Market of defaultable bonds driven by infinite dimensional Lévy processes

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Plan:

1. Introduction.

2. HJM condition for defaultable bonds.

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4. Pricing Bonds and CDS in the model with rating migration induced by Cox process.

• Fixed income market

A basic instrument is a zero-coupon bond, it is a financial security paying to its holder 1 unit of cash at a prespecified date θ in the future (the maturity time).
Let $P(t, \theta)$, $0 \leq t \leq \theta$ be the market price of a bond at moment t.

$$P(t, \theta) = e^{\int_\theta^t r(\sigma) d\sigma}.$$

Bond price models based on a specific short term interest rate process makes the problem of matching the initial term structure.

• Heath, Jarrow and Morton proposed to use the forward rate curve. It is a function $f(t,\theta)$ defined for $\theta \geq t$ and such that

$$P(t,\theta) = e^{-\int_t^\theta f(t,s)ds}.$$

$f(t,\theta)$ describe our expectation at the moment t of the value of short term

interest rate at the moment θ:

$$df(t,\theta) = \alpha(t,\theta)dt + < \sigma(t,\theta),\ dW(t) >$$

with W d-dimensional standard Wiener process.

We consider

$$df(t,\theta) = \alpha(t,\theta)dt + < \sigma(t,\theta),\ dZ(t) > . \qquad (1)$$

with Lévy process Z in a Hilbert space U.

We assume that the basic probability space $(\Omega, \mathcal{F}, \boldsymbol{P})$ is complete.
By μ we denote the measure associated to jumps of Z i.e i.e. for any $A \in \mathcal{B}(U)$ such that $\overline{A} \subset U \backslash \{0\}$ we have:

$$\mu([0,t], A) = \sum_{0<s\leq t} \mathbf{1}_A(\Delta Z(s)).$$

The measure ν defined by:

$$\nu(A) = E(\mu([0,1], A)),$$

is called Levy measure of process Z, stationarity of increments implies that we have also:

$$E(\mu([0,t], A)) = t\nu(A).$$

The Lévy-Khintchine formula shows that characteristic function of Lévy process has a form:

$$Ee^{i<\lambda,Z(t)>_U} = e^{t\psi(\lambda)},$$

where

$$\psi(\lambda) = i < a, \lambda >_U - \frac{1}{2} < Q\lambda, \lambda >_U + \int_U (e^{i\langle \lambda, x\rangle_U} - 1 - i < x, \lambda >_U \mathbf{1}_{[-1,1]}(|x|_U))\nu(dx),$$

and $a \in U$, Q is symmetric non negative nuclear operator on U, ν is a measure on U with $\nu(\{0\}) = 0$ and

$$\int_{\mathcal{U}} (|x|^2 \wedge 1)\nu(dx) < \infty. \tag{2}$$

Moreover Z has a well known Lévy-Itô decomposition:

$$Z(t) = at + W(t) + \int_0^t \int_{|y|_U \leq 1} y(\mu(ds, dy) - dt\nu(dy)) + \int_0^t \int_{|y|_U > 1} y\mu(ds, dy),$$

where W is a Wiener process with values in U and covariance operator Q.

For each θ the processes $\alpha(t, \theta)$, $\sigma(t, \theta)$, $t \leq \theta$ are assumed to be adapted processes with respect to a given filtration $(\mathcal{F}_t)$ and such that integrals in (1) are well defined.
For $\theta < t$ we put

$$\alpha(t, \theta) = \sigma(t, \theta) = 0. \tag{3}$$

It follows from (1) that for $t \leq \theta$,

$$f(t, \theta) = f(0, \theta) + \int_0^t \alpha(s, \theta) ds + \int_0^t < \sigma(s, \theta),\ dZ(s) >$$

and by (3) that for $t > \theta$

$$f(t,\theta) = f(0,\theta) + \int_0^\theta \alpha(s,\theta)ds + \int_0^\theta < \sigma(s,\theta),\ dZ(s) > .$$

Consequently for each $\theta > 0$, $f(t,\theta)$, $t > \theta$, is a process constant in t and should be identified with the short rate:

$$r(\theta) = f(0,\theta) + \int_0^\theta \alpha(s,\theta)ds + \int_0^\theta < \sigma(s,\theta),\ dZ(s) > . \qquad (4)$$

From now on we assume (1) and (3) and that the short rate is given by (4).

