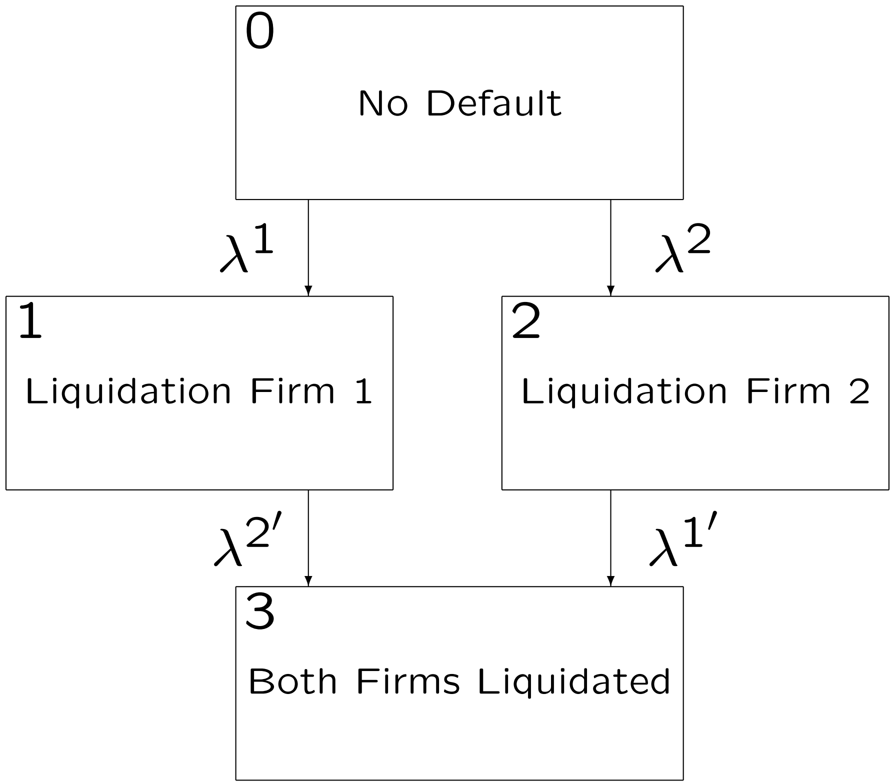
FOC: **Optimal bond demands** for economies with
Reorganization (Chapter 11) and **Liquidation (Chapter 7)**:

$$\pi^* = \frac{1 - \eta^{\frac{1}{\gamma-1}}}{-L}, \qquad \pi^* = \frac{1 - \eta^{\frac{1}{\gamma-1}\frac{f^1}{f^0}}}{-L}.$$

Proposition For positive default risk premiums, i.e. $\eta - 1 \geq 0$, we get $f^0 \leq f^1$ if $\gamma < 0$ and $f^0 \geq f^1$ if $\gamma > 0$

$\Longrightarrow$ The **bond demand** is **smaller** in an economy where **the firm is liquidated**.

Applications: Contagion



Contagion occurs if $\lambda^{1'} > \lambda^1$ or $\lambda^{2'} > \lambda^2$.

Applications: Contagion

For simplicity: Two corporate zeros with identical parameters, i.e. $\lambda := \lambda^1 = \lambda^2$, $\lambda' := \lambda^{1'} = \lambda^{2'}$, $\eta := \eta^1 = \eta^2$, $\eta' := \eta^{1'} = \eta^{2'}$, $L_1^{01} = L_2^{02}$, $L_1^{02} = L_2^{01}$.

