On exponentially affine martingales

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Outline

1. Semimartingale characteristics
2. Affine processes
3. Exponentially affine martingales
4. Applications

Semimartingale characteristics

- **Idea**: Local characterization of semimartingales
- **Deterministic**:
  Linear functions characterized by constant increments
  Derivative: local approximation by linear functions
- **Stochastic analogon**:
  Lévy processes characterized by independent, stationary increments
  Semimartingale characteristics: local approximation by Lévy processes

Semimartingale characteristics

- Linear function determined by slope $b \in \mathbb{R}$
- Distribution of Lévy process $(X_t)_{t \in \mathbb{R}_+}$ on $\mathbb{R}^d$ determined by **Lévy-Khintchine triplet** (b, c, F)
- (Differential) Semimartingale characteristics:

$$\partial X_t := (b_t(\omega), c_t(\omega), F_t(\omega, \cdot))$$

- Local Lévy-Khintchine triplet, time-dependent and random
- Connection to (integral) characteristics from Jacod & Shiryaev:

$$B_t = \int_0^t b_s ds, \quad C_t = \int_0^t c_s ds, \quad \nu([0,t] \times G) = \int_0^t F_s(G) ds$$

Semimartingale characteristics

- **Idea**: Modeling through local dynamics
- **Deterministic**: Ordinary differential equation

$$\frac{d}{dt} X_t = f(t, X_t)$$

- **Stochastic analogon**: Martingale problem, $\partial X = (b, c, F)$ with

$$\begin{aligned} b_t(\omega) &= \beta(t, X_{t-}(\omega)) \\ c_t(\omega) &= \gamma(t, X_{t-}(\omega)) \\ F_t(\omega, G) &= \varphi(t, G, X_{t-}(\omega)) \end{aligned}$$

- β, γ, φ constant: X Lévy process

Affine processes

- Affine process: Characteristics $\partial X = (b, c, F)$ affine in X_- :

$$b_t(\omega) = \beta_0 + \sum_{j=1}^{d} X_{t-}^j(\omega)\beta_j$$

$$c_t(\omega) = \gamma_0 + \sum_{j=1}^{m} X_{t-}^j(\omega)\gamma_j$$

$$F_t(G, \omega) = \varphi_0(G) + \sum_{j=1}^{m} X_{t-}^j(\omega)\varphi_j(G)$$

- $(\beta_j, \gamma_j, \varphi_j)$ given Lévy-Khintchine triplets

- Duffie, Filipović & Schachermayer (2003): Admissibility conditions for existence and uniqueness, Filipović (2005): time-inhomogeneous triplets

Affine processes

- **Example**: Stochastic volatility model of **Heston** (1993)
- $S_t = \mathscr{E}(X)_t$ asset price, v_t volatility, where (v, X) solves

$$\begin{aligned} dv_t &= (\kappa - \lambda v_t)dt + \sigma\sqrt{v_t}dZ_t \\ dX_t &= (\mu + \delta v_t)dt + \sqrt{v_t}dW_t \end{aligned}$$

- W, Z Brownian motions with constant correlation ϱ
- Characteristics:

$$\partial \begin{pmatrix} v \\ X \end{pmatrix}_t = \left(\begin{pmatrix} \kappa - \lambda v_t \\ \mu - \delta v_t \end{pmatrix}, \begin{pmatrix} \sigma^2 v_t & \sigma\varrho v_t \\ \sigma\varrho v_t & v_t \end{pmatrix}, 0 \right)$$

- affine in $v = v_-$

Affine processes

- **Example**: Model of **Carr**, **Geman**, **Madan** & **Yor** (2001)

$$\begin{aligned} X_t &= X_0 + L_{V_t} \\ dV_t &= v_{t-}dt, \\ dv_t &= -\lambda v_t dt + dZ_t \end{aligned}$$

- L, Z independent Lévy processes with Lévy-Khintchine triplets (b^L, c^L, F^L) and $(b^Z, 0, F^Z)$

- Characteristics:

$$\left(\begin{pmatrix} b^Z - \lambda v_{t-} \\ b^L v_{t-} \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & c^L v_{t-} \end{pmatrix}, \int 1_G(y,0) F^Z(dy) + \int 1_G(0,x) F^L(dx) v_{t-} \right)$$

- affine in v_-

Exponentially affine martingales

- X affine process, h truncation function
- **Goal**: Conditions for $\mathscr{E}(X^i)$ to be a martingale
- Jacod & Shiryaev (2003) $\Rightarrow$ $\mathscr{E}(X^i)$ is a local martingale, iff

$$\int h^i(x^i)x^i\varphi_j(dx) < \infty, \quad 0 \le j \le d$$

$$\beta_j^i + \int (x^i - h^i(x^i))\varphi_j(dx) = 0, \quad 0 \le j \le d$$

- Continuous case: Local martingale, iff $\beta_j^i = 0$, $0 \le j \le d$
- Condition on ∂X, easy to check in applications

Exponentially affine martingales

- Criterion for true martingale property?
- For Lévy process X:

$$\mathscr{E}(X) \text{ local martingale}, \mathscr{E}(X) \geq 0 \Rightarrow \mathscr{E}(X)\text{martingale}$$

- Does NOT hold in general for affine processes!
- Some general criteria either hard to check (e.g. Wong & Heyde (2004), Jacod & Shiryaev (2003), Kallsen & Shiryaev (2002))
- Others not even necessary in the Lévy case (e.g. Lepingle & Memin (1978))
- On the other hand: powerful theory of affine processes

