Issuance Costs and Stock Return Volatility*

Jean-Paul Décamps† Thomas Mariotti‡

Jean-Charles Rochet§ Stéphane Villeneuve¶

First Draft: July 2006
This draft: February 2007.

Abstract

Most of the theoretical literature in corporate finance neglects the costs that firms incur when they raise new capital. The paper shows that when these issuance costs are taken into account, stock prices naturally exhibit heteroskedasticity. Specifically we find that stock prices satisfy a property that Black (1976) has called the leverage effect: when the price of the stock falls, the volatility of its return increases. In the limit case where issuance costs are zero we obtain the constant volatility case of Black-Scholes-Merton.

*We thank Bruno Biais and Catherine Casamatta for many useful discussions.

†Université de Toulouse 1, GREMAQ (UMR CNRS 5604) and IDEI, 21 Allée de Brienne, 31000 Toulouse, France, and Europlace Institute of Finance. E-mail: decamps@cict.fr.

‡Université de Toulouse 1, GREMAQ (UMR CNRS 5604) and IDEI, 21 Allée de Brienne, 31000 Toulouse, France, and CEPR. E-mail: mariotti@cict.fr.

§Université de Toulouse 1, GREMAQ (UMR CNRS 5604) and IDEI, 21 Allée de Brienne, 31000 Toulouse, France, and CEPR. E-mail: rochet@cict.fr

¶Université de Toulouse 1, GREMAQ (UMR CNRS 5604), 21 Allée de Brienne, 31000 Toulouse, France, and Europlace Institute of Finance. E-mail: stephane.villeneuve@univ-tlse1.fr.

1. Introduction

A customary assumption in the theory of corporate finance is that the shareholders of a firm can costlessly inject new liquidity in the firm whenever they want to do so. This allows for example Black and Cox (1976) and Leland (1994)[1] to determine the value of corporate debt in a context where default is always strategic. In these models indeed, default only occurs when shareholders find it optimal to exercise their limited liability option. In the neighborhood of the closure threshold, this strategy implies that shareholders constantly inject new liquidity up to the point where current operating losses outweigh expected future profitability.

In practice, however, the only way in which a publicly traded firm can tap its shareholders is by organizing a seasoned equity offering (SEO), which is far from being a costless operation. For example, Lee et al. (1996) report the average costs of raising capital for U.S. corporations from 1990 to 1994, and find that the direct costs of SEOs vary from 13% of the proceeds of the issuing[2] (for small issues) to 3.1% (for large ones) with an average of 7.1%. As a result, such new issues of equity are relatively unfrequent, and typically involve substantial amounts.

The objective of the present paper is to show that when these issuance costs are taken into account, stock prices naturally exhibit heteroskedasticity even when the volatility of earnings is constant. In particular our model predicts that when the price of a stock falls, the volatility of its return should increase. The explanation is simple: the presence of issuance costs implies that the marginal value of cash within the firm increases after a fall in the price of the stock. Indeed, one more unit of cash within the firm decreases the risk of having to incur issuance costs in a near future, an event that becomes more likely after a fall in the stock price. By contrast, after an increase in the stock price (due for example to unexpected operating profits, and thus to an increase in cash reserves), the marginal value of cash within the firm decreases. Further shocks on profitability have therefore a larger impact on stock price following a negative initial shock than following a positive initial shock.

This is completely in line with the classical "leverage effect" first identified by Black (1976), who states p. 179: "when things go badly for the firm, its stock price will fall, and the volatility of the stock will go up". As pointed out by Black, this remains true even if the firm has no debt, as long as it has "operating leverage" i.e. a potential need to finance future operating costs by external funds.

In fact, our model predicts more than the usual leverage effect which asserts that volatility of stock returns increases after a negative shock. Indeed it predicts that the dollar volatility of stock prices increases after a negative shock on stock prices. This feature is documented by Black (1976), who provides no explanation for it. In our model, it is an immediate consequence of the fact that stock prices are a concave function of the level of cash reserves within the firm. The same shock on earnings has thus a greater impact on stock prices (and not only on stock returns) when the stock price is initially low than when it is high. This concavity provides also a natural explanation of why risk management activities might increase shareholder value. By contrast, traditional models without issuance costs typically predict (because of the limited liability option) the convexity of stock prices with respect to cash reserves with the unpalatable consequences that risk management activities always decrease shareholder value, and similarly that volatility of stock prices should decrease after

[1] See also Leland (1998), Leland and Toft (1996), and Mella-Barral and Perraudin (1997).

[2] These costs include the fees paid to the investment banks as well as other direct expenses such as legal and auditing costs.

a negative shock!

Interestingly, in the benchmark case where there are neither issuance costs nor solvency shocks (so that the limited liability option has no value) we find that the value of the firm is linear with respect to cash reserves, and thus that the volatility of stock prices is constant. This highlights a new, and somewhat unexpected connexion between the Black-Scholes-Merton option pricing model (that assumes constant volatility of stock returns) and the absence of transaction costs in financial markets. It is indeed usually pointed out that the arbitrage pricing methods that underlie the Black-Scholes-Merton formula are only valid in the absence of transaction costs on secondary markets. We point out here that the constant volatility assumption can only be consistent with the absence of transaction costs on primary markets.

There is of course a large literature on the relation between transaction costs and volatility on financial markets. But again, it focuses on secondary markets. Going back to the controversy on the (de)stabilizing role of speculation, two opposing strands of the literature have co-existed. Some, like Keynes and Tobin, argue that speculation may have destabilizing effect and thus that increasing transaction costs ("putting a grain of sand on the wheels of finance"...) might have beneficial effects by decreasing volatility. Others, like Friedman, claim on the contrary that, at least in the long run, higher transaction costs increase volatility on stock markets. Using recent data from a natural experiment on the French stock market, Hau (2006) finds indeed that transaction costs and volatility are positively related. Our model predicts a similar feature, but in the context of primary markets. We find that when transaction costs for issuing new securities are high, the survival of profitable firms may be jeopardized by liquidity problems. This is because the continuation value of a firm, even when it is profitable, may be insufficient to outweigh the costs of raising new capital. Even when transaction costs are low, and the firm's survival is not at stake, a further decrease in issuance costs will make the firm's value less dependent on current liquidity problems,[3] and thus its stock price less volatile.

Our paper is also related to the more mathematical literature on optimal liquidity management and dividend policy in the presence of frictions on financial markets, initiated by Karatzas et al. (1986). Our setup imbeds not only the pure dividend distribution model of Jeanblanc-Picqué and Shiryaev (1995) and Radner and Shepp (1996), but also the more recent dividend/issuance models of Sethi and Taksar (2002) and Løkka and Zervos (2005).

The rest of the paper is organized as follows. Section 2 presents the model. Section 3 introduces the first-best benchmark, that would be obtained in the absence of any transaction costs. Section 4 characterizes the optimal issuance and dividend policies. We obtain a system of variational inequalities that the value function must satisfy. Then we show that this system has a unique regular solution and we establish that this solution is indeed the optimal value function. Section 5 examines the implications of this optimal policy on the behavior of stock prices. Section 6 concludes.

2. The Model

The following notation will be maintained throughout the paper. Time is continuous, and labelled by $t \geq 0$. Uncertainty is modelled by a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ over

[3] One limitation of our model is that the technology is exogenous. The size of physical assets and the profitability of the firm are given. With an endogenous technology, a decrease in transaction costs might encourage firms to choose larger or riskier investment.

which is defined a standard Wiener process $W = \{W_t; t \geq 0\}$. We let $\mathcal{F}^W = \{\mathcal{F}_t^W; t \geq 0\}$ be the $\mathbb{P}$–augmentation of the filtration $\{\sigma(W_s; s \leq t); t \geq 0\}$ generated by W, and $\mathbb{E}$ be the expectation operator associated to $\mathbb{P}$.

A firm has a single and fixed size investment project that generates random cash-flows over time. The cumulative cash-flows $R = \{R_t; t \geq 0\}$ evolve according to an arithmetic Brownian motion with strictly positive drift μ and volatility σ,

$$R_0 = 0, \quad dR_t = \mu dt + \sigma dW_t \tag{1}$$

for all $t \geq 0$. Note in particular that the volatility of cash-flows is constant in our model. In the absence of financial frictions, this will imply a constant volatility of stock returns, see Section 3 below. Our analysis could be extended to more general diffusion processes for cash-flows, for instance along the lines of Shreve, Lehoczky and Gaver (1984). However, the main point of the paper is that issuance costs are a source of heteroskedasticity in stock returns, which is most manifest if we take as primitive a homoskedastic cash-flow process such as (1).

At each date, the project can be continued or liquidated. For simplicity, the liquidation value is set equal to 0. The firm is held by a diffuse basis of risk-neutral shareholders, with limited liability, and has no access to credit. As a result of this, its cash reserves must always remain non-negative. Equity holders discount future payments at the risk free interest rate rate $r > 0$.

Consider now the issuance policy of the firm. At each date, the firm can retain part of its earnings, or issue new equity. As discussed in the introduction, issuing equity typically involves substantial costs. Using detailed evidence on the flotation costs associated to various methods of raising new equity, Smith (1977) documents significant economies of scale in the issuance process: equity issues are very costly for small operations, and this is reflected in a declining average cost of financing. To capture these economies of scale in issuing activity, we posit a combination of variable and fixed issuance costs. First, as in Gomes (2001), Sethi and Taksar (2002), Hennessy and Whited (2005) or Løkka and Zervos (2005), equity issues have a constant marginal cost, which can for instance result from a proportional brokerage commission. Thus, for each dollar of new equity issued, the firm receives $1/p$ dollars in cash, where $p > 1$ measures the proportional transaction cost. In addition, a distinctive feature of our model is that each issue of equity involves a fixed transaction cost $f > 0$. The presence of this fixed cost implies that the firm will raise new equity in lumpy and infrequent issues. To obtain an order of magnitude for these issuance costs, one can fit a linear cost function to Smith's (1977) data, which cover equity issues in the U.S. between 1971 and 1975. A conservative estimate yields $p = 1.028$ and $f = 80,000$ dollars, see Gomes (2001).

Because of the presence of a fixed issuance cost, the firm's issuance policy can without loss of generality be described by an increasing sequence $(\tau_n)_{n\geq 1}$ of $\mathcal{F}^W$–adapted stopping times representing the successive dates at which new equity is issued, along with a sequence $(i_n)_{n\geq 1}$ of $(\mathcal{F}_{\tau_n}^W)_{n\geq 1}$–adapted non-negative random variables representing the total issuance proceeds at these dates. It should be noted that we allow the stopping times $(\tau_n)_{n\geq 1}$ to be infinite, which may for instance happen when the issuance costs are so high that the firm finds it optimal to never issue new equity.

At any date $t \geq 0$,

$$I_t = \sum_{n\geq 1} i_n 1_{\{\tau_n \leq t\}} \tag{2}$$

corresponds to the total issuance proceeds up to and including date t, while:

$$F_t = \sum_{n \geq 1} f 1_{\{\tau_n \leq t\}} \tag{3}$$

corresponds to the total fixed issuance costs incurred up to and including date t. We denote by $I = \{I_t; t \geq 0\}$ and $F = \{F_t; t \geq 0\}$ the processes defined by (2)–(3), which are $\mathcal{F}^W$–adapted by construction.