Let us recall that *HJM postulate* is the requirement that the discounted bond price processes $\widehat{P}(\cdot,\theta)$, $\theta \in [0,T]$:

$$\widehat{P}(t,\theta) = P(t,\theta)/B_t = e^{-\int_t^\theta f(t,s)ds} e^{-\int_0^t f(t,s)ds} = e^{-\int_0^\theta f(t,s)ds}$$

are local martingales.

Let b be the Laplace transform of the measure ν restricted to the complement of the ball $\{y : \ |y| \leq 1\}$,

$$b(u) = \int_{|y|>1} e^{-<u,y>} \nu(dy), \tag{5}$$

and B the set of those $u \in U$ for which the Laplace transform is finite:

$$B = \{u \in U : \ b(u) < \infty\}.$$

Assumption:

A1a: The processes α and σ are predictable and with probability one have bounded trajectories.

A1b: For arbitrary $r > 0$ the function b given by (5) is bounded on $\{u : |u| \leq r, b(u) < \infty\}$.

A2: For all $\theta \leq T^*$, $\boldsymbol{P}-$ almost surely holds

$$\int_t^\theta \sigma(t,v)\, dv \in B \tag{6}$$

for almost all $t \in [0,\theta]$.

It is convenient to express the HJM condition in terms of the logarithm of moment generating function of Lévy process Z, i.e. in terms of the functional $J : U \to \mathbb{R}$:

$$J(u) = -< u, a > + \frac{1}{2} < Qu, u > + \widehat{J}(u), \tag{7}$$

where

$$\begin{aligned} \widehat{J}(u) = & \int_{\{|y|\leq 1\}} \Big(e^{-<u,y>} - 1 + < u, y >)\nu(dy) \\ & + \int_{\{|y|>1\}} (e^{-<u,y>} - 1)\nu(dy). \end{aligned} \tag{8}$$

Theorem A. *i) Assume (A1) and (A2). Discounted bond prices are local martingales if and only if the following the HJM-type condition holds for each $\theta \in [0, T^*]$ and every $t \le \theta$:*

$$\int_t^\theta \alpha(t,v)dv = J\left(\int_t^\theta \sigma(t,v)dv \right). \tag{9}$$

ii) Assume (A1). If the HJM postulate holds then, for arbitrary $\theta \le T$, $\boldsymbol{P}$ -almost surely,

$$\int_t^\theta \sigma(t,v)\, dv \in B \tag{10}$$

for almost all $t \in [0, \theta]$.

- Our aim is to derive the HJM-type conditions for the market containing a risk free bond and defaultable bonds.

The payoff of the defaultable bond is as follows:

$$D(\theta,\theta) = \mathbf{1}_{\{\tau>\theta\}} + \mathbf{1}_{\{\tau\leq\theta\}} \cdot \text{ \textit{recovery payment} },$$

where τ is the moment of default.

Recovery payment can take different forms:

• $\delta_t D(\tau-,\theta)\frac{B_\theta}{B_\tau}$ - *fractional recovery of market value*

$$D(\theta,\theta) = \mathbf{1}_{\{\tau>\theta\}} + \mathbf{1}_{\{\tau\leq\theta\}} \cdot \delta_\tau D(\tau-,\theta)\frac{B_\theta}{B_\tau}$$

where δ_t is $\mathbb{F}$ predictable and takes values in $[0,1]$,

• δ - *fractional recovery of Treasury value*:

$$D^\delta(\theta,\theta) = \mathbf{1}_{\{\tau>\theta\}} + \mathbf{1}_{\{\tau\leq\theta\}} \cdot \delta,$$

• $\frac{\delta B_\theta}{B_\tau}$ - *fractional recovery of par value*:

$$D^{\triangle}(\theta,\theta) = \mathbf{1}_{\{\tau>\theta\}} + \mathbf{1}_{\{\tau\le\theta\}} \cdot \delta \frac{B_\theta}{B_\tau}.$$

• *fractional recovery with multiple defaults* - defaultable bonds such that their face values, at each default time τ_i, is reduced by fraction L_{τ_i}, where L_s is $\mathbb{F}$ - predictable process taking values in $[0,1]$:

$$D^m(\theta,\theta) = \prod_{\tau_i\le\theta} (1 - L_{\tau_i})$$

τ_i are moments of jumps of Cox process N_t with stochastic intensity process $(\lambda_t)_{t\ge 0}$.
Note that $1 - L_t$ can be interpreted as a recovery process and therefore we will denote it by δ_t. Thus $\delta_t = 1 - L_t$.

