Model-free pricing and hedging of exotic claims related to the local time

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Model-free pricing and hedging

Standard approach:

$$\text{Model} \implies \text{Calibration} \implies \text{Exotics' prices \& hedges}$$

Model-free approach:

$$\text{Market data} \stackrel{\textit{no arbitrage}}{\implies} \text{Price range + robust super-replication}$$

The methodology in short

Aim: price and hedge $F_T = F(S_t : t \leq T)$ – European type exotic claim (typically pathwise increasing, ex. knock-in barrier).

Assumptions: – S_t is a martingale under the risk-neutral measure
– we trust liquid market data

Analysis:

- From call prices at maturity T we can read the distribution $\mu \sim S_T$. Indeed: $\mu(dx) = C''(x)dx$, where $C(x) = \mathbb{E}(S_T - x)^+$.
- $S_t = B_{\Gamma_t}$ is a time-changed Brownian Motion, $B_0 = S_0$. In particular $B_{\Gamma_T} = S_T \sim \mu$ and thus Γ_T is a solution to the *Skorokhod embedding problem*.
- Conversely, for any τ with $B_\tau \sim \mu$, $(B_{t\wedge\tau})$ UI, the process $S_t := B_{\tau\wedge[t/(T-t)]}$ gives a market model consistent with observed call prices.
- The price of F_T consistent with *no arbitrage* is bounded above by $\sup_\tau \mathbb{E}[F(B_t : t \leq \tau)]$, $\tau : B_\tau \sim \mu$, and the bound attained by $S_t := B_{\tau^*\wedge[t/(T-t)]}$.

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The methodology in practice

New aim: Identify the interval of possible prices for F_T and model-independent super-replicating strategies.

Key steps:

- Given μ, find τ such that $B_\tau \sim \mu$.
  (Throughout) we impose that $(B_{t\wedge\tau})$ is a UI martingale, so that $S_t := B_{\tau\wedge[t/(T-t)]}$ is a valid market model.
  Any such τ is a solution to the *Skorokhod embedding problem.*
- Identify the optimal solutions $\rightsquigarrow$ option's price range.
  Root, Azéma-Yor, Perkins...
- Gain pathwise understanding $\rightsquigarrow$ super-replication.

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Approach via pathwise inequalities I

Let $F_T^S = F(S_t : t \leq T)$ be the payoff. Denote μ the law of S_T. We look for a convex H and a predictable (Δ_t^S) such that

$$(*) \qquad F_T^S \leq H(S_T) + \int_0^T \Delta_t^S dS_t, \quad a.s.$$

The payoff F_T is then super-replicated with a portfolio of calls and puts $H(S_T)$ and dynamic trading in the underlying. The cost of the strategy is $\int H(x)\mu(dx)$ and gives an upper bound on the price of F_T^S.

Re-write $(*)$ for Brownian motion using

- the time-change $S_t = B_{\Gamma_t}$, and assuming
- $F_T^S = F_{\Gamma_T}^B$ and $\Delta_t^S = \Delta_{\Gamma_t}^B$, say $\Delta_t^S = \Delta(S_t, F_t^S)$,

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Approach via pathwise inequalities II

- Given the payoff functional F, we thus look for a convex H and $\Delta : \mathbb{R}^2 \to \mathbb{R}$ such that

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  and that for any τ s.t. $(B_{u\wedge\tau})$ is UI we have $\mathbb{E}\int_0^\tau \Delta(B_s, F_s^B) dB_s = 0$.

- We look for an optimal stopping time τ_* such that $B_{\tau^*} \sim \mu$ and $F_{\tau^*}^B = H(B_{\tau^*}) + \int_0^{\tau^*} \Delta(B_u, F_u^B) dB_u$ a.s.

  Then, for $S_t^* := B_{\tau_*\wedge t/(T-t)}$ we have $\mathbb{E}F_T^{S^*} = \int H(x)\mu(dx)$ and thus the upper bound on the price $\mathbb{E}F_T^S \leq \int H(x)\mu(dx)$ is tight.

  A natural candidate is given by: $\tau^* = \inf\{u : B_u \notin (\varphi_-(F_u^B), \varphi_+(F_u^B))\}$.