What is not retained from earnings is paid out as dividends. Let $L = \{L_t; t \geq 0\}$ be the cumulative dividend process. It is assumed that L is $\mathcal{F}^W$–adapted and right-continuous, with $L_0 = 0$, and that it is non-decreasing, reflecting the shareholders' limited liability. Thus the firm can distribute (positive) dividends at no cost, but has to incur issuance costs if it wants to collect new funds from (new or old) shareholders. The net cash inflow from (or to) shareholders is thus $\frac{1}{p}\, dI_t - dF_t - dL_t$, while it would just be $dI_t - dL_t$ in the absence of transaction costs (that is when $p = 1$ and $f = 0$).

In addition to these issuance costs, we also introduce a second type of friction, in the spirit of the free cash-flow theory of Jensen (1986). We assume that managers can divert a (constant) fraction λ of cash reserves per unit of time. This cash flow diversion assumption is standard in the theoretical literature, which has investigated the features of remuneration contracts that provide managers with the incentives not to divert cash (for recent examples of this approach, see Oded 2005 or Biais et al. 2007). In the present paper, we do not delve into these difficult governance issues, and simply take this cash diversion property as given. Its consequence is that it is costly for shareholders to leave idle cash reserves within the firm. The optimal level of cash reserves results from a trade-off between transaction costs on primary markets (that create a precautionary demand for cash, of the Baumol Tobin type) and the agency costs within the firm (that generate a cost of holding cash). As we see below, the liquidity management problem would become trivial in the absence of either type of costs. Without issuance costs it would become optimal to hoard no cash at all, and to distribute all net earnings as (positive or negative) dividends. Without agency costs à la Jensen, it would become optimal to fully retain earnings and never to distribute any dividends. The cash reserves $M = \{M_t; t \geq 0\}$ of the firm then evolve according to:

$$M_{0-} = m, \quad dM_t = (r - \lambda) M_t dt + dR_t + \frac{1}{p}\, dI_t - dF_t - dL_t \tag{4}$$

for all $t \geq 0$. The processes R, I and F are defined by (1)–(3), and $m \geq 0$ represents the initial cash holdings of the firm. Cash reserves are invested in money market securities (or a bank account) that are remunerated at the riskless rate r. The parameter $\lambda > 0$ is a constant exogenous agency cost that represents the fraction of reserves that is diverted by managers.

Since the firm must always hold non-negative cash reserves, (4) represents the dynamics of the cash reserves up to the time:

$$\tau_B = \inf\{t \geq 0 \,|\, M_t < 0\} \tag{5}$$

at which the firm goes eventually bankrupt. It should be noted that we allow the stopping time τ_B to be infinite, which may for instance happen when the issuance costs are so low that the firm finds it optimal to always issue new equity when it runs out of cash, thereby avoiding bankruptcy.

Given an issuance policy $((\tau_n)_{n\geq 1}, (i_n)_{n\geq 1})$, a dividend policy L, and initial cash reserves m, the value of the firm can be computed as:

$$v(m; (\tau_n)_{n\geq 1}, (i_n)_{n\geq 1}, L) = \mathbb{E}^m \left[\int_0^{\tau_B} e^{-rt} \left(dL_t - dI_t \right) \right], \tag{6}$$

where $(\tau_n)_{n\geq 1}$, $(i_n)_{n\geq 1}$, I, L and τ_B are related by (1)–(5), and $\mathbb{E}^m$ is the expectation operator induced by the process M starting at m. Note that, by construction,

$$\mathbb{E}^m \left[\int_0^{\tau_B} e^{-rt} dI_t \right] = \mathbb{E}^m \left[\sum_{n\geq 1} e^{-r\tau_n} i_n 1_{\{\tau_n \leq \tau_B\}} \right].$$

We define the corresponding value function as:

$$V^*(m) = \sup_{(\tau_n)_{n\geq 1}, (i_n)_{n\geq 1}, L} \{ v(m; (\tau_n)_{n\geq 1}, (i_n)_{n\geq 1}, L) \} \tag{7}$$

for all $m \geq 0$, where the supremum in (7) is taken over all admissible issuance and dividend policies. It is technically convenient to extend the value function V^* to $(-\infty, 0)$ by setting $V^*(m) = 0$ for all $m < 0$. This allows us to put no restrictions on the issuance proceeds $(i_n)_{n\geq 1}$ besides that they remain non-negative.

Our objective in the remainder of the paper is twofold. First, we characterize the optimal value function, as well as the optimal issuance and dividend policies that maximize the value of the firm. Second, we show how, in the presence of issuance costs, these optimal policies translate into a uniquely determined stock price process whose dynamics we fully characterize, allowing us to derive a rich set of testable asset pricing implications.

3. The First-Best Benchmark

Before considering how issuance costs affect the firm's issuance and dividend policies, as well as the dynamics of stock prices, we examine a benchmark case in which such costs are absent, that is $p = 1$ and $f = 0$ (but agency costs à la Jensen are still present: $\lambda > 0$). In this first-best environment, the firm is never liquidated, and its value at date 0 is simply the sum of its initial cash reserves and of the present value of its future cash-flows:

$$\hat{V}(m) = m + \mathbb{E}^m \left[\int_0^{\infty} e^{-rt} \left(\mu dt + \sigma dW_t \right) \right] = m + \frac{\mu}{r}. \tag{8}$$

In the absence of issuance costs, hoarding cash reserve does not create any value for shareholders while it is costly due to cash diversion by the manager. It is therefore optimal for the firm to distribute all its initial cash reserves m as a special dividend at date 0, and to hold no cash reserves beyond that date. In the absence of other financial frictions, the Modigliani and Miller (1958) logic applies, so that we have many degrees of freedom in designing issuance and dividend processes $\hat{I} = \{\hat{I}_t; t \geq 0\}$ and $\hat{L} = \{\hat{L}_t; t \geq 0\}$ that deliver the value (8). Indeed, as can be seen from (6), the only variable that matters is the difference $\hat{L} - \hat{I}$. To illustrate this point, suppose for instance that, after date 0, the flow of dividends stays constant per unit of time. Since the firm distributes all its cash reserves m at date 0, this means that the dividend process can be written as:

$$\hat{L}_t = m 1_{\{t=0\}} + lt \tag{9}$$

for all $t \geq 0$, where $l > 0$ is some arbitrary constant. Allowing for share repurchases, that is, for a non-monotonic issuance process $\hat{I}$, and taking advantage of (4) with $p = 1$ and $f = 0$, the requirement that cash reserves be constant and equal to 0 after date 0 yields:

$$\hat{I}_t = (l - \mu)t - \sigma W_t \tag{10}$$

for all $t \geq 0$. Formula (10) just means that new shares are issued (or repurchase) to offset exactly the difference between dividends (ldt) and earnings ($\mu dt + \sigma dW_t$), so that cash reserves are maintained at zero.

Applying formula (6) to $(\hat{L}, \hat{I})$ as defined by (9) and (10), and noting that the integral on the right-hand side of (6) includes $\hat{L}_0 = m$, it is immediate to check that the pair $(\hat{I}, \hat{L})$ defined by (9)–(10) delivers the first-best value (8), independently of the dividend flow l.

Now turn to the dynamics of stock prices in this frictionless market. Let $\hat{S} = \{\hat{S}_t; t \geq 0\}$ be the process describing the ex-dividend price of a share in the firm, and $\hat{N} = \{\hat{N}_t; t \geq 0\}$ the process modelling the number of outstanding shares of the firm. Without loss of generality, one can set $\hat{N}_0 = 1$. After date 0, the market capitalization $\hat{N}_t \hat{S}_t$ of the firm stays constant at a level μ/r. Hence, at any date $t > 0$, the following relation holds:

$$d\hat{I}_t = d(\hat{N}_t \hat{S}_t) - \hat{N}_t d\hat{S}_t = -\hat{N}_t d\hat{S}_t = -\frac{\mu}{r} \frac{d\hat{S}_t}{\hat{S}_t}, \tag{11}$$

where the first and second equalities reflect that the flow of funds into the firm, whether positive or negative, is entirely absorbed by existing shareholders, as issuing new equity has no impact on the value of the firm.

Assuming as above a constant flow of dividend $l > 0$ per unit of time after date 0, it follows from (10)–(11) that:

$$\frac{d\hat{S}_t}{\hat{S}_t} = r\left(1 - \frac{l}{\mu}\right)dt + \frac{\sigma r}{\mu}\, dW_t \tag{12}$$

for all $t > 0$. Using the fact that, after date 0, the dividend per share and per unit of time is given by $l/\hat{N}_t = lr\hat{S}_t/\mu$, which is strictly positive as $l > 0$, the log-normal dynamics (12) for the stock price implies that:

$$\hat{S}_t = \mathbb{E}\left[\int_t^\infty e^{-r(s-t)}\, \frac{lr\hat{S}_s}{\mu}\, ds \,|\, \mathcal{F}_t\right] = \mathbb{E}\left[\int_t^\infty e^{-r(s-t)}\, \frac{l}{\hat{N}_s}\, ds \,|\, \mathcal{F}_t\right] \tag{13}$$

for all $t > 0$. That is, the stock price is simply the present value of future dividends per share, reflecting the fact that shareholders are risk-neutral in our model.

Formulas (12)–(13) can be generalized to more general dividend processes. Indeed, fix some non-decreasing process $\hat{L}$ such that $\hat{L}_0 = m$, which for simplicity we shall suppose continuous. To obtain the analogue of formula (13), we must ensure that the stock price exhibits no bubble, in the sense that it grows at an expected rate strictly lower than r as in (12). This will be the case if the dividend process $\hat{L}$ grows at a fast enough rate. One has the following result.

Proposition 1. *Suppose that there are no financial frictions, that is $p = 1$ and $f = 0$. Consider a dividend process $\hat{L}$ such that:*

$$\lim_{T\to\infty} \left\{\mathbb{E}\left[\exp\left(-\frac{1}{2}\left(\frac{\sigma r}{\mu}\right)^2 T + \frac{\sigma r}{\mu}\, W_T - \frac{r}{\mu}\, \hat{L}_T\right)\right]\right\} = 0. \tag{14}$$

Then, at any date $t > 0$, the market capitalization of the firm is:

$$\hat{N}_t \hat{S}_t = \frac{\mu}{r}, \tag{15}$$

the instantaneous return on stocks satisfies:

$$\frac{d\hat{S}_t + d\hat{L}_t / \hat{N}_t}{\hat{S}_t} = r dt + \frac{\sigma r}{\mu} \, dW_t, \tag{16}$$

and the stock price is the present value of future dividends per share:

$$\hat{S}_t = \mathbb{E}\left[\int_t^\infty e^{-r(s-t)} \, \frac{1}{\hat{N}_s} \, d\hat{L}_s \,|\, \mathcal{F}_t \right]. \tag{17}$$

The stock price dynamics (12) and its generalization (16) are consistent with the log-normal specification postulated by Black and Scholes (1973) and Merton (1973). It should be noted that, while the dividend process is indeterminate, the constant volatility of stock returns is a direct implication of the fact that the market capitalization of the firm stays constant over time. As we shall see in Section 4, stock prices no longer exhibit this feature when issuance costs are taken into account.