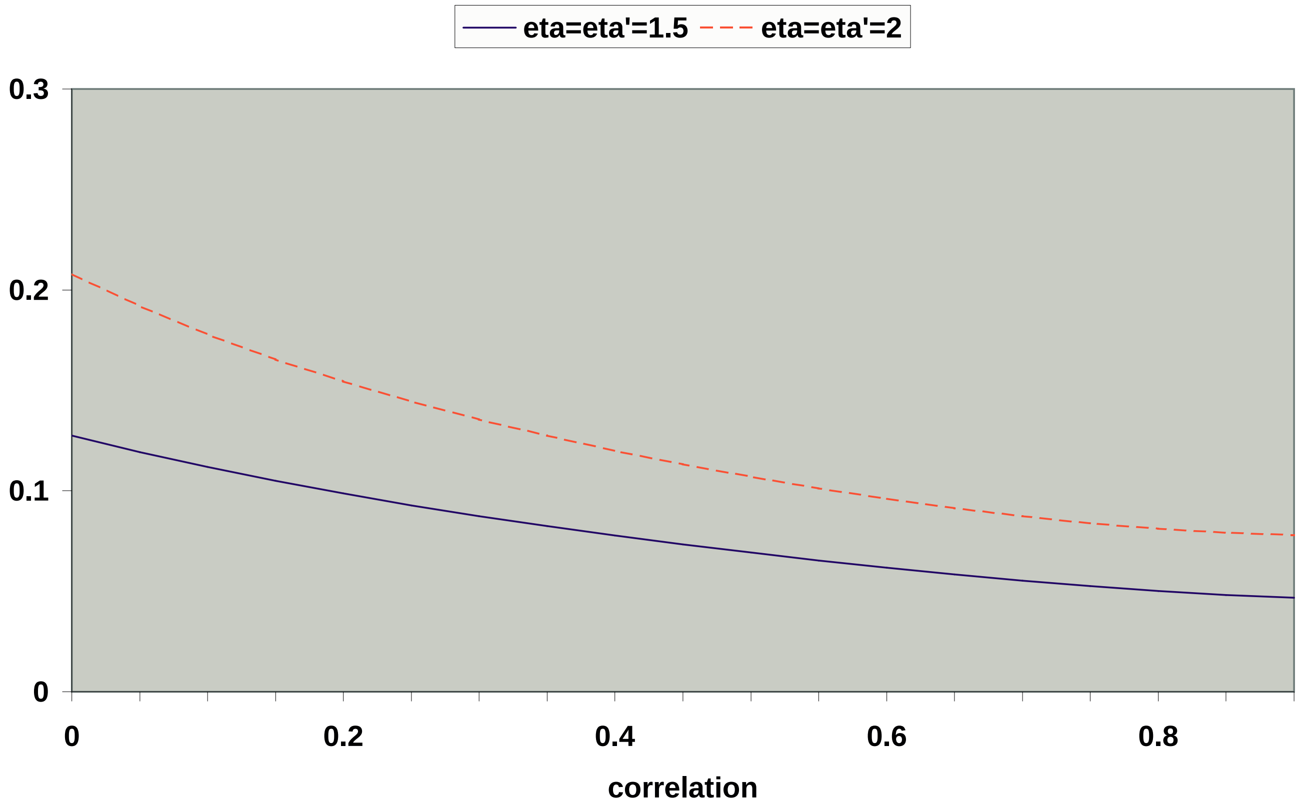
Optimal demands

$$\pi_1^* = \frac{1 - \eta^{\frac{1}{\gamma-1}} \frac{f^1}{f^0}}{-(\tilde{L}_1^{01} + \tilde{L}_1^{02})}.$$

The effect of f^1/f^0 on the optimal demands is small compared to the effect of $\tilde{L}_1^{02}$.

Interpretation: If we have a **positive contagion effect** (under the risk-neutral measure), i.e. $\eta\lambda = \lambda_Q < \lambda'_Q = \eta'\lambda'$, then $\tilde{L}_1^{02} < 0$ and the **demand decreases**, whereas the opposite is true for a negative contagion effect (competition effect).

Applications: Contagion



Optimal Demands for Different Default Correlations

Conclusion

- We present a sophisticated framework covering various realistic features of defaultable bonds (e.g. Chapter 7 vs. Chapter 11)
- We are able to solve various involved portfolio problems with corporate bonds.
- In particular, we are able to *quantify* the impact of contagion on bond demands.
- This is important because contagion reduces the investor's ability to diversify his portfolio.
- Our approach can be applied to several other situations: ratings, default vs. bankruptcy, exogenous (re)issuing of bonds

$$
\begin{aligned}
\rho_{1,2}(t,T) &= \frac{\mathsf{Cov}_t(\mathbf{1}_{\{\tau_1 \le T\}}, \mathbf{1}_{\{\tau_2 \le T\}})}{\sqrt{\mathsf{Var}_t(\mathbf{1}_{\{\tau_1 \le T\}})}\sqrt{\mathsf{Var}_t(\mathbf{1}_{\{\tau_2 \le T\}})}} \\
&= \frac{p^{03} - (p^{01} + p^{03})(p^{02} + p^{03})}{\sqrt{(p^{01} + p^{03})(1 - p^{01} - p^{03})}\sqrt{(p^{02} + p^{03})(1 - p^{02} - p^{03})}}
\end{aligned}
$$