We assume that the moment of default τ is a $\mathbb{G}$ stopping time, and that our filtration $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$, where $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ and $\mathbb{H} = (\mathcal{H}_t)_{t \geq 0}$ are filtrations generated by Levy processes and observing default time i.e. $\mathcal{H}_t = \sigma(\{\tau \leq u\} : u \leq t)$, respectively.

We assume that τ admits an $\mathbb{F}$ intensity $(\lambda_t)_{t \geq 0}$ which is an $\mathbb{F}$ adapted process such that for $H_t = \mathbf{1}_{\{\tau \leq t\}}$, process M_t given by the formula

$$M_t = H_t - \int_0^{t \wedge \tau} \lambda_u du = H_t - \int_0^t (1 - H_u) \lambda_u du$$

follows a $\mathbb{G}$–martingale.

All result are proved under

Hypothesis (H): We say that *hypothesis (H)* holds for filtrations $\mathbb{F}$ and $\mathbb{G}$, with $\mathbb{F} \subseteq \mathbb{G}$, if every an $\mathbb{F}$-local martingale is a $\mathbb{G}$-local martingale.

We denote by $g_1(t,u)$ the pre-default forward rate corresponding to the pre-default term structure observed on the market. We postulate here that

$$dg_1(t,\theta) = \alpha_1(t,\theta)dt + < \sigma_1(t,\theta), dZ_1(t) >,$$

where Z_1 is a Lévy process with values in U, which has the Lévy-Itô decomposition with parameters a_1, μ_1 ν_1. If

$$D_1(t,\theta) = e^{-\int_t^\theta g_1(t,u)du},$$

then applying Itô lemma

Theorem 1. *Dynamics of the process* $\widehat{D}_1(t,\theta) = D_1(t,\theta)/B_t$ *is given by*

$$d\widehat{D}_1(t,\theta) = \widehat{D}_1(t-,\theta)\Bigg(\Big(g_1(t,t) - f(t,t) + \bar{a}_1(t,\theta)\Big)dt +$$

$$\int_U \left[e^{-<\int_t^\theta \sigma_1(t,v)dv,y>} - 1\right](\mu_1(dt,dy) - dt\nu_1(dy)) - < \int_t^\theta \sigma_1(t,v)dv, dW_1(t)$$

where

$$\bar{a}_1(t,\theta) = -\int_t^\theta \alpha_1(t,v)dv + J_1(\int_t^\theta \sigma_1(t,v)dv).$$

• Fractional recovery of market value

$$D(\theta,\theta) = \mathbf{1}_{\{\tau>\theta\}} + \mathbf{1}_{\{\tau\le\theta\}} \cdot \delta_\tau D(\tau-,\theta)\frac{B_\theta}{B_\tau}$$

and for $t \le \theta$ we model a value of defaultable bond by

$$D(t,\theta) = \mathbf{1}_{\{\tau>t\}} e^{-\int_t^\theta g_1(t,u)du} + \mathbf{1}_{\{\tau\le t\}} \delta_\tau D(\tau-,\theta)\frac{B_t}{B_\tau}, \qquad (11)$$

where $g_1(t,u)$ is the pre-default forward rate corresponding to pre-default term structure.

Using the process H_t we can represent D as

$$D(t,\theta) = (1-H_t)D_1(t,\theta) + H_t\delta_\tau D_1(\tau-,\theta)\frac{B_t}{B_\tau}.$$

Theorem 2. *(HJM drift condition for $D(t,\theta)$) Discounted prices of defaultable bonds with fractional recovery of market value are local martingales if and only if the following conditions hold:*
for all $\theta \in [0, T^*]$ and for almost all $t \leq \theta$ on the set $\{\tau > t\}$:

$$g_1(t,t) = f(t,t) + (1-\delta_t)\lambda_t, \tag{12}$$

$$\int_t^\theta \alpha_1(t,v)dv = J_1\left(\int_t^\theta \sigma_1(t,v)dv\right). \tag{13}$$

• Fractional recovery of treasury

$$D^{\delta}(\theta,\theta) = \mathbf{1}_{\{\tau>\theta\}} + \mathbf{1}_{\{\tau\le\theta\}} \cdot \delta.$$