4. The Optimal Issuance and Dividend Policies

In this section, we characterize the optimal issuance and dividend policies when there are issuance costs. We first derive heuristically a system of variational inequalities for the value function V^*. We then prove that this system has a solution satisfying appropriate regularity conditions. Finally, a verification argument establishes that this solution coincides with V^*, from which the optimal issuance and dividend policies can be inferred.

4.1. A Heuristic Derivation of the Value Function

To derive the system of variational inequalities satisfied by V^*, suppose for the moment that V^* is twice continuously differentiable over $(0, \infty)$, with a uniformly bounded derivative, and that for all $m \geq 0$ there exists an optimal policy that attains the supremum in (7). Fix some $m > 0$. The policy that consists in distributing $l \in (0, m)$ worth of dividends, and then immediately executing the optimal policy associated with cash reserves $m - l$ must yield no more than the optimal policy:

$$V^*(m) \geq V^*(m - l) + l.$$

Subtracting $V^*(m - l)$ from both sides of this inequality, dividing through by l and letting l go to 0 yields that:

$$V^{*\prime}(m) \geq 1 \tag{18}$$

for all $m > 0$, as is usual in dividend distribution models. Next, the policy that consists in issuing $i > 0$ worth of equity, and then immediately executing the optimal policy associated with cash reserves $m + i/p - f$ must yield no more than the optimal policy:

$$V^*(m) \geq V^*\left(m + \frac{i}{p} - f \right) - i.$$

Thus denoting $m + i/p$ by m', one must have:

$$V^*(m) \geq \sup_{m' \in [m,\infty)} \left\{ V^*(m' - f) - p(m' - m) \right\} \tag{19}$$

for all $m > 0$. Finally, consider the policy that consists in abstaining from issuing new equity and from distributing any dividends for $t \wedge \tau_B \equiv \min\{t, \tau_B\}$ units of time, where $t > 0$, after which the optimal policy associated to the cash reserves $m + \int_0^{t\wedge\tau_B} [(\mu + (r - \lambda)M_s)ds + \sigma dW_s]$ is executed. Again, this policy must yield no more than the optimal policy:

$$\begin{aligned} V^*(m) &\geq \mathbb{E}^m \left[e^{-r(t\wedge\tau_B)}\, V^* \left(m + \int_0^{t\wedge\tau_B} [(\mu + (r - \lambda)M_s)ds + \sigma dW_s] \right) \right] \\ &= V^*(m) + \mathbb{E}^m \left[\int_0^{t\wedge\tau_B} e^{-rs} \left[-rV^*(M_s) + (\mu + (r - \lambda)M_s)V^{*\prime}(M_s) + \frac{\sigma^2}{2}\, V^{*\prime\prime}(M_s) \right] ds \right], \end{aligned}$$

where the second inequality follows from Itô's Lemma. Letting t go to 0 results in:

$$-rV^*(m) + \mathcal{L}V^*(m) \leq 0 \tag{20}$$

for all $m > 0$, where the infinitesimal generator $\mathcal{L}$ is defined as:

$$\mathcal{L}u(m) = (\mu + (r - \lambda)m)u'(m) + \frac{\sigma^2}{2}\, u''(m). \tag{21}$$

We shall refer to (18)–(20) as the fundamental system of variational inequalities satisfied by V^*. To move forward, we make the following guess about the optimal strategy. Consider first the issuance policy. Because of the fixed transaction cost associated with equity issues, it is natural to expect that these should be delayed as much as possible. This suggests that, if any issuance activity takes place at all, it must be at the times when the cash reserves hit 0 so as to avoid bankruptcy. Because of the stationarity of the model, we postulate that the optimal issuance policy then consists in issuing a constant dollar amount of equity, or in abstaining from issuing equity altogether, which triggers bankruptcy. As a result of this, the value of the firm when it runs out of cash is:

$$V^*(0) = \left[\max_{i \in [0,\infty)} \left\{ V^* \left(\frac{i}{p} - f \right) - i \right\} \right]^+, \tag{22}$$

where $x^+ \equiv \max\{x, 0\}$. Denote by i^* a solution to the maximization problem in (22). It will turn out that i^* is uniquely determined at the optimum. It may be that $i^* = 0$, in which case the firm abstains from issuing equity, and $V^*(0) = 0$.[4] Whenever $i^* > 0$, the firm issues new equity when it runs out of cash, and $V^*(0) = V^*(i^*/p - f) - i^* > 0$. The quantity $m_0^* = i^*/p - f > 0$ then represents the post issuance level of the cash reserves.

Consider now the dividend policy. In line with standard dividend distribution models, it is natural to expect dividends to be distributed as soon as cash reserves hit or exceed a boundary $m_1^* > 0$. This implies that, for all $m \geq m_1^*$,

$$V^{*\prime}(m) = 1. \tag{23}$$

[4] Remember our convention that $V^*(m) = 0$ when $m < 0$.

Since V^* is postulated to be twice continuously differentiable over $(0,\infty)$, (23) implies that, in addition, the following super contact condition[5] holds at the dividend boundary m_1^*:

$$V^{*\prime\prime}(m_1^*) = 0. \tag{24}$$

When cash reserves lie in $(0, m_1^*)$, no issuance or dividend activity take place, and (20) holds as an equality. It then follows from (21) and (23)–(24) that $V^*(m_1^*) = (\mu + rm_1^*)/r$. We are thus led to the problem of finding a function V, along with a threshold $m_1 > 0$, that solve the following variational system:

$$V(m) = 0; \quad m < 0, \tag{25}$$

$$V(0) = \left[\max_{m\in[-f,\infty)} \{V(m) - p(m+f)\}\right]^+; \tag{26}$$

$$-rV(m) + \mathcal{L}V(m) = 0; \quad 0 < m < m_1, \tag{27}$$

$$V(m) = \frac{\mu + (r-\lambda)m_1}{r} + m - m_1; \quad m \geq m_1. \tag{28}$$

Note that V may be discontinuous, with a positive jump at 0. However, the maximum in (26) is always attained since V is upper semicontinuous. We shall then proceed as follows. First, we prove that there exists a unique solution V to (25)–(28) that is twice continuously differentiable over $(0,\infty)$. It is then easy to check that V satisfies the variational inequalities (18)–(20) over $(0,\infty)$. One can finally infer from this that V coincides with the value function V^* for problem (7).

4.2. Solving the Variational Inequalities

We solve (25)–(28) as follows. First fix some $m_1 > 0$, and consider the following boundary value problem over $[0, m_1]$:

$$-rV(m) + \mathcal{L}V(m) = 0; \quad 0 \leq m \leq m_1, \tag{29}$$

$$V'(m_1) = 1, \tag{30}$$

$$V''(m_1) = 0. \tag{31}$$

Standard existence results for linear second-order differential equations yield that (29)–(31) has a unique solution over $[0, m_1]$, which we denote by V_{m_1}. By construction, this solution satisfies $V_{m_1}(m_1) = (\mu + (r-\lambda)m_1)/r$. Extending linearly V_{m_1} to $[m_1, \infty)$ as in (28), we obtain a twice continuously differentiable function over $[0,\infty)$, which we denote again by V_{m_1}. The following lemma establishes key monotonicity and concavity properties of V_{m_1}.

Lemma 1. *$V'_{m_1} > 1$ and $V''_{m_1} < 0$ over $[0, m_1)$.*

[5] See Dumas (1991) for an insightful discussion of the super contact condition as an optimality condition for singular control problems.

Now observe that if there exists a solution V to (25)–(28) that is twice continuously differentiable over $(0,\infty)$, then by construction, V must coincide with some V_{m_1} over $[0,\infty)$ for an appropriate choice of m_1. This choice is in turn dictated by the boundary condition (26) that V must satisfy at 0. It is therefore crucial to examine the behavior of V_{m_1} and V'_{m_1} at 0. One has the following result.

Lemma 2. *$V_{m_1}(0)$ is a strictly decreasing and concave function of m_1, and $V'_{m_1}(0)$ is a strictly increasing and convex function of m_1.*

Since $\lim_{m_1 \downarrow 0} V_{m_1}(0) = \mu/r > 0$ and $\lim_{m_1 \downarrow 0} V'_{m_1}(0) = 1 < p$, it follows from Lemma 2 that there exists a unique $\hat{m}_1 > 0$ such that $V_{\hat{m}_1}(0) = 0$, and that there exists a unique $\tilde{m}_1 > 0$ such that $V'_{\tilde{m}_1}(0) = p$. It is easy to verify that $\hat{m}_1 > \tilde{m}_1$ if and only if $V'_{\hat{m}_1}(0) > p$. Lemma 1 along with the fact that $V'_{m_1}(m_1) = 1$ further implies that if $m_1 \geq \tilde{m}_1$, there exists a unique $m_p(m_1) \in [0, m_1)$ such that $V'_{m_1}(m_p(m_1)) = p$. This corresponds to the unique maximum over $[0,\infty)$ of the function $m \mapsto V_{m_1}(m) - p(m+f)$. Note that, by construction, $m_p(\tilde{m}_1) = 0$. There are now two cases to consider.

Case 1. Suppose first that:

$$\max_{m \in [-f,\infty)} \{V_{\hat{m}_1}(m) - p(m+f)\} = 0. \tag{32}$$

Condition (32) holds if $\hat{m}_1 \leq \tilde{m}_1$, in which case the proportional cost p of issuance is so high that $V'_{\hat{m}_1}(0) \leq p$, or if $\hat{m}_1 > \tilde{m}_1$ and the fixed cost f of issuance is so high that $V_{\hat{m}_1}(m_p(\hat{m}_1)) - p[m_p(\hat{m}_1) + f] \leq 0$. Define a function $\hat{V}$ by:

$$\hat{V}(m) = \begin{cases} 0 & m < 0, \\ V_{\hat{m}_1}(m) & m \geq 0. \end{cases} \tag{33}$$

Note that, by construction, $\hat{V}(0) = 0$. Furthermore, condition (32) implies that the function $m \mapsto \hat{V}(m) - p(m+f)$ reaches its maximum over $[-f,\infty)$ at $-f$. It is then easy to check that $(V, m_1) = (\hat{V}, \hat{m}_1)$ solves the variational system (25)–(28).

Case 2. Suppose next that:

$$\max_{m \in [-f,\infty)} \{V_{\hat{m}_1}(m) - p(m+f)\} > 0. \tag{34}$$

Condition (34) holds whenever p and f are low enough, so that $\hat{m}_1 > \tilde{m}_1$ or equivalently $V'_{\hat{m}_1}(0) > p$, and $V_{\hat{m}_1}(m_p(\hat{m}_1)) - p[m_p(\hat{m}_1) + f] > 0$. One then has the following lemma.