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One-touch option (Hobson 1998; Brown, Hobson and Rogers 2001)

Consider a one-touch option which pays $O_T = \mathbf{1}_{\sup_{u \le T} S_u \ge b}$.
Hobson '98 devised the pathwise inequality:

$$\mathbf{1}_{\sup_{t \le T} S_t \ge b} \le \underbrace{\frac{(S_T - K)^+}{b - K}}_{\text{calls}} + \underbrace{\frac{(b - S_T)}{b - K} \mathbf{1}_{\sup_{t \le T} S_t \ge b}}_{\text{martingale}},$$

which holds for any $K < b$.
In consequence, a one-touch option is superhedged by: *buy calls with strike* K and *if the stock price reaches b sell forwards*.
Taking optimal $K^* = \Psi_\mu^{-1}(b)$, where $\Psi_\mu(y) = \frac{1}{\mu([y,\infty))} \int_y^\infty x\mu(dx)$ is the barycentre function, yields $\mathbb{E}O_T \le \mathbb{E}(S_T - K^*)^+/(b - K^*) = \mu([\Psi_\mu^{-1}(b)))$.
This bound is attained in the market $S_t := B_{\tau_{AY} \wedge [t/(T-t)]}$, where
$\tau_{AY} = \inf\{u : \Psi_\mu(B_u) \ge \sup_{s \le u} B_s\}$ is the Azéma-Yor solution to the SEP.

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Local time options - introduction

We consider an option paying $F(L_T^S)$, where F is a positive convex function and (L_t^S) is the local time of (S_t) (at a given level, say S_0):

$$L_t^S = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_0^t \mathbf{1}_{0 \le S_u - S_0 \le \epsilon} d\langle S \rangle_u .$$

- Local time can be seen as an approximation of number of downcrossings of an interval or of a *corridor variance swap*: $\epsilon L_t^S \approx \int_0^t \mathbf{1}_{0 \le S_u - S_0 \le \epsilon} d\langle S \rangle_u$.
- Local time calls can secure against big losses from using the naïve hedging such as the *Stop-loss start-gain strategy* (Seidenverg '88): borrow K€ and keep K€ or one stock whichever is worth more. At T pay back K€ and end up with $(S_T - K)^+$... The paradox explained by accumulation of the local time at level K (Carr and Jarrow '90) - that is by the Itô-Tanaka formula.
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Pathwise inequalities with local time (symmetric)

Our method is shift-invariant so we put $S_0 = 0$.
Suppose μ and H are symmetric and let φ be increasing. We have $H(b) \geq H'(a)(|b| - a) + H(a)$ so that

$$\begin{aligned} H(B_t) &\geq H'(\varphi(L_t))|B_t| - H'(\varphi(L_t))\varphi(L_t) + H(\varphi(L_t)) \\ &= H'(\varphi(L_t))|B_t| - \int_0^{L_t} H'(\varphi(x))dx + \theta_H(L_t) \\ &= \int_0^t H'(\varphi(L_u))sgn(B_u)dB_u + \theta_H(L_t) \end{aligned}$$

The aim now is to find H, φ such that $\theta_H = F$ and $\tau_\varphi = \inf\{t : |B_t| = \varphi(L_t)\}$ embeds μ: $B_{\tau_\varphi} \sim \mu$. We then have

$$\mathbb{E}F(L_\tau) \leq \mathbb{E}F(L_{\tau_\varphi}) = \int H(x)\mu(dx), \quad \text{for all } \tau : B_\tau \sim \mu.$$

Explicit solution for the symmetric case

This is achieved explicitly:

$$\varphi^{-1}(x) = \int_0^x \frac{s}{\overline{\mu}(s)} \mu(ds); \quad H'(b) = \frac{1}{\overline{\nu}(\varphi^{-1}(b))} \int_{\varphi^{-1}(b)}^{\infty} F'(m)\nu(dm),$$

where $\overline{\nu}(l) = \exp\left(-\int_0^l \frac{dm}{\varphi(m)}\right)$.

Furthermore, for any stopping time τ with $B_\tau \sim \mu$ and $(B_{t\wedge\tau})$ UI martingale $\mathbb{E}\int_0^\tau H'(\varphi(L_u))sgn(B_u)dB_u = 0$.

Example:

Suppose $\overline{\mu}(l) = e^{-\alpha l}/2$ and $F(l) = \kappa l^{3/2}$. Then $H_{F,\phi}(x) = \kappa(3\sqrt{\alpha}/4\sqrt{2})x^2 + \kappa(3/2\sqrt{2\alpha})|x|$.