So $$D^{\delta}(t,\theta) = \mathbf{1}_{\{\tau>t\}} e^{-\int_t^\theta g_1(t,u)du} + \mathbf{1}_{\{\tau\le t\}} \cdot \delta \cdot B(t,\theta).$$

Therefore

$$D^{\delta}(t,\theta) = (1 - H_t) D_1(t,\theta) + H_t \delta B(t,\theta). \tag{14}$$

Theorem 3. *(HJM drift condition for $D^{\delta}(t,\theta)$) The processes of discounted defaultable bond prices with fractional recovery of treasury are local martingales if and only if the following condition holds:*
for all $\theta \in [0, T^*]$ and for almost all $t \le \theta$ on the set $\{\tau > t\}$:

$$g_1(t,t) = f(t,t) + (1-\delta)\lambda_t, \tag{15}$$

$$\int_t^\theta \alpha_1(t,v)dv = J_1\left(\int_t^\theta \sigma_1(t,v)dv\right) + \delta\left(\frac{B(t-,\theta)}{D_1(t-,\theta)} - 1\right)\lambda_t. \tag{16}$$

- Fractional recovery of par

The payoff at maturity has form

$$D^{\triangle}(\theta,\theta) = \mathbf{1}_{\{\tau>\theta\}} + \mathbf{1}_{\{\tau\le\theta\}} \cdot \delta \frac{B_\theta}{B_\tau},$$

and before maturity has form

$$D^{\triangle}(t,\theta) = \mathbf{1}_{\{\tau>t\}} D_1(t,\theta) + \mathbf{1}_{\{\tau\le t\}} \cdot \delta \frac{B_t}{B_\tau}.$$

Theorem 4. *(HJM drift condition for $D^{\triangle}(t,\theta)$)*
Discounted prices of defaultable bond with fractional recovery of par are local martingales if and only if the following conditions hold for all $\theta \in [0, T^*]$ and for almost all $t \leq \theta$ on the set $\{\tau > t\}$:

$$g_1(t,t) = f(t,t) + (1-\delta)\lambda_t, \tag{17}$$

$$\int_t^\theta \alpha_1(t,v)dv = J_1\left(\int_t^\theta \sigma_1(t,v)dv\right) + \delta\left(\frac{1}{D_1(t-,\theta)} - 1\right)\lambda_t. \tag{18}$$

• Fractional recovery with multiple defaults

A holder of such defaultable bond receives, at maturity θ,

$$D^m(\theta,\theta) = \prod_{\tau_i \le \theta} (1 - L_{\tau_i}).$$

If we introduce process V_t by the formula:

$$V_t = \prod_{\tau_i \le t} (1 - L_{\tau_i}),$$

then $D^m(\theta,\theta) = V_\theta$ and for $t \le \theta$:

$$D^m(t,\theta) = V_t e^{-\int_t^\theta g_1(t,u)du} = V_t D_1(t,\theta).$$

Moreover, we assume that τ_i are moments of jumps of Cox process N_t (doubly stochastic Poisson process) with stochastic intensity process $(\lambda_t)_{t \ge 0}$. It can be shown that V_t solves the following SDE:

$$dV_t = -V_{t-} L_t dN_t, \tag{19}$$

and the process

$$M_t = N_t - \int_0^t \lambda_u du \tag{20}$$

follows $\mathbb{G}$ - martingale.

Theorem 5. *Discounted prices of defaultable bonds with multiple defaults and fractional recovery are local martingales if and only if the following conditions hold*

for all $\theta \in [0, T^*]$ *and for almost all* $t \leq \theta$ *on the set* $\{V_{t-} > 0\}$*:*

$$g_1(t,t) = f(t,t) + (1-\delta_t)\lambda_t, \tag{21}$$

$$\int_t^\theta \alpha_1(t,v)dv = J_1\bigg(\int_t^\theta \sigma_1(t,v)dv\bigg). \tag{22}$$

These results can be generalize to the rating migration case.

The set of rating classes $\mathcal{K}$ is identical with $= \{1, \ldots, K\}$, where the state $i = 1$ represents the highest rank and the state $i = K$ the default event. The credit rating migration process will be denoted by C^1 and assumed to be a conditional Markov chain with absorbtion state K.