Lemma 3. *If (32) holds, there exists a unique $\bar{m}_1 \in (\tilde{m}_1, \hat{m}_1)$ such that:*

$$V_{\bar{m}_1}(0) = V_{\bar{m}_1}(m_p(\bar{m}_1)) - p[m_p(\bar{m}_1) + f]. \tag{35}$$

Define a function $\bar{V}$ by:

$$\bar{V}(m) = \begin{cases} 0 & m < 0, \\ V_{\bar{m}_1}(m) & m \geq 0. \end{cases} \tag{36}$$

Note that Lemma 2 along with $\bar{m}_1 < \hat{m}_1$ implies that $\bar{V}(0) > 0$. Furthermore, since $\bar{m}_1 > \tilde{m}_1$, the function $m \mapsto \bar{V}(m) - p(m+f)$ reaches its maximum over $[-f, \infty)$ at $m_p(\bar{m}_1)$. It is then easy to check that $(V, m_1) = (\bar{V}, \bar{m}_1)$ solves the variational system (25)–(28).

Note that, in either case, the function $m \mapsto V(m) - p(m+f)$ reaches its maximum at a single point, m_0. In Case 1, $m_0 = -f$, while in Case 2, $m_0 = m_p(m_1)$. The following proposition summarizes our findings.

Proposition 2. *There exists a unique solution V to the variational system (25)–(28) that is twice continuously differentiable over $(0, \infty)$. Moreover, V satisfies the variational inequalities (18)–(20) over $(0, \infty)$.*

4.3. The Verification Argument

In this subsection, we establish that the solution V to (25)–(28) coincides with the value function V^* for problem (7). Our first result is that V is an upper bound for V^*.

Lemma 4. *For any admissible issuance and dividend policy $((\tau_n)_{n\geq 1}, (i_n)_{n\geq 1}, L)$,*

$$V(m) \geq v(m; (\tau_n)_{n\geq 1}, (i_n)_{n\geq 1}, L); \quad m \geq 0.$$

We now construct an admissible policy whose value coincides with V. Given Lemma 4, this establishes that $V^* = V$, and thereby provides the optimal issuance and dividend policy. Define $m_0^* = m_0^+$ and $m_1^* = m_1$, where m_0 and m_1 are given by the solution to the variational system (25)–(28). To construct the optimal policy, we rely on the theory of reflected diffusion processes, initiated by Skorokhod (1961). The intuition is that, at the optimum, the cash reserve process can be modelled as a diffusion process that is reflected back each time it hits m_1^*, and that is either absorbed at 0 or jumps to m_0^* each time it hits 0, according to whether Case 1 or 2 holds. The precise formulation of this process is given by the solution to the following version of Skorokhod's problem:

$$M_t^* = m + \int_0^t (\mu + (r - \lambda)M_s^*)\, ds + \sigma W_t + \sum_{n\geq 1} m_0^* 1_{\{T_n^* \leq t\}} - L_t^*, \tag{37}$$

$$M_t^* \leq m_1^*, \tag{38}$$

$$L_t^* = \int_0^t 1_{\{M_s^* = m_1^*\}}\, dL_s^*, \tag{39}$$

for all $t \in [0, \tau_B^*]$, where $\tau_B^* = \inf\{t \geq 0 \,|\, M_t^* < 0\}$ and the sequence of stopping times $(T_n^*)_{n\geq 1}$ is recursively defined by:

$$T_n^* = \inf\{t \geq T_{n-1}^* \,|\, M_{t-}^* = 0\}; \quad n \geq 1 \tag{40}$$

where $T_0^* = 0$. Standard results on Skorokhod's problem along with the strong Markov property imply that there exists a pathwise unique solution $(M^*, L^*) = \{(M_t^*, L_t^*); t \geq 0\}$ to (37)–(40). Condition (39) requires that L^* increases only when M^* hits the boundary m_1^*, while (37)–(38) express that this causes M^* to be reflected back at m_1^*. As for the behavior of M^* at 0, two cases can arise. If (32) holds, $m_0^* = (-f)^+ = 0$, so that $\tau_B^* = T_1^*$ $\mathbb{P}$–almost surely. This corresponds to a situation in which the project is liquidated as soon as the firm runs out of cash. By contrast, if (34) holds, $m_0^* = m_p(m_1^*) > 0$. In that case, the process M^* discontinuously jumps to m_0^* each time it hits 0, so that $\tau_B^* = \infty$ $\mathbb{P}$–almost surely. This corresponds to a situation in which an amount $i^* = p(m_0^* + f)$ of new equity is issued when the firm runs out of cash. One has the following result.

Proposition 3. *The value function V^* for problem (7) coincides with the unique solution V to the variational system (25)–(28) that is twice continuously differentiable over $(0, \infty)$. The optimal issuance and dividend policy is given by $((\tau_n^*)_{n \geq 1}, (i_n^*)_{n \geq 1}, L^*)$, where:*

$$\tau_n^* = \infty, \quad i_n^* = 0; \quad n \geq 1 \quad \textit{if condition (32) holds,}$$

$$\tau_n^* = T_n^*, \quad i_n^* = i^*; \quad n \geq 1 \quad \textit{if condition (34) holds.}$$

According to Proposition 3, the firm's optimal dividend policy consists in retaining all its earnings until accumulated cash reserves exceed the threshold m_1^*. When this arises, the firm pays all the excess over m_1^* as dividends. Regarding the firm's issuance policy, two situations can arise. If condition (32) holds, which intuitively arises when the issuance costs p and f are high, the firm never resorts to outside financing. The model is then essentially equivalent to that of Jeanblanc-Picqué and Shiryaev (1995) or Radner and Shepp (1996), the only difference being that we allow for cash remuneration. By contrast, if issuance costs are low and condition (34) holds, the firm avoids liquidation by issuing new equity when its cash reserves are depleted. Although the firm is never liquidated, its value $V^*(m)$ falls short of the first-best value $\hat{V}(m) = m + \mu/r$ because of the presence of issuance costs. The concavity of V^* over $[0, \infty)$ reflects that the value of firm reacts less to changes in the level of cash reserves when past performance has been high. This is because high accumulated cash reserves allow the firm to postpone the time at which it will have to raise new equity and incur the corresponding issuance costs. By contrast, following unfavorable cash-flow realizations, cash reserves are low, and the value of the firm reacts strongly to performance and ensuing changes in cash reserves. The value function V^* is illustrated on Figure 1 below.

—Insert Figure 1 here—

Two limiting cases of our analysis are worth mentioning. If $p = 1$, equity issuances involve no proportional cost. It is then easy to see that, if f is small enough so as to ensure that condition (34) is fulfilled, the dividend boundary m_1^* coincides with the post issuance level of cash reserves of the firm, $m_0^* = m_1^*$. The intuition is that since issuances involve only a fixed cost f, it is optimal for the firm to raise as much equity as possible from the market. In that case, equity issuances are tied to dividend distribution: following an equity issuance and a favorable cash-flow realization, the firm immediately distributes the excess of cash over m_1^* as dividends. By contrast, if f tends to 0, the lump sum amounts of equity issued tend to 0, and in the limit we have $V^{*\prime}(0) = p$ as in the model of Løkka and Zervos (2005). In that case, the optimal issuance policy is no longer described by an impulse control as in Proposition 3, with discontinuous jumps in the cash reserves when the firm runs out of cash, but rather by a singular control similar to the optimal dividend process. Equity issuances would then occur in infinitesimal amounts and would typically be highly clustered in time. In practice, equity issuances are rarely followed by dividend distributions, and firms undertake equity adjustments in lumpy and infrequent issues (Bazdresch (2005), Leary and Roberts (2005)). This is consistent with a combination of fixed and proportional issuance costs such as the one we have postulated.

The characterization of the value function V^* provided in Proposition 2 allows us to study the impact of an increase in issuance costs on the sensitivity of the value of the firm to changes in its cash reserves. To focus on an interesting case, we assume in the following result that issuance costs remain low enough so as to guarantee that condition (34) holds and hence that the firm does resort to outside financing at the optimum.

Corollary 1. *The elasticity of the value of the firm with respect to its cash reserves,*

$$\epsilon^*(m) = \frac{mV^{*\prime}(m)}{V^*(m)}; \quad m \geq 0, \tag{41}$$

is an increasing function of the issuance costs p and f.

The proof of this result proceeds as follows. An increase in issuance costs obviously results in a fall in the firm's value, which mechanically raises the elasticity (41). This fall in value is tied to an increase in the dividend boundary m_1^*, reflecting the intuitive fact that as issuance costs increase, the firm must accumulate more liquidities before distributing dividends. Using the non-crossing property of the solutions to (29)–(31), it is then easy to establish that this implies that the marginal value of cash increases with issuance costs, which further raises the elasticity (41).

For a given firm, the concavity of V^* guarantees that the semi-elasticity $m\epsilon^*(m) = V^{*\prime}(m)/V^*(m)$ is a decreasing function of the level m of cash reserves. What Corollary 1 establishes is that this effect is magnified by issuance costs. Intuitively, the percentage change in firm value per percentage change in cash reserves is larger when issuance costs are relatively high, because allowing the firm to postpone a costly new equity issuance is more valuable in this situation. Conversely, the holding of liquid assets is less important when the firm has access to cheap outside financing. A testable implication of this is that firms' valuations should be more responsive to changes in their cash reserves on markets with high costs of issuing equity. Alternatively, a reduction in issuance costs triggered for instance by a capital market deregulation should reduce the responsiveness of firms' valuations to changes in their cash reserves.

5. Stock Prices

We are now ready to derive the implications of our theory for stock prices. To focus on the case where the firm does resort occasionally to outside financing, we suppose thereafter that condition (34) holds. We denote by $S^* = \{S_t^*; t \geq 0\}$ the process describing the ex-dividend price of a share in the firm, and by $N^* = \{N_t^*; t \geq 0\}$ the process modelling the number of shares issued by the firm. Thus at any date $t \geq 0$, S_t^* does not include dividends distributed at date t, while N_t^* includes new shares issued at date t. We assume that N^* is a non-decreasing process and we adopt the normalization $N_{0-}^* = 1$. A key observation is that issuance and payout decisions critically depend on the amount of liquidities accumulated by the firm. As a result of this, the stock price and the number of outstanding shares are contingent on the current level of cash reserves. At any date $t \geq 0$, the value of the firm satisfies:

$$V^*(M_t^*) = N_t^* S_t^*. \tag{42}$$

Now turn to the optimal issuance process I^*. At any date $t \geq 0$,

$$dI_t^* = d[V^*(M_t^*)] - N_t^* dS_t^* = d(N_t^* S_t^*) - N_t^* dS_t^* = S_t^* dN_t^*, \tag{43}$$

where the first equality reflects the fact that part of the change in the value of the firm due to new equity issuance is absorbed by existing shareholders, and the third inequality follows from the fact that N^* is an increasing process, so that $d\langle N^*, S^* \rangle_t = 0$ for all $t \geq 0$. The following lemma holds.

Lemma 5. *For each $n \geq 1$, $S^*_{\tau_n^*} = S^*_{\tau_n^*-}$.*

Lemma 5 expresses the fact that the stock price does not jump at the optimal equity issuance dates. This is because the issuance process is predictable in our model: the firm raises new equity as it runs out of cash, an event that is observable by all participants to the market. The fact that the stock price does not react to new equity issuances follows then simply from the absence of arbitrage opportunities. In particular, equity issuances do not convey bad news about the profitability of the firm, unlike what typically happens when firms have private information about future profitability (Myers and Majluf (1984)).

