Convexity theory for the term structure equation

Erik Ekström[1] and Johan Tysk[2]

[1]*Dept of Mathematics, Uppsala University, Box 480, 751 06 Uppsala, Sweden*
erik.ekstrom@math.uu.se
[2]*Dept of Mathematics, Uppsala University, Box 480, 751 06 Uppsala, Sweden*
johan.tysk@math.uu.se

ABSTRACT

Already in the seminal paper [3], convexity of the option price in the underlying is discussed. In the last decade this issue has attracted renewed interest in the literature starting with [1]. In this paper we conduct a study of convexity of solutions to the term structure equation using techniques developed in [2]. We assume that the short rate is modeled under some given risk-neutral probability measure as a stochastic process $X = (X_t)_{t\geq 0}$ with dynamics $dX_t = \beta(X_t, t)\,dt + \sigma(X_t, t)\,dB_t$, where B is a standard Brownian motion and β and σ are given functions of time and the current short rate. We show that if the drift β satisfies $\beta_{xx} \leq 2$, then the bond prices are convex in the current short rate, increasing in the volatility of the short rate and decreasing in the drift. Similar results hold for call options written on bond prices. For models with regular coefficients, the condition $\beta_{xx} \leq 2$ is also a necessary condition for preservation of convexity. We also have a general comparison theorem: if a model has smaller drift and larger volatility than another model, and at least one of them has a drift satisfying the condition above, then the first model has the larger bond prices. For bond call options the analogous result holds.
We also study convexity properties of the logarithm of a bond price corresponding to the relative sensitivity of the bond price to changes in the short rate. This relative sensitivity is often described by the duration, i.e. the negative of the derivative of the logarithm. We show that if the drift β is concave and the square σ^2 of the volatility is convex, then bond prices are log-convex (a decreasing duration). Similarly, if β is convex and σ^2 is concave, then bond prices are log-concave (an increasing duration). We also note that if we demand that the price is log-convex *and* log-concave, we recover the well-known sufficient conditions for a model to admit an affine term structure.

References

[1] Bergman, Y., Grundy, B. and Wiener, Z. *General properties of option prices.* J. Finance 51 (1996), 1573-1610.

[2] Janson, S. and Tysk, J. *Preservation of convexity of solutions to parabolic equations*, J. Differential Equations 206 (2004), 182-226.

[3] Merton R. *Theory of rational option pricing.* Bell J. Econom. and Management Sci. 4 (1973), 141-183.