By $f(t)$ we denote the forward process associated with risk free bond and by $g_1, g_2, ..., g_{K-1}$ the pre-default term structures associated with ratings $1, 2, ..., K-1$. The pre-default term structure g is thus given by the formula

$$g(t,u) = g_{C^1(t)}(t,u) = 1_{\{C^1(t)=1\}} g_1(t,u) + \ldots + 1_{\{C^1(t)=K-1\}} g_{K-1}(t,u).$$

To avoid arbitrage it is reasonable to assume that

$$g_{K-1}(t,\theta) > g_{K-2}(t,\theta) > \ldots > g_1(t,\theta) > f(t,\theta)$$

for all $t \in [0,\theta]$ and all $\theta \in [0, T^*]$.

Recovery payment depends on credit rating before default i.e.

$$\delta_t = \delta_{C^2(t)}(t) = 1_{\{C^2(t)=1\}}\delta_1(t)+1_{\{C^2(t)=2\}}\delta_2(t)+\ldots+1_{\{C^2(t)=K-1\}}\delta_{K-1}$$

where δ_i is a recovery payment connected with $i-$th rating class and C^2 is so called process of the previous ratings:

$$C^2(t) = C^1(\tau_k), \qquad t \in [\tau_k, \tau_{k+1}),$$

where $\tau_1, \tau_2, \tau_3, \ldots$ denote the consecutive moments of jumps of credit migration process C^1.

Hypothesis (H1): We assume that for $(\tau_k)_{k\geq 0}$ the consecutive times of jumps of credit migration process and for all $\theta \in [0, T^*]$ we have

$$\boldsymbol{P}\left(\Delta B(\tau_k,\theta) \neq 0\right) = 0, \qquad \boldsymbol{P}\left(\Delta D_i(\tau_k,\theta) \neq 0\right) = 0 \ \forall i = 1, \ldots K-1$$

We follow approach from Bielecki and Rutkowski book.
The conditional infinitesimal generator of the process C^1 describing credit rating migration, at time t given $\mathbb{G}_t$ has the form

$$\Lambda(t) = \begin{pmatrix} \lambda_{1,1}(t) & \lambda_{1,2}(t) & \cdots & \lambda_{1,K-1}(t) & \lambda_{1,K}(t) \\ \lambda_{2,1}(t) & \lambda_{2,2}(t) & \cdots & \lambda_{2,K-1}(t) & \lambda_{2,K}(t) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \lambda_{K-1,1}(t) & \lambda_{K-1,2}(t) & \cdots & \lambda_{K-1,K-1}(t) & \lambda_{K-1,K}(t) \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

where off-diagonal processes $\lambda_{i,j}(t)$, $i \neq j$ are non-negative processes adapted to $\mathbb{G}$ and diagonal elements are negative and are determined by off-diagonals by the formula $\lambda_{i,i}(t) = -\sum_{j\in\mathcal{K}\setminus\{i\}} \lambda_{i,j}(t)$. We can regard $p_{i,j}(t) = -\frac{\lambda_{i,j}(t)}{\lambda_{i,i}(t)}$ as a probability of jumping from the state i to the state j given that we jump-off the state i.

Auxiliary processes. Define process $H_i(t) = 1_{\{i\}}(C^1(t))$ and for $i \neq j$

$$H_{i,j}(t) = \sum_{0<u\leq t} H^i(u-)H^j(u), \forall t \in \mathbb{R}_+.$$

Then the the processes

$$M_i(t) = H_i(t) - \int_0^t \lambda_{C^1(u),i}(u)du,$$

$$M_{i,j}(t) = H_{i,j}(t) - \int_0^t \lambda_{i,j}(u)H_i(u)du = H_{i,j}(t) - \int_0^t \lambda_{C^1(u),j}(u)H_i(u)du,$$

and

$$M_K(t) = H_K(t) - \int_0^t \sum_{i=1}^{K-1} \lambda_{i,K} H_i(u)du = H_K(t) - \int_0^t \lambda_{C^1(u),K}(1-H_K(u))du,$$

are $\mathbb{G} = \mathbb{F} \vee \mathbb{F}^{C^1} \vee \mathbb{F}^{\Lambda}$-martingales.

HJM condition.