We are now ready to derive the dynamics of the processes N^* and S^*. Our first result is a direct implication of the fact that, when an equity issuance occurs, the value of the firm discontinuously jumps from $V^*(0)$ to $V^*(m_0^*)$, while the stock price itself is unaffected according to Lemma 5.

Proposition 4. *The process N^* modelling the number of outstanding shares is punctual and defined by*:

$$N_t^* = \begin{cases} 1 & 0 \leq t < \tau_1^*, \\ \left[\dfrac{V^*(m_0^*)}{V^*(0)}\right]^n & \tau_n^* \leq t < \tau_{n+1}^*. \end{cases} \tag{44}$$

According to (44), each time new equity is raised, the ratio of new shares to outstanding shares is constant and equal to $[V^*(m_0^*) - V^*(0)]/V^*(0)$, which corresponds to a constant dilution factor. The number of shares is constant between two consecutive issuance dates. Thus, for all $n \geq 0$ and $t \in [\tau_n^*, \tau_{n+1}^*)$, one has $dS_t^* = d[V^*(M_t^*)]/N^*_{\tau_n^*}$. Using Itô's Lemma along with (27), together with the facts that the dividend process L^* increases only at m_1^* and that $V^*(m_1^*) = (\mu + (r - \lambda)m_1^*)/r$ and $V^{*\prime}(m_1^*) = 1$, it is easy to derive the following result.

Proposition 5. *Between two consecutive issuance dates τ_n^* and τ_{n+1}^*, the stock price process S^* evolves according to*:

$$\frac{dS_t^* + dD_t^*}{S_t^*} = r dt + \sigma^*(N^*_{\tau_n^*} S_t^*) dW_t, \tag{45}$$

where

$$\sigma^*(v) \equiv \sigma \frac{V^{*\prime}\left[(V^*)^{-1}(v)\right]}{v}$$

and D_t^* denotes the cumulative dividend per share process.

Along with the characterization of the value function V^* provided in Section 3, this result implies that the dynamics of the stock price S^* differs in three important ways from the log-normal specification postulated by Black and Scholes (1973) and Merton (1973), and derived in equation (12) in the first-best benchmark. First, the stock price is reflected back each time dividends are distributed, which occurs when the process $V^*(M^*) = N^*S^*$ hits m_1^*. As a result of this, the stock price cannot take arbitrarily large values. Second, since the function V^* is strictly increasing and strictly concave over $[0, \infty)$, the volatility $\sigma V^{*\prime}((V^*)^{-1}(N^*S^*))/(N^*S^*)$ of stock returns is a decreasing function of S^*, so that changes in the volatility of stock returns are negatively correlated with stock price movements. That is, between two consecutive issuance dates, volatility tends to rise in response to bad news, and to fall in response to good news. Therefore our model predicts heteroscedasticity in stock prices, as documented for instance by Black (1976), Christie (1982) and Nelson (1991). While this "leverage effect" that ties stock returns and volatility changes cannot be attributed to financial leverage, as our firm is 100% equity financed, one can argue following Black (1976) that the firm has "operating leverage" as it must occasionally resort to costly outside financing to continue its activity. When earnings fall, the likelihood that these expenses will have to be incurred in the near future raises. As the value of the firm declines, it becomes more volatile, as small changes in earnings result in large changes in the difference between earnings and anticipated financial costs. Finally, the last difference between the stock price process (45) and the standard log-normal specification is that the dynamics of stock prices is path dependent, since it is modified by the successive equity issuances. As more stocks are issued, the number of outstanding shares N^* increases, which modifies both the stock price threshold m_1^*/N^* at which dividends are distributed and the volatility $\sigma V^{*\prime}((V^*)^{-1}(N^*S^*))/(N^*S^*)$ of stock returns. As more equity issuances take place, both the stock prices and the volatility on their return tend to fall. Another testable implication of our model is that the value of the firm is always more volatile than the cash-flows, $\sigma V^{*\prime}((V^*)^{-1}(N^*S^*)) \geq \sigma$, with equality only at the dividend boundary.

It should be noted that, while the stock price processes in the first-best benchmark and in the presence of issuance costs are qualitatively very different, there is nevertheless some formal analogy between (16) and (45). Indeed, in the absence of issuance costs, the value of the firm as given by (8) has a slope equal to 1, the market capitalization of the firm stays constant, $\hat{N}\hat{S} = \mu/r$, and the firm holds no cash reserves, which can be heuristically expressed by saying that the dividend boundary m_1^* is equal to 0. Substituting formally in (45), one retrieve exactly formula (16).

It is instructive to compare the stock price process (45) with that arising in the dynamic agency models of Biais, Mariotti, Plantin and Rochet (2004) or DeMarzo and Sannikov (2004). Much like in our framework, these models predict that stock return volatility tends to increase in response to bad performance. However, the mechanism that leads to this result is different. Agency costs typically make it optimal to liquidate the project as soon as the firm runs short of cash. This is what generates a concavity of the firm value and of the stock price in the level of liquidities that the firm has accumulated. In the implementation of the optimal contract, it is never optimal to issue new securities as the firm becomes illiquid. By contrast, time-varying volatility arises in our model precisely because raising new funds from the market is costly.

In line with Corollary 1, it is easy to characterize the impact of an increase in issuance costs on the volatility of stock returns. Again, we assume that condition (34) holds so that the firm does resort to outside financing at the optimum.

Corollary 2. *The volatility of stock returns as a function of the firm's valuation,*

$$\sigma^*(v) = \sigma \frac{V^{*\prime}((V^*)^{-1}(v))}{v}; \quad V^*(0) \leq v \leq V^*(m_1^*), \tag{46}$$

is an increasing function of the issuance costs p and f.

The proof follows from the fact that V^* is a decreasing function of p and f, while $V^{*\prime}$ is an increasing function of p and f. A testable implication of this result is that a reduction in issuance costs triggered by a capital market liberalization should lead to a fall in the volatility of stock returns.

6. Conclusion

This paper has shown that the introduction of exogenous issuance costs was enough to generate heteroskedasticity of stock prices, even when earnings are i.i.d. It would be important to extend this result in several directions. Two of them are random profitability and endogenous technology. Indeed, if the profitability of the firm evolves randomly, it should be possible to extend the analysis to a more complex Markov model, with two state variables that correspond to the two most important dimensions of the financial policy of firms: liquidity and solvency. On the other hand introducing new investment opportunities would also modify the analysis in an important way. A decrease in issuance costs is likely to encourage firms to invest more and to choose more risky projects, which may counteract the stabilizing effect identified in this paper. These, and related questions must await for future work.

Appendix

Proof of Proposition 1. From (4), the requirement that cash reserves be constant and equal to 0 after date 0 yields the following analogue of (10):

$$\hat{I}_t = \hat{L}_t - \mu t - \sigma W_t \tag{A.1}$$

for all $t \geq 0$. Using (11) along with (A.1) then yields:

$$\frac{d\hat{S}_t}{\hat{S}_t} = r dt + \frac{\sigma r}{\mu} dW_t - \frac{r}{\mu} d\hat{L}_t \tag{A.2}$$

for all $t > 0$. The value of the firm at any date $t > 0$ is the present value of future cash-flows, hence (15). This allows in turn to rewrite (A.2) as (16). One has the following lemma.

Lemma A.1. *Given an initial condition $\hat{S}_{0+} > 0$, the stochastic differential equation (A.2) has a unique strong solution, given by:*

$$\hat{S}_t = \hat{S}_{0+} \exp\left(\left[r - \frac{1}{2} \left(\frac{\sigma r}{\mu} \right)^2 \right] t + \frac{\sigma r}{\mu} W_t - \frac{r}{\mu} \hat{L}_t \right) \tag{A.3}$$

for all $t > 0$.

Proof. From the generalized Itô's formula (Dellacherie and Meyer (1982, Theorem VIII.27)), it is easy to check that the process $\hat{S} = \{\hat{S}_t; t > 0\}$ defined by (A.3) solves the stochastic differential equation (A.2). Consider now another solution $\tilde{S} = \{\tilde{S}_t; t > 0\}$ to (A.2) with the same initial condition $\hat{S}_{0+}$ as $\hat{S}$. Applying again the generalized Itô's formula, one can verify that, for any $t > 0$,

$$\begin{aligned} \mathbb{E}\left[(\hat{S}_t - \tilde{S}_t)^2\right] &= \left(2r + \frac{\sigma^2 r^2}{2\mu^2}\right) \int_0^t \mathbb{E}\left[(\hat{S}_s - \tilde{S}_s)^2\right] ds - \frac{2r}{\mu} \mathbb{E}\left[\int_0^t (\hat{S}_s - \tilde{S}_s)^2 \, d\hat{L}_s\right] \\ &\leq \left(2r + \frac{\sigma^2 r^2}{2\mu^2}\right) \int_0^t \mathbb{E}\left[(\hat{S}_s - \tilde{S}_s)^2\right] ds \\ &\leq 0, \end{aligned}$$

where the first inequality follows from the fact that $\hat{L}$ is a non-decreasing process, and the second from Gronwall's lemma. Thus one has $\hat{S}_t = \tilde{S}_t$ with probability 1 for all $t > 0$. Since the processes $\hat{S}$ and $\tilde{S}$ are continuous, the result follows. □

Note that the initial condition for $\hat{S}$ in Lemma A.1 is stipulated at time 0^+, that is, immediately after the special dividend distribution m at time 0. Letting $\hat{N}_{0+} = 1$ without loss of generality, (15) yields $\hat{S}_{0+} = \mu/r$. To conclude the proof, we need only to check that (17) holds. This requires the following lemma.

Lemma A.2. *Suppose that condition (14) holds. Then,*

$$\lim_{T\to\infty} \left\{ \mathbb{E}\left[\exp\left(-\frac{1}{2} \left(\frac{\sigma r}{\mu} \right)^2 T + \frac{\sigma r}{\mu} W_T - \frac{r}{\mu} \hat{L}_T \right) | \mathcal{F}_t \right] \right\} = 0 \tag{A.4}$$

for all $t \geq 0$, $\mathbb{P}$–almost surely.

