Exponentially affine martingales

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Outline

1. Semimartingale characteristics
2. Affine processes
3. Exponentially affine martingales
4. Applications

Semimartingale characteristics

- **Idea**: Local characterization of semimartingales
- **Deterministic**:
  Linear functions characterized by constant increments
  Derivative: local approximation by linear functions
- **Stochastic analogon**:
  Lévy processes characterized by independent, stationary increments
  Semimartingale characteristics: local approximation by Lévy processes

Semimartingale characteristics

- Linear function determined by slope $b \in \mathbb{R}$
- Distribution of Lévy process $(X_t)_{t\in\mathbb{R}_+}$ on $\mathbb{R}^d$ determined by **Lévy-Khintchine triplet** (b, c, F), $b \in \mathbb{R}^d$, $c \in \mathbb{R}^{d\times d}$, F measure on $\mathbb{R}^d$
- (Differential) Semimartingale characteristics:

$$\partial X_t := (b_t(\omega), c_t(\omega), F_t(\omega, \cdot))$$

- local Lévy-Khintchine triplet
- Time-dependent and random

Semimartingale characteristics

- X Lévy process: Lévy-Khintchine triplet
- X differentiable: Derivative
- Connection to (integral) characteristics (B, C, ν) from Jacod & Shiryaev (2003):

$$B_t = \int_0^t b_s ds, \quad C_t = \int_0^t c_s ds, \quad \nu([0,t]\times G) = \int_0^t F_s(G)ds, \quad \forall G \in \mathscr{B}^d$$

- For details see e.g. Jacod & Shiryaev (2003) and Kallsen (2006)

Semimartingale characteristics

- **Idea**: Modeling through local dynamics
- **Deterministic**: Ordinary differential equation

$$\frac{d}{dt}X_t = f(t, X_t)$$

- **Stochastic analogon**: Martingale problem, $\partial X = (b, c, F)$ with

$$\begin{aligned} b_t(\omega) &= \beta(t, X_{t-}(\omega)) \\ c_t(\omega) &= \gamma(t, X_{t-}(\omega)) \\ F_t(\omega, G) &= \varphi(t, G, X_{t-}(\omega)) \end{aligned}$$

- **Example**: β, γ, φ constant $\Rightarrow$ X Lévy process

Affine processes

- Affine process: Characteristics $\partial X = (b, c, F)$ affine in X_- :

$$b_t(\omega) = \beta_0 + \sum_{j=1}^{d} X^j_{t-}(\omega)\beta_j$$

$$c_t(\omega) = \gamma_0 + \sum_{j=1}^{d} X^j_{t-}(\omega)\gamma_j$$

$$F_t(G, \omega) = \varphi_0(G) + \sum_{j=1}^{d} X^j_{t-}(\omega)\varphi_j(G)$$

- $(\beta_j, \gamma_j, \varphi_j)$ given Lévy-Khintchine triplets
- Duffie, Filipović & Schachermayer (2003): Admissibility conditions for existence and uniqueness, Filipović (2005): time-inhomogeneous triplets

Affine processes

- **Example 1**: **Heston's** stochastic volatility model
- $S_t = \mathscr{E}(X)_t$ asset price, v_t volatility, where (v, X) solves

$$dv_t = (\kappa - \lambda v_t)dt + \sigma\sqrt{v_t}dZ_t$$
$$dX_t = (\mu + \delta v_t)dt + \sqrt{v_t}dW_t$$

- W, Z Brownian motions with constant correlation ϱ
- Characteristics:

$$\partial \begin{pmatrix} v \\ X \end{pmatrix}_t = \left(\begin{pmatrix} \kappa - \lambda v_t \\ \mu - \delta v_t \end{pmatrix}, \begin{pmatrix} \sigma^2 v_t & \sigma\varrho v_t \\ \sigma\varrho v_t & v_t \end{pmatrix}, 0 \right)$$

- affine in $v = v_-$

Affine processes

- **Example 2**: Stochastic volatility modeled by time-changed Lévy models of **Carr, Geman, Madan & Yor**:

$$\begin{aligned} X_t &= X_0 + L_{V_t} \\ dV_t &= v_{t-}dt, \\ dv_t &= -\lambda v_t dt + dZ_t \end{aligned}$$