In the case of fractional recovery of par value the holder of defaultable bond receives 1 unit cash if there is no default prior to maturity and if bond has defaulted a fixed fraction δ of par value is paid at default time. Therefore the payoff at maturity has form

$$D(\theta,\theta) = 1_{\{\tau > \theta\}} + 1_{\{\tau \leq \theta\}} \delta_{C^2(t)} B_\theta B_\tau,$$

hence

$$D(t,\theta) = \sum_{i=1}^{K-1} \Big(H_i(t) D_i(t,\theta) + H_{i,K}(t) \delta_i B_t B_\tau \Big).$$

Theorem 6. *Assume (H1). The processes of discounted prices of defaultable bond with fractional recovery of par value are local martingales if and only if the following two conditions hold:*
for all $\theta \in [0, T^*]$ *and for almost all* $t \leq \theta$ *we have on the set* $\{C^1(t) \neq K\}$*:*

$$
\begin{aligned}
g_{C^1(t)}(t,t) =& f(t,t) + (1 - \delta_{C^1(t)})\lambda_{C^1(t),K}(t), \\
\int_t^\theta \alpha_{C^1(t)}(t,u)du =& J_{C^1(t)}\left(\int_t^\theta \sigma_{C^1(t)}(t,v)dv\right) \\
& + \delta_{C^1(t)}\left[\frac{1}{D_{C^1(t)}(t-,\theta)} - 1\right]\lambda_{C^1(t),K}(t) \\
& + \sum_{j=1, j\neq C^1(t)}^{K-1}\left[\frac{D_j(t-,\theta)}{D_{C^1(t)}(t-,\theta)} - 1\right]\lambda_{C^1(t),j}(t).
\end{aligned}
$$

Consistency conditions

We investigate the form of the HJM type conditions under consistency conditions analogous to that in Bielecki and Rutkowski. In the case of fractional recovery of par value with rating migrations the consistency condition has the form

$$\sum_{i=1, i\neq C^1(t)}^{K-1} \Big(D_i(t-,\theta) - D_{C^1(t)}(t-,\theta)\Big)\lambda_{C^1(t),i}(t) + \tag{23}$$

$$(\delta_{C^1(t)}(t) - D_{C^1(t)}(t-,\theta))\lambda_{C^1(t),K}(t) + \lambda_{C^1(t),C^1(t)}(t) D_{C^1(t)}(t-,\theta) = 0.$$

Theorem 7. *The HJM-type conditions for defaultable bonds with credit migrations and fractional recovery of par value for which the consistency conditions (23) holds, have the following form :*
for all $\theta \in [0, T^*]$ *and all* $t \leq \theta$

$$g_{C^1(t)}(t,t) = f(t,t) + \lambda_{C^1(t),C^1(t)}(t),$$

$$\int_t^\theta \alpha_{C^1(t)}(t,u)du = J_{C^1(t)}\left(\int_t^\theta \sigma_{C^1(t)}(t,v)dv\right).$$

Pricing Bonds and CDS in the model with rating migration induced by Cox process.

We define ratings migration process by setting:

$$C(t) := \bar{C}_{N_t},$$

where N is the Cox process with $\mathbb{F}$ adapted (filtration $\mathbb{F}$ contains market information without credit migration) intensity process λ and $(\bar{C})$ is a discrete time (homogenous) Markov chain with values in set $\mathcal{K} = \{1, \ldots, K\}$ independent of $\mathbb{F}$ with one-step transition matrix P of following form:

$$P = \begin{pmatrix} 0 & p_{1,2} & p_{1,3} & \cdots & p_{1,K} \\ p_{2,1} & 0 & p_{2,3} & \cdots & p_{2,K} \\ p_{3,1} & p_{3,2} & 0 & \cdots & p_{3,K} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ p_{K-1,1} & p_{K-1,2} & p_{K-1,2} & \cdots & p_{K-1,K} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

.

Consider bonds with fractional recovery of par value

$$D(t,T) = B_t \mathbf{E}\left(\frac{1}{B_T} 1_{\{\tau > T\}} + \frac{\delta_{C_{\tau-}}}{B_\tau} 1_{\{t<\tau\le T\}} | \mathcal{G}_t\right).$$

Theorem 8. *The price of defaultable bond with fractional recovery of par value is given by*

$$D(t,T)1_{\{C(t)\neq K\}} = \sum_{i=1}^{K-1} 1_{\{C_t=i\}}\Bigg(B(t,T) - \sum_{j=1}^{K-1} B_t \frac{\mathbf{E}\left(\int_t^T \left(\frac{1}{B_T} - \frac{\delta_j}{B_u}\right) dG_{t,i,j}(u) | \mathcal{F}_t\right)}{\mathbf{P}(C_t = i|\mathcal{F}_t)}\Bigg),$$

where

$$G_{t,i,j}(v) := \mathbf{P}(\tau \le v, C_t = i, C_{\tau-} = j|\mathcal{F}_v).$$

We are also interested in pricing credit derivatives connected with such defaultable bond, for example in pricing of CDS on such bond. Credit Default Swap is an agreement between two parties protection seller and protection buyer. This contracts have two legs.