Proof. Denote by $\{X_T; T \geq 0\}$ the random variables within the expectations in (14). We first show that the random variables $\mathbb{E}[X_T \,|\, \mathcal{F}_t]$, $T \geq t$, have a $\mathbb{P}$–almost surely well-defined limit as T goes to ∞. For any $t \geq 0$, define:

$$Z_t = \exp\left(-\frac{1}{2}\left(\frac{\sigma r}{\mu}\right)^2 t + \frac{\sigma r}{\mu}\, W_t\right).$$

The process $Z = \{Z_t; t \geq 0\}$ is a martingale, and $\mathbb{E}[Z_t] = 1$ for all $t \geq 0$. Now suppose that $T_2 \geq T_1 \geq t$. Then one has:

$$\begin{aligned}
\mathbb{E}[X_{T_2} \,|\, \mathcal{F}_t] &= \mathbb{E}\left[Z_{T_2} \exp\left(-\frac{r}{\mu}\,\hat{L}_{T_2}\right) |\, \mathcal{F}_t\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[Z_{T_2} \exp\left(-\frac{r}{\mu}\,\hat{L}_{T_2}\right) |\, \mathcal{F}_{T_1}\right] |\, \mathcal{F}_t\right] \\
&\leq \mathbb{E}\left[\mathbb{E}[Z_{T_2} \,|\, \mathcal{F}_{T_1}] \exp\left(-\frac{r}{\mu}\,\hat{L}_{T_1}\right) |\, \mathcal{F}_t\right] \\
&= \mathbb{E}\left[Z_{T_1} \exp\left(-\frac{r}{\mu}\,\hat{L}_{T_1}\right) |\, \mathcal{F}_t\right] \\
&= \mathbb{E}[X_{T_1} \,|\, \mathcal{F}_t]
\end{aligned}$$

$\mathbb{P}$–almost surely, where the inequality follows from the fact that $\hat{L}$ is a non-decreasing process, and the third equality from the fact that Z is a martingale. It follows that the random variables $\mathbb{E}[X_T \,|\, \mathcal{F}_t]$, $T \geq t$, decrease as a function of T. Since they are positive, they have a $\mathbb{P}$–almost surely well-defined limit as T goes to ∞, as claimed. We now show that this limit is $\mathbb{P}$–almost surely 0, which concludes the proof. Since the process $\hat{L}$ is non-negative, one has:

$$\mathbb{E}[X_T \,|\, \mathcal{F}_t] \leq \mathbb{E}[Z_T \,|\, \mathcal{F}_t] \leq Z_t,$$

where the second inequality follows from the fact that Z is a martingale. Since $\mathbb{E}[Z_t] = 1$, the positive random variables $\mathbb{E}[X_T \,|\, \mathcal{F}_t]$ are uniformly bounded above by an integrable random variable. Since they converge $\mathbb{P}$–almost surely to a well defined limit as T goes to ∞, one has:

$$\mathbb{E}\left[\lim_{T\to\infty}\{\mathbb{E}[X_T \,|\, \mathcal{F}_t]\}\right] = \lim_{T\to\infty}\{\mathbb{E}[\mathbb{E}[X_T \,|\, \mathcal{F}_t]]\} = \lim_{T\to\infty}\{\mathbb{E}[X_T]\} = 0,$$

where the first equality follows from Lebesgue's dominated convergence theorem, and the last from (14). Since $\lim_{T\to\infty}\{\mathbb{E}[X_T \,|\, \mathcal{F}_t]\}$ is a non-negative random variable, the result follows. □

We are now ready to complete the proof. Using (15) and (A.2), it is easy to verify that:

$$e^{-rt}\hat{S}_t = \mathbb{E}[e^{-rT}\hat{S}_T \,|\, \mathcal{F}_t] + \mathbb{E}\left[\int_t^T e^{-rs}\,\frac{1}{\hat{N}_s}\, d\hat{L}_s \,|\, \mathcal{F}_t\right]$$

for each $t \geq 0$ and $T \geq t$. It follows from Lemmas A.1 and A.2 that the first term on the right-hand side of this equation goes to 0 as T goes to ∞. Since $\hat{L}$ is an increasing process and $\hat{N}$ remains strictly positive, the result follows from the monotone convergence theorem. ■

Proof of Lemma 1. Since V_{m_1} is smooth over $[0, m_1)$, differentiating (29) and using the definition (21) of $\mathcal{L}$ yields that $-\lambda V'_{m_1} + \mathcal{L}V'_{m_1} = 0$ over $[0, m_1)$. Using this along with (30)–(31), we obtain that

$V'''_{m_1-}(m_1) = 2(r-r)/\sigma^2 > 0$. Since $V''_{m_1}(m_1) = 0$ and $V'_{m_1}(m_1) = 1$, it follows that $V''_{m_1} < 0$ and thus $V'_{m_1} > 1$ over an interval $(m_1 - \varepsilon, m_1)$ for $\varepsilon > 0$. Now suppose by way of contradiction that $V'_{m_1}(m) \leq 1$ for some $m \in [0, m_1 - \varepsilon]$, and let $\tilde{m} = \sup\{m \in [0, m_1 - \varepsilon] \,|\, V'_{m_1}(m) \leq 1\} < m_1$. Then $V'_{m_1}(\tilde{m}) = 1$ and $V'_{m_1} > 1$ over $(\tilde{m}, m_1)$, so that $V_{m_1}(m_1) - V_{m_1}(m) > m_1 - m$ for all $m \in (\tilde{m}, m_1)$. Since $V_{m_1}(m_1) = (\mu + rm_1)/r$, this implies that for any such m,

$$
\begin{aligned}
V''_{m_1}(m) &= \frac{2}{\sigma^2}\left[rV_{m_1}(m) - (\mu + (r-\lambda)m)V'_{m_1}(m)\right] \\
&< \frac{2}{\sigma^2}\left\{r[m - m_1 + V_{m_1}(m_1)] - (\mu + (r-\lambda)m)\right\} \\
&= \frac{2}{\sigma^2}\,\lambda(m - m_1) \\
&< 0,
\end{aligned}
$$

which contradicts the fact that $V'_{m_1}(\tilde{m}) = V'_{m_1}(m_1) = 1$. Therefore $V'_{m_1} > 1$ over $[0, m_1)$, from which it follows as above that $V''_{m_1} < 0$ over $[0, m_1)$. Hence the result. ■

Proof of Lemma 2. Consider the solutions H_0 and H_1 to be the linear second-order differential equation $-rH + \mathcal{L}H = 0$ over $[0, \infty)$ that are characterized by the initial conditions $H_0(0) = 1$, $H'_0(0) = 0$, $H_1(0) = 0$ and $H'_1(0) = 1$. We first show that H'_0 and H'_1 are strictly positive over $(0, \infty)$. Consider H'_0. Since $H_0(0) = 1$ and $H'_0(0) = 0$, one has $H''_0(0) = 2r/\sigma^2 > 0$, so that $H'_0 > 0$ over an interval $(0, \varepsilon)$ for $\varepsilon > 0$. Suppose that $\tilde{m} = \inf\{m \geq \varepsilon \,|\, H'_0(m) \leq 0\} < \infty$. Then $H'_0(\tilde{m}) = 0$ and $H''_0(\tilde{m}) \leq 0$. Since $-rH_0 + \mathcal{L}H_0 = 0$, it follows that $H_0(\tilde{m}) \leq 0$, which stands in contradiction with the facts that $H_0(0) = 1$ and that H_0 is strictly increasing over $[0, \tilde{m}]$. Thus $H'_0 > 0$ over $(0, \infty)$, as claimed. The proof for H'_1 is similar, and is therefore omitted. Note that both H_0 and H_1 remain strictly positive over $(0, \infty)$. Next, let $W_{H_0,H_1} = H_0H'_1 - H_1H'_0$ be the Wronskian of H_0 and H_1. One has $W_{H_0,H_1}(0) = 1$ and:

$$
\begin{aligned}
W'_{H_0,H_1}(m) &= H_0(m)H''_1(m) - H_1(m)H''_0(m) \\
&= \frac{2}{\sigma^2}\left\{H_0(m)[rH_1(m) - (\mu + (r-\lambda)m)H'_1(m)] - H_1(m)[rH_0(m) - (\mu + (r-\lambda)m)H'_0(m)]\right\} \\
&= -\frac{2(\mu + (r-\lambda)m)}{\sigma^2}\,W_{H_0,H_1}(m)
\end{aligned}
$$

for all $m \geq 0$, from which Abel's identity follows by integration:

$$
W_{H_0,H_1}(m) = \exp\left(-\frac{2\mu m + (r-\lambda)m^2}{\sigma^2}\right) \tag{A.5}
$$

for all $m \geq 0$. Since $W_{H_0,H_1} > 0$, H_0 and H_1 are linearly independent. As a result of this, (H_0, H_1) is a basis of the 2-dimensional space of solutions to the equation $-rH + \mathcal{L}H = 0$. It follows in particular that for any $m_1 > 0$, one can represent V_{m_1} as:

$$
V_{m_1} = V_{m_1}(0)H_0 + V'_{m_1}(0)H_1
$$

over $[0, m_1]$. Using the boundary conditions $V_{m_1}(m_1) = (\mu + (r-\lambda)m_1)/r$ and $V'_{m_1}(m_1) = 1$, one can solve for $V_{m_1}(0)$ and $V'_{m_1}(0)$ as follows:

$$V_{m_1}(0) = \frac{H'_1(m_1)(\mu + (r-\lambda)m_1)/r - H_1(m_1)}{W_{H_0,H_1}(m_1)}, \tag{A.6}$$

$$V'_{m_1}(0) = \frac{H_0(m_1) - H'_0(m_1)(\mu + (r-\lambda)m_1)/r}{W_{H_0,H_1}(m_1)}. \tag{A.7}$$

Using the explicit expression (A.5) for W_{H_0,H_1} along with the fact that H_0 and H_1 are solutions to $-rH + \mathcal{L}H = 0$, it is easy to verify from (A.6)–(A.7) that:

$$\begin{aligned}
\frac{dV_{m_1}(0)}{dm_1} &= -\left(1 - \frac{(r-\lambda)}{r}\right)\exp\left(\frac{2\mu m_1 + (r-\lambda)m_1^2}{\sigma^2}\right)H'_1(m_1),\\
\frac{d^2V_{m_1}(0)}{dm_1^2} &= -\frac{2}{\sigma^2}\,\lambda\exp\left(\frac{2\mu m_1 + (r-\lambda)m_1^2}{\sigma^2}\right)H_1(m_1),\\
\frac{dV'_{m_1}(0)}{dm_1} &= \left(1 - \frac{(r-\lambda)}{r}\right)\exp\left(\frac{2\mu m_1 + (r-\lambda)m_1^2}{\sigma^2}\right)H'_0(m_1).\\
\frac{d^2V'_{m_1}(0)}{dm_1^2} &= \frac{2}{\sigma^2}\,\lambda\exp\left(\frac{2\mu m_1 + (r-\lambda)m_1^2}{\sigma^2}\right)H_0(m_1).
\end{aligned}$$

The result then follows immediately from the fact that $\lambda > 0$ and that H_0, H'_0, H_1 and H'_1 are strictly positive over $\mathbb{R}_{++}$. ■