Exponentially affine martingales

Theorem: *∂X affine relative to admissible Lévy-Khintchine triplets $(\beta_j, \gamma_j, \varphi_j)$, $0 \le j \le d$. Then $\mathscr{E}(X^i)$ is a martingale, if*

1. $\mathscr{E}(X_i) \geq 0$,
2. $\int h^i(x^i)x^i\varphi_j(dx) < \infty, \quad 0 \le j \le d$
3. $\beta^i_j + \int (x^i - h^i(x))\varphi_j(dx) = 0, \quad 0 \le j \le d$
4. $\int_{\{|x^k|>1\}} |x^k||1 + x^i|\varphi_j(dx) < \infty, \quad 1 \le k, j \le d$

- Similar criterion for $\exp(X^i)$ instead of $\mathscr{E}(X^i)$
- Extension to time-inhomogeneous affine processes possible

Exponentially affine martingales

- Continuous case (e.g. Heston (1993)):

$$\mathscr{E}(X) \text{ positive local martingale} \Rightarrow \text{ martingale}$$

- Also holds for **CGMY** asset price S
- S positive, if $F^L((-\infty, -1)) = 0$
- S local martingale, if

$$\int (yh(y))F^L(dy) < \infty, \quad b^L + \int (y - h(y))F^L(dy) = 0$$

- For PII X: $\mathscr{E}(X)$ positive local martingale $\Rightarrow$ martingale

Application 1: Absolutely continuous change of measure

- X, Y semimartingales with affine characteristics
- **Goal**: Criterion for $P^Y \overset{\text{loc}}{\ll} P^X$
- **Application**: X model for asset price under physical , Y under risk neutral measure. Equivalence for arbitrage theory
- **Idea**: Define appropriate candidate Z for density process
- Show: Z exponentially affine, local martingale $\Rightarrow$ martingale
- Define $Q \overset{\text{loc}}{\ll} P^X$ via density process $Z \Rightarrow Q = P^Y$ by Girsanov and uniqueness of affine martingale problems

Application 1: Absolutely continuous change of measure

- Similar results by Cheridito, Filipović and Yor (2005) in more general setup
- Here, the moment conditions are often less restrictive though
- **Example**: Esscher change of measure
- Condition in Cheridito, Filipović, & Yor (2005):

$$\int_{\{|x|>1\}} (H^\top x) e^{H^\top x} \varphi_j(dx) < \infty, \quad 0 \leq j \leq d$$

- Condition here:

$$\int_{\{|x|>1\}} e^{H^\top x} \varphi_j(dx) < \infty, \quad 0 \leq j \leq d$$

Application 2: Exponential moments

- X $\mathbb{R}^d$-valued affine process
- Duffie, Filipović and Schachermayer (2003): conditional characteristic function given by

$$\mathbb{E}(e^{iu^\top X_T}|\mathscr{F}_t) = \exp(\Phi(T-t, iu) + \Psi(T-t, iu)^\top X_t),$$

- Φ and Ψ solve integro-differential equations with initial values 0, iu
- If analytic extension to open set $\mathscr{U}$ exists,

$$\mathbb{E}(e^{p^\top X_T}|\mathscr{F}_t) = \exp(\Phi(T-t, p) + \Psi(T-t, p)^\top X_t), \quad \forall p \in \mathscr{U}$$

- Problem: construction of analytic extension is often tedious

Application 2: Exponential moments

Alternative approach:

- Assume solutions Φ and Ψ to integro-differential equations with initial values 0 and $p \in \mathbb{R}^d$ exist
- Define $N_t := \Phi(T-t) + \Psi(T-t)X_t$
- Show: (X, N) is affine, $\exp(N)$ is a local martingale
- Results on time-inhomogeneous exponentially affine martingales, (mild) condition on the big jumps of $X \Rightarrow$ martingale
- Martingale property yields

$$\mathbb{E}(e^{p^\top X_T}|\mathscr{F}_t) = \mathbb{E}(e^{N_T}|\mathscr{F}_t) = e^{N_t} = \exp(\Phi(T-t) + \Psi(T-t)^\top X_t)$$

Application 3: Portfolio optimization

- **Goal**: Find trading strategy φ, such that

$$\mathbb{E}(u(V_T(\varphi))) \geq \mathbb{E}(u(V_T(\psi))), \quad \forall \psi$$

- u utility function
- Example: Power utility, i.e. $u(x) = x^{1-p}/(1-p)$
- φ, ψ admissible, i.e. $V(\varphi), V(\psi) \geq 0$
- Asset price modeled as an affine process, e.g. Heston (1993) or Carr, Geman, Madan & Yor (2001)

Application 3: Portfolio optimization

- **Sufficient criterion for optimality**: If there exists a positive martingale Z, such that

  1. $(ZS)^T$ is a local martingale
  2. $Z_T = u'(V_T(\varphi))$
  3. $(ZV(\varphi))^T$ is a martingale

  we have
  $$\mathbb{E}(u(V_T(\varphi))) \geq \mathbb{E}(u(V_T(\psi))), \quad \forall \psi$$

- **Observation**: S, $V(\varphi)$ and $u'(V(\varphi))$ exponentially affine

Application 3: Portfolio optimization

- **Idea**: Make an exponentially affine ansatz for Z as well!
- Computation of ansatz functions through drift conditions
- Verification of the candidate processes with results on exponentially affine martingales
- Approach works for the models proposed by Heston (1993) and Carr, Geman, Madan & Yor (2001) among others

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