Premium Leg: Protection buyer agrees to pay fixed amount κ (CDS spread) at given dates $\mathcal{T} = \{T_1 < T_2 < \ldots < T_n\}$ provided that default didn't happened before or at T_n. For $t \leq T_1$ we have:

$$B_t \mathbf{E} \left(\sum_{k=1}^{n} \frac{\kappa}{B_{T_k}} 1_{\{\tau > T_k\}} | \mathcal{G}_t \right).$$

Default Leg: Protection seller agrees to cover all losses on bond provided that loss occurred before T_n, the protection horizon. For $t \leq T_1$ value of this leg is equal to

$$B_t \mathbf{E}\left(\frac{1-\delta_{C_{\tau-}}}{B_\tau} 1_{\{t<\tau\le T_n\}} | \mathcal{G}_t\right).$$

Provided that we know value of spread κ, at time t value of CDS is difference between premium leg and default leg:

$$\mathrm{CDS}(t,\mathcal{T},\kappa) = B_t \mathbf{E}\left(\sum_{k=1}^{n} \frac{\kappa}{B_{T_k}} 1_{\{\tau>T_k\}} - \frac{(1-\delta_{C_{\tau-}})}{B_\tau} 1_{\{t<\tau\le T_n\}} | \mathcal{G}_t\right).$$

CDS spread κ which is agreed at contracts inception (time $t \le T_1$) is chosen in such a way that value of contract (at inception date) is equal to 0: $\mathrm{CDS}(t,\mathcal{T},\kappa) = 0$.

Pricing of CDS is the mainly the issue of determining a CDS spread κ. To find κ we must compute value of two legs, since *fair* CDS spread is given as:

$$\kappa(t,\mathcal{T})1_{\{C(t)\neq K\}} = 1_{\{C(t)\neq K\}}\frac{\mathbf{E}\left(\frac{B_t}{B_\tau}(1-\delta_{C_{\tau-}})1_{\{t<\tau\leq T_n\}}|\mathcal{G}_t\right)}{\mathbf{E}\left(\sum_{k=1}^{n}\frac{B_t}{B_{T_k}}1_{\{\tau>T_k\}}|\mathcal{G}_t\right)}.$$

Theorem 9. *Value of Default leg*

$$B_t\mathbf{E}\left(\frac{1-\delta_{C_{\tau-}}}{B_\tau}1_{\{t<\tau\leq U\}}|\mathcal{G}_t\right) =$$

$$\sum_{i=1}^{K-1} 1_{\{C_t=i\}}\left(\sum_{j=1}^{K-1}(1-\delta_j)\frac{B_t\mathbf{E}\left(\int_t^U \frac{1}{B_u}dG_{t,i,j}(u)|\mathcal{F}_t\right)}{\mathbf{P}(C_t=i|\mathcal{F}_t)}\right).$$

Value of premium leg:

$$\kappa B_t \mathbf{E}\left(\sum_{k=1}^{n} \frac{1}{B_{T_k}} 1_{\{\tau > T_k\}} | \mathcal{G}_t\right) =$$

$$\sum_{i=1}^{K-1} 1_{\{C_t = i\}} \sum_{k=1}^{n} \left(B(t, T_k) - \frac{B_t \mathbf{E}\left(\frac{1}{B_{T_k}} F_{t,i}(T_k) | \mathcal{F}_t\right)}{\mathbf{P}(C_t = i | \mathcal{F}_t)} \right),$$

where

$$F_{t,i}(v) := \mathbf{P}(\tau \leq v, C_t = i \neq K | \mathcal{F}_v) .$$

Useful conditional expectations

Theorem 10. *Let X be a bounded $\mathcal{F}_\infty$ measurable random variable. Then for all $u \geq t$ and all $j \in \mathcal{K}$ we have*

$$\mathbf{E}(X 1_{\{C_u=j\}} | \mathcal{F}_t \vee \mathcal{F}_t^C) = \mathbf{E}(X 1_{\{C_u=j\}} | \mathcal{F}_t \vee \sigma(C_t)) =$$

$$= \sum_{i=1}^{K} 1_{\{C_t=i\}} \frac{\mathbf{E}(X 1_{\{C_u=j, C_t=i\}} | \mathcal{F}_t)}{\mathbf{P}(C_t = i | \mathcal{F}_t)} .$$