Proof of Lemma 3. Equation (35) can be rewritten as $\varphi(\bar{m}_1) = 0$, where:

$$\varphi(m_1) = V_{m_1}(m_p(m_1)) - V_{m_1}(0) - p[m_p(m_1) + f].$$

If $\hat{m}_1 > \tilde{m}_1$, the function φ is well defined and continuous over $[\tilde{m}_1, \hat{m}_1]$, while $\varphi(\tilde{m}_1) = -pf < 0$ and $\varphi(\hat{m}_1) = V_{\hat{m}_1}(m_p(\hat{m}_1)) - p[m_p(\hat{m}_1) + f] > 0$ if the second half of condition (34) holds. Thus φ has at least a zero over $(\tilde{m}_1, \hat{m}_1)$. To prove that it is unique, we show that φ is strictly increasing over $(\tilde{m}_1, \hat{m}_1)$. Using the Envelope Theorem to evaluate the derivative of φ, this amounts to:

$$\frac{\partial W}{\partial m_1}(m_p(m_1), m_1) > \frac{\partial W}{\partial m_1}(0, m_1)$$

for all $m_1 \in (\tilde{m}_1, \hat{m}_1)$, where $W(m, m_1) = V_{m_1}(m)$ for all $(m, m_1) \in [0,\infty)\times(\tilde{m}_1, \hat{m}_1)$. Since $m_p(m_1) \in (0, m_1)$ for all $m_1 \in (\tilde{m}_1, \hat{m}_1)$, all that needs to be established is that for any such m_1, $(\partial W/\partial m_1)(\cdot, m_1)$ is strictly increasing over $[0, m_1]$. From (29)–(31), it is easy to check that $(\partial W/\partial m_1)(\cdot, m_1)$ solves the following boundary value problem over $[0, m_1]$:

$$-r\,\frac{\partial W}{\partial m_1}(m, m_1) + \mathcal{L}\,\frac{\partial W}{\partial m_1}(m, m_1) = 0; \quad 0 \le m \le m_1, \tag{A.8}$$

$$\frac{\partial^2 W}{\partial m \partial m_1}(m_1, m_1) = 0, \tag{A.9}$$

$$\frac{\partial^3 W}{\partial^2 m \partial m_1}(m_1, m_1) = -\frac{2}{\sigma^2}\,\lambda. \tag{A.10}$$

We are interested in the sign of $(\partial^2 W/\partial m \partial m_1)(m, m_1)$ for $m \in [0, m_1)$. As $(\partial^2 W/\partial m \partial m_1)(m_1, m_1) = 0$ and $(\partial^3 W/\partial^2 m \partial m_1)(m_1, m_1) < 0$, $(\partial^2 W/\partial m \partial m_1)(\cdot, m_1) > 0$ over an interval $(m_1 - \varepsilon, m_1)$ for $\varepsilon > 0$.

Now suppose by way of contradiction that $(\partial^2 W/\partial m \partial m_1)(m, m_1) \leq 0$ for some $m \in [0, m_1 - \varepsilon]$, and let $\tilde{m} = \inf\{m \in [0, m_1 - \varepsilon] \mid (\partial^2 W/\partial m \partial m_1)(m, m_1) \leq 0\}$. Then $(\partial^2 W/\partial m \partial m_1)(\tilde{m}, m_1) = 0$ and $(\partial^2 W/\partial m \partial m_1)(m, m_1) > 0$ for all $m \in (\tilde{m}, m_1)$, so that $(\partial W/\partial m_1)(m, m_1) < 0$ for all $m \in (\tilde{m}, m_1)$ as $(\partial W/\partial m_1)(m_1, m_1) = -\lambda/r < 0$ by (A.8)–(A.10). This implies that for any such m,

$$\frac{\partial^3 W}{\partial^2 m \partial m_1}(m, m_1) = \frac{2}{\sigma^2}\left[r \frac{\partial W}{\partial m_1}(m, m_1) - (\mu + (r - \lambda)m) \frac{\partial^2 W}{\partial m \partial m_1}(m, m_1)\right] < 0,$$

which contradicts the fact that $(\partial^2 W/\partial m \partial m_1)(\tilde{m}, m_1) = (\partial^2 W/\partial m \partial m_1)(m_1, m_1) = 0$. Therefore $(\partial^2 W/\partial m \partial m_1)(\cdot, m_1) > 0$ over $[0, m_1)$, and the result follows. Note for further reference that the above argument shows that $(\partial W/\partial m_1)(\cdot, m_1) < 0$ over $[0, m_1]$. ■

Proof of Proposition 2. We first establish uniqueness. As explained in the text, any solution V to (25)–(28) that is twice continuously differentiable over $(0, \infty)$ must coincide with some V_{m_1} over $[0, \infty)$. Since $V(0)$ must be non-negative by (26), one must have $m_1 \leq \hat{m}_1$. Suppose first that $\hat{m}_1 \leq \tilde{m}_1$, and that $m_1 < \hat{m}_1$. Then $V(0) = V_{m_1}(0) > 0$. But since $m_1 < \tilde{m}_1$, one has $V'_+(0) = V'_{m_1}(0) < p$. It follows that the maximum of the mapping $m \mapsto V(m) - p(m + f)$ over $[f, \infty)$ is either attained at $-f$, for a value of 0, or at 0, for a value of $V(0) - pf$. In either case, this is inconsistent with condition (26). It follows that $m_1 = \hat{m}_1$, and thus $V = \hat{V}$ as given by (33). Suppose next that $\hat{m}_1 > \tilde{m}_1$. The above argument can be used to show that necessarily $m_1 > \tilde{m}_1$. Two cases must be distinguished. If $V_{\hat{m}_1}(m_p(\hat{m}_1)) - p[m_p(\hat{m}_1) + f] > 0$, then Lemma 3 establishes the uniqueness of a value $\bar{m}_1$ of $m_1 \in (\tilde{m}_1, \hat{m}_1)$ consistent with condition (26). It follows that $m_1 = \bar{m}_1$, and thus $V = \bar{V}$ as given by (36). Suppose finally that $V_{\hat{m}_1}(m_p(\hat{m}_1)) - p[m_p(\hat{m}_1) + f] \leq 0$. Defining φ as in the proof of Lemma 3, and using the fact that φ is strictly increasing over $(\tilde{m}_1, \hat{m}_1)$, we obtain that φ has no zeros over $(\tilde{m}_1, \hat{m}_1)$. Thus condition (26) cannot be satisfied for $m_0 = m_p(m_1)$ and $m_1 \in (\tilde{m}_1, \hat{m}_1)$. It follows that the maximum of the mapping $m \mapsto V(m) - p(m + f)$ over $[f, \infty)$ must be attained at $-f$, for a value of 0. The only choice of m_1 that is then consistent with (26) is $m_1 = \hat{m}_1$, and thus $V = \hat{V}$ as given by (33).

We now verify that our solution V to (25)–(28) satisfies the variational inequalities (18)–(20) over $(0, \infty)$. Inequality (18) follows from (28) and Lemma 1, while inequality (20) follows from (27)–(28) along with the fact that $\lambda > 0$. As for (19), two cases must be distinguished. Suppose first that $\hat{m}_1 \leq \tilde{m}_1$, and hence $V'_+(0) \leq p$. For any $m \geq 0$, the mapping $m' \mapsto V(m' - f) - p(m' - m)$ is then strictly decreasing over $[m, \infty)$, and thus (19) holds as $V(m) \geq V(m - f)$ for any such m. Suppose next that $\hat{m}_1 > \tilde{m}_1$, and hence $V'_+(0) > p$. If $m \geq m_p(m_1) + f$, the same reasoning as above applies and (19) holds. If $m_p(m_1) + f > m \geq 0$, the maximum of the mapping $m' \mapsto V(m' - f) - p(m' - m)$ over $[m, \infty)$ is attained at $m_p(m_1) + f$, and we must therefore check that:

$$V(m) - pm \geq V(m_p(m_1)) - p[m_p(m_1) + f] \tag{A.11}$$

for any such m. The mapping $m \mapsto V(m) - pm$ is strictly increasing over $[0, m_p(m_1)]$, and strictly decreasing over $[m_p(m_1), m_p(m_1) + f]$. Thus we need only to check that (A.11) holds at $m = 0$ and at $m = m_p(m_1) + f$. The latter point is immediate. For the former, two cases must be distinguished. If (32) holds, then $m_1 = \hat{m}_1$, and (A.11) holds at $m = 0$ since the right-hand side is non-positive while the left-hand side is equal to 0 as $\hat{V}(0) = 0$. If (34) holds, then $m_1 = \bar{m}_1$, and (A.11) holds as an equality at $m = 0$ as $\bar{V}(0) = \bar{V}(m_p(\bar{m}_1)) - p[m_p(\bar{m}_1) + f]$. The result follows. ■

Proof of Lemma 4. Fix an admissible policy $((\tau_n)_{n \geq 1}, (i_n)_{n \geq 1}, L)$, from which processes I, F and M and the bankruptcy date τ_B can be obtained as in (2)–(5). Let us decompose the process L as

$L_t = L_t^c + \Delta L_t$ for all $t \geq 0$, where L^c is the pure continuous part of L. The generalized Itô's formula (Dellacherie and Meyer (1982, Theorem VIII.27)) yields:

$$\begin{aligned} e^{-r(T\wedge\tau_B)}V(M_{T\wedge\tau_B}) \;=\; & V(m) + \int_0^{T\wedge\tau_B -} e^{-rt}[-rV(M_t) + \mathcal{L}V(M_t)]\,dt \\ & + \sigma \int_0^{T\wedge\tau_B -} e^{-rt}V'(M_t)\,dW_t - \int_0^{T\wedge\tau_B -} e^{-rt}V'(M_t)\,dL_t^c \\ & + \sum_{t\in[0,T\wedge\tau_B]} e^{-rt}[V(M_t) - V(M_{t-})] \end{aligned} \tag{A.12}$$

for all $T \geq 0$. Since V satisfies (18) by Proposition 2, it follows that for each $t \in [0, T \wedge \tau_B]$,

$$\begin{aligned} V(M_t) - V(M_{t-}) \;=\; & V\left(M_{t-} + \frac{\Delta I_t}{p} - \Delta F_t - \Delta L_t\right) - V(M_{t-}) \\ \leq\; & V\left(M_{t-} + \frac{\Delta I_t}{p} - \Delta F_t\right) - \Delta L_t - V(M_{t-}). \end{aligned}$$

Plugging into (A.12) and using again inequality (18) yields:

$$\begin{aligned} e^{-r(T\wedge\tau_B)}V(M_{T\wedge\tau_B}) \;\leq\; & V(m) + \int_0^{T\wedge\tau_B -} e^{-rt}[-rV(M_t) + \mathcal{L}V(M_t)]\,dt \\ & + \sigma \int_0^{T\wedge\tau_B -} e^{-rt}V'(M_t)\,dW_t - \int_0^{T\wedge\tau_B -} e^{-rt}\,dL_t \\ & + \sum_{n\geq 1} e^{-r\tau_n} i_n 1_{\{\tau_n \leq T\wedge\tau_B\}} \\ & + \sum_{n\geq 1} e^{-r\tau_n}\left[V\left(M_{\tau_n -} + \frac{i_n}{p} - f\right) - i_n - V(M_{\tau_n -})\right]1_{\{\tau_n \leq T\wedge\tau_B\}}. \end{aligned} \tag{A.13}$$