- L, Z independent Lévy processes with Lévy-Khintchine triplets (b^L, c^L, F^L) and $(b^Z, 0, F^Z)$ respectively

- Characteristics:

$$\left(\begin{pmatrix} b^Z - \lambda v_{t-} \\ b^L v_{t-} \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & c^L v_{t-} \end{pmatrix}, \int 1_G(y,0) F^Z(dy) + \int 1_G(0,x) F^L(dx) v_{t-} \right)$$

- affine in v_-

Exponentially affine martingales

- X affine process, h truncation function
- **Goal**: Conditions on ∂X, such that $\mathscr{E}(X^i)$ is a martingale
- Jacod & Shiryaev (2003) $\Rightarrow \mathscr{E}(X^i) \in \mathscr{M}_{\mathrm{loc}}$, iff

$$\int h^i(x^i) x^i \varphi_j(dx) < \infty, \quad 0 \leq j \leq d$$

$$\beta^i_j + \int (x^i - h^i(x^i))\varphi_j(dx) = 0 \quad 0 \leq j \leq d$$

- Continuous case: Local martingale, iff $\beta^i_j = 0$, $0 \leq j \leq d$
- Extension of this criterion to true martingale property?

Exponentially affine martingales

- For Lévy process X:

$$\mathscr{E}(X^i) \in \mathscr{M}_{\text{loc}}, \Delta(X^i) > -1 \Rightarrow \mathscr{E}(X^i)\text{martingale}$$

- Generally wrong for affine processes!

- True, however, if

$$\int_{\{|x^k|>1\}} |x^k||1 + x^i|\varphi_j(dx) < \infty, \quad 1 \leq k, j \leq d$$

- Satisfied for bounded jumps, in particular in the continuous case

Exponentially affine martingales

Theorem: *∂X affine relative to admissible Lévy-Khintchine triplets $(\beta_j, \gamma_j, \varphi_j)$, $0 \leq j \leq d$. Then $\mathscr{E}(X^i)$ is a martingale, if*

1. $\Delta(X_i) > -1$,
2. $\int h^i(x^i) x^i \varphi_j(dx) < \infty, \quad 0 \leq j \leq d$
3. $\beta^i_j + \int (x^i - h^i(x))\varphi_j(dx) = 0, \quad 0 \leq j \leq d$
4. $\int_{\{|x^k|>1\}} |x^k||1 + x^i|\varphi_j(dx) < \infty, \quad 1 \leq k, j \leq d$

- Similar criterion for $\exp(X^i)$ instead of $\mathscr{E}(X^i)$
- Extension to time-inhomogeneous affine processes possible

Exponentially affine martingales

- Continuous case: $\beta^i_j = 0 \Leftrightarrow \mathscr{E}(X^i) \in \mathscr{M}_{\mathrm{loc}} \Leftrightarrow \mathscr{E}(X^i)$martingale
- Asset price $S = \mathscr{E}(X)$ in **Heston's** model: characteristics

$$\partial \begin{pmatrix} v \\ X \end{pmatrix}_t = \left(\begin{pmatrix} \kappa - \lambda v_t \\ \mu + \delta v_t \end{pmatrix}, \begin{pmatrix} \sigma^2 v_t & \sigma \varrho v_t \\ \sigma \varrho v_t & v_t \end{pmatrix}, 0 \right)$$

- Martingale, iff $\mu = 0$, $\delta = 0$
- **CGMY** asset price: martingale, if $F^L(\{y \in \mathbb{R}; 1 + y < 1\}) = 0$ and

$$\int h(y) y F^L(dy) < \infty$$

$$b^L + \int (y - h(y)) F^L(dy) = 0$$

Application 1: Absolutely continuous change of measure

- Y, Z semimartingales with affine characteristics
- **Goal**: Criterion for $P^Z \overset{\text{loc}}{\ll} P^Y$
- **Application**: Y model for asset price under physical , Z under risk neutral measure. Equivalence for arbitrage theory
- **Idea**: Define candidate Z for density process. Show: Z exponentially affine and local martingale $\Rightarrow$ martingale. Then apply Girsanov's Theorem
- Yields the following result (time-inhomogeneous extension possible):