Theorem 11. *Let Z be a bounded $\mathbb{F}$ predictable stochastic process. Then we have*

$$\mathbf{E}(Z_\tau 1_{\{t<\tau\le u, C_{\tau-}=j\}}|\mathcal{F}_t \vee \mathcal{F}_t^C) = \mathbf{E}(Z_\tau 1_{\{t<\tau\le u, C_{\tau-}=j\}}|\mathcal{F}_t \vee \sigma(C_t))$$

$$= \sum_{i=1}^{K-1} 1_{\{C_t=i\}} \frac{\mathbf{E}(\int_t^u Z_v dG_{t,i,j}(v)|\mathcal{F}_t)}{\mathbf{P}(C_t = i|\mathcal{F}_t)},$$

where $\qquad G_{t,i,j}(v) := \mathbf{P}(\tau \le v, C_t = i, C_{\tau-} = j|\mathcal{F}_v).$

Theorem 12. *We have following formula*

$$\mathbf{P}(t < \tau \le T, C_{\tau-} = j|\mathcal{F}_\infty \vee \sigma(C_t))1_{\{C_t=i\}} =$$

$$= 1_{\{C_t=i\}} e^{-\int_t^T \lambda(u)du} P_{j,K} \sum_{l=1}^{\infty} \left(\sum_{m=1}^{l} (P^{m-1})_{i,j} \right) \frac{(\int_t^T \lambda(u)du)^l}{l!}.$$

Example 1. *We will consider now special case $K = 3$, and transition matrix P of following form*

$$P = \begin{pmatrix} 0 & a & (1-a) \\ b & 0 & (1-b) \\ 0 & 0 & 1 \end{pmatrix},$$

where $a > 0, b > 0$.
Let

$$A := \{\tau \le T, C_{\tau-} = j, C_0 = i\}, \quad \Lambda(T) := \int_0^T \lambda(s)ds$$

. For $i < j$ the conditional probability of A equals

$$\mathbf{P}(A|\mathcal{F}_\infty \vee \sigma(C_0))1_{\{C_0=i\}} = 1_{\{C_0=i\}} e^{-\Lambda(T)} P_{j,K} \frac{a}{1-ab} \quad \cdot$$

$$\left(e^{\Lambda(T)} - 1 - \Lambda(T) - \frac{\cosh(\sqrt{ab}\Lambda(T)) - 1}{ab} - \frac{\sinh(\sqrt{ab}\Lambda(T))}{(\sqrt{ab})^3} + \frac{\Lambda(T)}{ab} \right)$$

and similarly for $j > i$ we have

$$\mathbf{P}(A|\mathcal{F}_\infty \vee \sigma(C_0))1_{\{C_0=i\}} = 1_{\{C_0=i\}} e^{-\Lambda(T)} P_{j,K} \frac{b}{1-ab} \cdot$$

$$\left(e^{\Lambda(T)} - 1 - \Lambda(T) - \frac{\cosh(\sqrt{ab}\Lambda(T)) - 1}{ab} - \frac{\sinh(\sqrt{ab}\Lambda(T))}{(\sqrt{ab})^3} + \frac{\Lambda(T)}{ab} \right).$$

For $i = j$

$$\mathbf{P}(A|\mathcal{F}_\infty \vee \sigma(C_0))1_{\{C_0=i\}} = 1_{\{C_0=i\}} e^{-\Lambda(T)} P_{j,K} \frac{1}{1-ab} \cdot$$

$$\left(e^{\Lambda(T)} - 1 + \Lambda(T) - \frac{\cosh(\sqrt{ab}\Lambda(T)) - 1}{ab} - \frac{1}{\sqrt{ab}} \sinh(\sqrt{ab}\Lambda(T)) \right).$$

The talk is based on

• Jacek Jakubowski and Mariusz Niewęgłowski, Defaultable bonds with infinite number of Lévy factors, preprint

• Jacek Jakubowski and Mariusz Niewęgłowski, Valuation of credit default swap in the model with rating migration induced by Cox process, in preparation

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