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Pathwise inequalities with local time (general)

Let now μ be any centered probability measure and H a convex positive function. Consider $\varphi_- : \mathbb{R}_+ \to \mathbb{R}_-$ and $\varphi_+ : \mathbb{R}_+ \to \mathbb{R}_+$ decreasing and increasing functions respectively.
Put $\gamma_\pm(l) = H'(\varphi_\pm(l))$ and $\theta_\pm(l) = H(\varphi_\pm(l)) - \varphi_\pm(l)\gamma_\pm(l)$. Then

$$\begin{aligned} H(B_t) &\geq \gamma_+(L_t)B_t^+ - \gamma_-(L_t)B_t^- + \theta_+(L_t)\mathbf{1}_{B_t\geq 0} + \theta_-(L_t)\mathbf{1}_{B_t<0} \\ &= M_t^{H,\varphi} + \Gamma(L_t) + \theta_+(L_t)\mathbf{1}_{B_t\geq 0} + \theta_-(L_t)\mathbf{1}_{B_t<0}, \end{aligned}$$

where $\Gamma(l) = \int_0^l(\gamma_+(m) - \gamma_-(m))/2\; dm$ and
$M_t^{H,\phi} = \gamma_+(L_t)B_t^+ - \gamma_-(L_t)B_t^- - \Gamma(L_t)$ is a local martingale.
Given μ and F we are able to spell out explicitly $H, \varphi_\pm$ such that $\theta_- = \theta_+ = \theta$ and $F(l) = \Gamma(l) + \theta(l)$. In consequence

$$H(B_t) - M_t^{H,\varphi} \geq F(L_t), \quad t \geq 0 \; a.s.$$

An equality achieved at Vallois' stopping time
$\tau_\varphi = \inf\{t > 0 : B_t \notin (\varphi_-(L_t), \varphi_+(L_t))\}$, and $B_{\tau_\varphi} \sim \mu$.

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Pricing argument revisited

The forward price S_t is a martingale and thus $S_t = B_{\Gamma_t}$, $B_0 = S_0$, with $S_T = B_{\Gamma_T} \sim \mu$. In consequence

$$\mathbb{E}F(L_T^S) \leq \mathbb{E}H(S_T) = \int H(x)\mu(dx) =: \Theta$$

and the upper bound is attained for $S_t := B_{\tau_\varphi \wedge t/(T-t)}$.
$F(L_T^S)$ is hedged with a static position in calls and puts $H(S_T) = \int_{S_0}^{\infty}(S_T - K)^+ H''(dK) + \int_0^{S_0}(K - S_T)^+ H''(dK)$ and dynamic trading V_t with $dV_t = -\Delta_t dS_t$,

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A selling price $\tilde{\Theta} < \Theta$ can only be justified if the forward price process is known to belong to some subclass of models. Even in this case the seller can still use the hedging mechanism described above and be certain that his potential loss is bounded below by $\Theta - \tilde{\Theta}$ regardless of all other factors.

Pricing argument revisited

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Questions:

- Sparse data: finite family of strikes, intermediate data
- Similar solution for options paying a convex function of realized volatility
- Quantitative comparison with Markovian setup

What else can we do?

We developed a.s. inequalities of the form

$$\begin{aligned} H(B_t) &\geq M_t + F(L_t) \\ H(B_\tau) &= M_\tau + F(L_\tau), \quad B_\tau \sim \mu \end{aligned}$$

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I. Introduction (*understanding the construction*):
μ & $H \Longrightarrow F$

II. Finance and Skorokhod embeddings:
μ & $F \Longrightarrow H$

III. Optimal stopping problem:
F & $H \Longrightarrow \tau$

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Optimal stopping: In the class of τ s.t. $\mathbb{E} M_\tau = M_0$,

$$\sup_\tau \mathbb{E}\left[F(L_\tau) - \int_0^\tau \beta(B_s) ds\right] = \sup_\tau \mathbb{E}\left[F(L_\tau) - H(B_\tau)\right] \leq -M_0,$$

where $2\beta(x) = H''(x)$.

The exact solution

Theorem

Suppose $F \in C^1$ *and* $F'(l) > 0$, $l > 0$, *and that* $\beta(\cdot) > 0$ *is continuous. Define*

$$V = \sup_{\tau \in \mathcal{T}_\beta} \mathbb{E}\left[F(L_\tau) - \int_0^\tau \beta(B_s)\, ds\right]. \tag{1}$$

Suppose there exists a solution (φ_-, φ_+) *on* $\mathbb{R}$ *to*

$$\begin{gathered} \varphi'_\pm(l) = \frac{\frac{1}{2}(H'(\varphi_+(l)) - H'(\varphi_-(l))) - F'(l)}{\varphi_\pm(l) H''(\varphi_\pm(l))}, \\ H(\varphi_+(0)) - \varphi_+(0)H'(\varphi_+(0)) = H(\varphi_-(0)) - \varphi_-(0)H'(\varphi_-(0)). \end{gathered} \tag{2}$$

Then there is a minimal solution (φ_-, φ_+) *which does not hit the origin and*

$$V = F(0) + \varphi_+(0)H'(\varphi_+(0)) - H(\varphi_+(0))$$

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If every solution to (2) *hits the origin and* $\int_{\mathbb{R}_\pm} |z|\beta(z)dz = \infty$ *then* $V = \infty$.

THANK YOU FOR STAYING TILL SATURDAY