Since V' is bounded over $(0, \infty)$, the third term of the left hand side of (A.13) is a square integrable martingale. Using inequalities (19) and (20) along with the fact that V is non-negative by construction, we can take expectations in (A.13) to obtain:

$$V(m) \geq \mathbb{E}^m\left[\int_0^{T\wedge\tau_B -} e^{-rt}(dL_t - dI_t)\right], \tag{A.14}$$

from which the result follows by letting T go to ∞. ■

Proof of Proposition 3. Assume that (34) holds, so that $\tau_B = \infty$ $\mathbb{P}$–almost surely, and suppose without loss of generality that $m \in [0, m_1^*]$. The process M^* has paths that are continuous except at the dates $(\tau_n^*)_{n\geq 1}$ at which new equity is issued, in which case $V(M^*_{\tau_n^*}) - V(M^*_{\tau_n^* -}) = V(m_0^*) - V(0) = i^*$ by construction. Proceeding as in the proof of Proposition 2, we obtain that:

$$\begin{aligned} \mathbb{E}^m\left[e^{-rT}V(M_T)\right] \;=\; & V(m) - \mathbb{E}^m\left[\int_0^{T-} e^{-rt}V'(M_t^*)\,dL_t^*\right] + \mathbb{E}^m\left[\sum_{n\geq 1} e^{-r\tau_n^*} i^* 1_{\{\tau_n^* \leq T\}}\right] \\ =\; & V(m) - \mathbb{E}^m\left[\int_0^{T-} e^{-rt}\,(dL_t^* - dI_t^*)\right] \end{aligned} \tag{A.15}$$

for all $T \geq 0$, where the process $I^* = \{I_t^*; t \geq 0\}$ is defined as in (2) with $i_n^* = i^*$ for all $n \geq 1$, and the second inequality follows from (39) along with the fact that $V'(m_1^*) = 1$. To conclude the proof, we need only to check that $\lim_{T\to\infty} \mathbb{E}^m[e^{-rT}V(M_T)] = 0$ in (A.15). Since V is non-negative with bounded derivatives, one has:

$$0 \leq e^{-rT}V(M_T) \leq e^{-rT}C(1 + M_T^*) \leq e^{-rT}C(1 + m_1^*)$$

for some positive constant C, where the second and third inequality follow from the fact that the process M^* never leaves the interval $[0, m_1^*]$. Taking expectations and letting T go to ∞ yields the result. The proof for the case in which (32) holds is similar, and therefore omitted. ■

Proof of Corollary 1. To establish this result, we show that V^* is a decreasing function of p and f, and that $V^{*\prime}$ is an increasing function of p and f. To prove the first claim, start without loss of generality from a situation in which p and f are such that condition (34) holds, and consider the impact of a decrease in p or f, $p' \leq p$ and $f' \leq f$ with at least one strict inequality. Then the firm can keep the same dividend policy L^*, while adjusting its issuance policy so as to maintain the same dynamics for cash reserves (37) as when the issuance costs are p and f. Indeed, to do so, it needs only to issue amounts $i' = p'(m_0^* + f')$ worth of equity instead of $i^* = p(m_0^* + f)$, at the same dates $(\tau_n^*)_{n\geq1}$. That is, the new issuance and dividend policy of the firm is $((\tau_n^*)_{n\geq1}, (i'_n)_{n\geq1}, L^*)$ with $i'_n = i' < i^*$ for all $n \geq 1$. Since the dividend policy and the dynamics of cash reserves are the same as in the initial situation, while the amounts of equity issued are strictly lower, this policy yields a strictly higher value for the firm than in the initial situation. Thus V^* is a decreasing function of p and f, as claimed. Now, using the notation of the proof of Lemma 3, one has $V^* = W(\cdot, m_1^*)$ over $\mathbb{R}_+$. Since $(\partial W/\partial m_1)(\cdot, m_1) < 0$ over $[0, m_1]$, the above argument implies that an increase in either p or f leads to an increase in m_1^*. Since $(\partial^2 W/\partial m \partial m_1)(\cdot, m_1) > 0$ over $[0, m_1)$, it follows that $V^{*\prime}$ is an increasing function of p and f. Hence the result. ■

Proof of Lemma 5. Proposition 3 along with (42) implies that for each $n \geq 1$,

$$S_{\tau_n^*}^* N_{\tau_n^*}^* - S_{\tau_n^*-}^* N_{\tau_n^*-}^* = V^*(M_{\tau_n^*}^*) - V^*(M_{\tau_n^*-}^*) = V^*(m_0^*) - V^*(0) = p(m_0^* + f) = i^*. \tag{A.16}$$

Next, it follows from (43) that the issuance proceeds at date τ_n^* are given by:

$$i^* = I_{\tau_n^*}^* - I_{\tau_n^*-}^* = S_{\tau_n^*}^*(N_{\tau_n^*}^* - N_{\tau_n^*-}^*). \tag{A.17}$$

It then follows immediately from (A.16)–(A.17) that $S_{\tau_n^*}^* = S_{\tau_n^*-}^*$, as claimed. ■

References

[1] Bazdresch, S. (2005): "Financial Lumpiness and Investment," mimeo, Yale University.

[2] Biais, B., T. Mariotti, G. Plantin, and J.-C. Rochet (2006): "Dynamic Security Design: Convrgence to Continuous-Time and Asset Pricing Implications," forthcoming, *Review of Economic Studies.*

[3] Black, F. (1976): "Studies of Stock Market Volatility Changes," *Proceedings of the 1976 Meetings of the American Statistical Association*, *Business and Economic Statistics Section*, 177–181.

[4] Black, F., and M. Scholes (1973): "The Pricing of Options and Corporate Liabilities," *Journal of Political Economy*, 81, 637–654.

[5] Christie, A.A. (1982): "The Stochastic Behavior of Common Stock Variances : Value, Leverage and Interest Rate Effects," *Journal of Financial Economics*, 10, 407–432.

[6] Dellacherie, C., and P.-A. Meyer (1982): *Probabilities and Potential*, Volume B, Amsterdam, North-Holland.

[7] DeMarzo, P.M., and Y. Sannikov (2004): "Optimal Security Design and Dynamic Capital Structure in a Continuous-Time Agency Model," *Journal of Finance*, forthcoming.

[8] Dumas, B. (1991): "Super Contact and Related Optimality Conditions," *Journal of Economic Dynamics and Control*, 15, 675–685.

[9] Gomes, J.F. (2001): "Financing Investment," *American Economic Review*, 91, 1263–1285.

[10] Hau, H. (2006): "The Role of Transaction Costs for Financial Volatility: Evidence from the Paris Bourse," *Journal of the European Economic Association*, 4, 862–890.

[11] Hennessy, C.A., and T.M. Whited (2005): "Debt Dynamics," *Journal of Finance*, 60, 1129–1165.

[12] Jeanblanc-Picqué, M., and A.N. Shiryaev (1995): "Optimization of the Flow of Dividends," *Russian Mathematical Surveys*, 50, 257–277.

[13] Jensen, M., (1986): "Agency Costs of Free Cash Flow, Corporate Finance, and Takeovers," *American Economic Review*, 76, 323–329.

[14] Karatzas, I., and S.E. Shreve (1991): *Brownian Motion and Stochastic Calculus*, New York, Springer.

[15] Leary, M.T., and M.R. Roberts (2005): "Do Firms Rebalance their Capital Structure?" *Journal of Finance*, 60, 2575–2619.

[16] Lee, I., S. Lochhead, J. Ritter, and Q. Zhao (1996): "The Costs of Raising Capital," *Journal of Financial Research*, 19(1): 59–74.

[17] Løkka, A., and M. Zervos (2005): "Optimal Dividend and Issuance of Equity in the Presence of Proportional Costs," Mimeo, Department of Mathematics, King's College, London.

[18] Mella-Barral, P., and W. Perraudin (1997): "Strategic Debt Service," *Journal of Finance*, 52, 531–556.

[19] Merton, R.C. (1973): "The Theory of Rational Option Pricing," *Bell Journal of Economics*, 4, 141–183.

[20] Modigliani, F., and M. Miller (1958): "The Cost of Capital, Corporation Finance, and the Theory of Investment," *American Economic Review*, 48, 261–297.

[21] Myers, S., and N. Majluf (1984): "Corporate Financing and Investment Decisions when Firms have Information that Investors Do not Have," *Journal of Financial Economics*, 13, 187–221.

[22] Nelson, D.B. (1991): "Conditional Heteroskedasticity in Asset Returns: A New Approach," *Econometrica*, 59, 347–370.

[23] Oded, J., (2005): "Payout Policy, Financial Flexibility, and Agency Costs of Free Cash Flow," Working paper, Sloan School of Management, Boston.

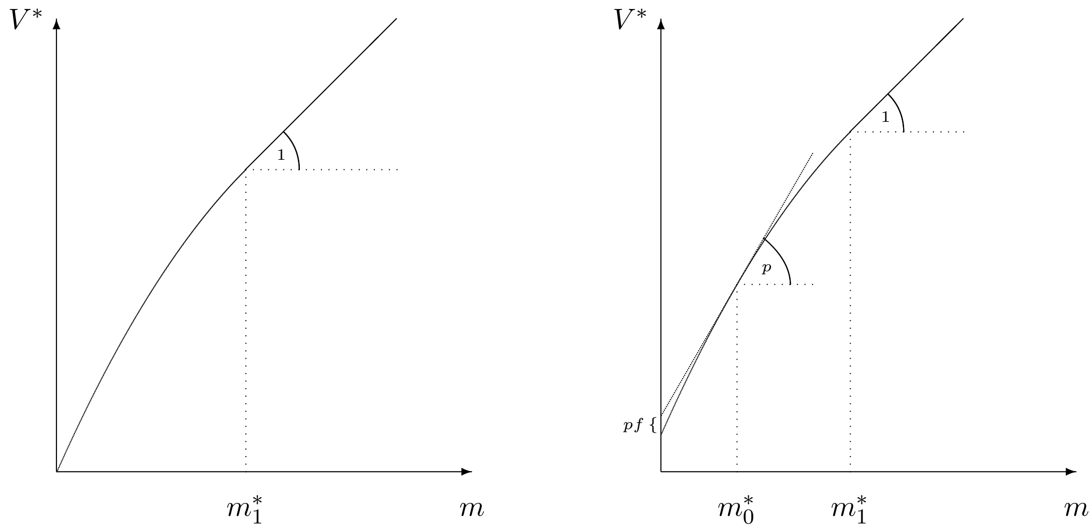
[24] Radner, R., and L. Shepp (1996): "Risk vs. Profit Potential: A Model for Corporate Strategy," *Journal of Economic Dynamics and Control*, 20, 1373–1393.

[25] Sethi, S.P., and M.I. Taksar (2002): "Optimal Financing of a Corporation Subject to Random Returns," *Mathematical Finance*, 12, 155–172.

[26] Shreve, S.E., J.P. Lehoczky, and D.P. Gaver (1984): "Optimal Consumption for General Diffusions with Absorbing and Reflecting Barriers," *SIAM Journal of Control and Optimization*, 22, 55–75.

[27] Skorokhod, A.V. (1961): "Stochastic Equations for Diffusion Processes in a Bounded Region," *Theory of Probability and Its Applications*, 6, 264–274.

[28] Smith, C.W., Jr. (1977): "Alternative Methods for Raising Capital: Rights versus Underwritten Offerings," *Journal of Financial Economics*, 5, 273–307.



Condition (32) holds   Condition (34) holds

FIGURE 1. The value function V^*.