Application 1: Absolutely continuous change of measure

Theorem: *∂Y, ∂Z affine relative to admissible Lévy-Khintchine triplets $(\beta_j, \gamma_j, \varphi_j)$ and $(\widetilde{\beta}_j, \widetilde{\gamma}_j, \widetilde{\varphi}_j)$, $0 \le j \le d$. Assume there exist $H \in \mathbb{R}^d$ and a Borel function $W : \mathbb{R}^d \to [0, \infty)$, such that, for $0 \le j \le d$,*

1. $\int (1 - \sqrt{W(x)})^2 \varphi_j(dx) < \infty$,
2. $\widetilde{\varphi}_j(G) = \int 1_G(x) W(x) \varphi_j(dx)$,
3. $\int |h(x)(W(x) - 1)| \varphi_j(dx) < \infty$,
4. $\widetilde{\beta}_j = \beta_j + H^\top \gamma_j + \int h(x)(W(x) - 1)\varphi_j(dx)$,
5. $\widetilde{\gamma}_j = \gamma_j$.

Then $P^Z \overset{\text{loc}}{\ll} P^Y$

Application 2: Exponential moments

- X $\mathbb{R}^d$-valued affine process, $X_0 = x \in \mathbb{R}^d$
- Duffie, Filipović and Schachermayer (2003): conditional characteristic function given by

$$\mathbb{E}(e^{iu^\top X_T} | \mathscr{F}_t) = \exp(\Phi(T-t, iu) + \Psi(T-t, iu)^\top X_t),$$

- Φ and Ψ solve integro-differential equations with initial value iu
- If analytic extension to open set $\mathscr{U}$ exists,

$$\mathbb{E}(e^{p^\top X_T} | \mathscr{F}_t) = \exp(\Phi(T-t, p) + \Psi(T-t, p)^\top X_t), \quad \forall p \in \mathscr{U}$$

- Problem: construction of analytic extension is often tedious

Application 2: Exponential moments

- **Alternative approach**: Use results on exponentially affine martingales
- Solve integro-differential equations with initial value p
- Under mild conditions on the big jumps of X, solutions Φ, Ψ satisfy

$$\mathbb{E}(e^{p^\top X_T} | \mathscr{F}_t) = \exp(\Phi(T-t) + \Psi(T-t)^\top X_t),$$

- Conditions met automatically e.g. in the continuous case or in the model of CGMY

Application 3: Portfolio optimization

- **Goal**: Find trading strategy φ, such that
$$\mathbb{E}(u(V_T(\varphi))) \geq \mathbb{E}(u(V_T(\psi))), \quad \forall \psi$$
- u utility function, e.g. $u(x) = x^{1-p}/(1-p)$, $u(x) = \log(x)$ or $u(x) = 1 - (1/p)\exp(-px)$
- φ, ψ admissible, i.e. $V(\varphi), V(\psi) \geq 0$
- Asset price modeled by stochastic volatility model, e.g. Heston or CGMY

Application 3: Portfolio optimization

- **Sufficient criterion for optimality**: If there exists a positive martingale Z, such that

  1. $(ZS)^T \in \mathscr{M}_{\mathrm{loc}}$
  2. $Z_T = u'(V_T(\varphi))$
  3. $(ZV(\varphi))^T$ is a martingale

  we have
  $$\mathbb{E}(u(V_T(\varphi))) \geq \mathbb{E}(u(V_T(\psi))), \quad \forall \psi$$

- **Idea**: Exponentially affine ansatz for Z, computation of ansatz functions through drift conditions, verification with results on exponentially affine martingales

References

- Duffie, Filipović & Schachermayer (2003): Affine processes and applications in finance, *The Annals of Applied Probability*, 13(3), 984-1053
- Filipović (2005): Time-inhomogeneous affine processes, *Stochastic Processes and their Applications* 115(4), 639-659
- Jacod & Shiryaev (2003): *Limit theorems for stochastic processes*, Springer Verlag, Berlin, second edition
- Kallsen (2006): A didactic note on affine stochastic volatility models, in *From stochastic calculus to mathematical finance*, 343-368, Springer, Berlin