Growth rate optimization under transaction costs

Jan Palczewski[*] Łukasz Stettner[†]

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Abstract

This paper studies portfolio optimization in a discrete-time model, where tradeable asset prices and economic factors form a time-homogeneous Markov process. There are transaction costs of a general form, which covers the case of costant and proportional costs. We prove existence of a self-financing trading strategy maximizing the average growth rate. We show that this strategy has a Markovian form. The highlights of this paper are: transaction costs with a constant term and a general form of dependence of asset prices on economic factors. Our result is obtained by a generalization of a vanishing discount method to discontinuous transition operators and by large deviations estimates on empirical measures of the price process.

Keywords: transaction costs, portfolio optimization, growth rate, logarithmic return, Markov process, impulsive strategy, vanishing discount

1. Introduction

On a given probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a discrete filtration $(\mathcal{F}_t)_{t=0,1,\ldots}$, where $\mathcal{F}_0$ is trivial, consider a market modeled by a time homogeneous Markov process $\big(S(t), Z(t)\big)_{t=0,1,\ldots}$, where $S(t) = \big(S^1(t), \ldots, S^d(t)\big) \in (0,\infty)^d$ denotes prices of d assets and $Z(t) \in (E, \mathcal{E})$ models economic factors (E is a separable complete metric space with Borel σ-algebra $\mathcal{E}$). Markovian models with economic factors has recently gained popularity (see eg. [5], [6], [12], [25], [26], [31]), as they extend applicability of Markovian frameworks. Examples cover models with varying market trends and stochastic volatility. Existing literature concentrates on continuous-time diffusion models. In Bielecki et al. [6] economic factors are independent of Brownian motion governing the price process and they affect only the drift of the price process. Fleming and Sheu

[*]School of Mathematics, University of Leeds, Leeds LS2 9JT, UK and Faculty of Mathematics, Warsaw University, Banacha 2, 02-097 Warszawa, Poland (e-mail: J.Palczewski@mimuw.edu.pl)

[†]Institute of Mathematics, Polish Academy of Sciences, Sniadeckich 8, 00-950 Warszawa, Poland, (e-mail: stettner@impan.gov.pl).

[12] allow both processes to have dependent Brownian motions but their diffusions are of a special form. In this paper we consider a discrete time model, in which we assume a general form of dependence of asset prices on the economic factors (see Section 2). A discrete time framework allows us to overcome limitations and technicalities of the existing theory of continous time Markov processes and impulsive control. For optimal investment in a continuous-time general Markovian model with economic factors and transaction costs see Palczewski and Stettner [25].

The market consists of d assets, whose prices are, in general, interdependent. We assume that their prices are positive (they cannot go bankrupt). We impose costs of performing transactions. These costs, in the simplest, consist of a constant part, independent on the transaction, and the proportional part, depending on the volume and type of assets sold and purchased (see 4, 5 and the following discussion). This type of transaction costs prevents continuous trading in continuous time models (see e.g. [25]) and emulates market mechanisms quite well. The framework of this paper allows for a much more general transaction costs structure (see Section 2).

In our model with economic factors and transaction costs, we consider maximisation of the functional

$$J(\Pi) = \liminf_{T\to\infty} \frac{1}{T}\mathbb{E}\, \ln X^{\Pi}(T), \tag{1}$$

where $X^{\Pi}(T)$ is the wealth of the portfolio Π at time T. This functional computes an average growth rate of the portfolio Π as can be seen from the following reformulation of the above formula:

$$J(\Pi) = \liminf_{T\to\infty} \frac{1}{T}\mathbb{E} \sum_{k=0}^{T-1} \ln \frac{X^{\Pi}(k+1)}{X^{\Pi}(k)}. \tag{2}$$

Optimisation with the logarithmic utility function is widely used in the economic and financial community. Optimal portfolios are refered to as log-optimal or growth-optimal. For a broader treatment see textbooks [10], [23]. In mathematical context the research goes back to Kelly (see [32]) and has continued in discrete time ([3]) and continuous time ([1], [2]) up to today ([13], [17], [27]). Functional (1) can be also seen as a risk sensitive functional and the literature is here broad as well ([6], [21], [31]). It should be stressed that the majority of papers considers continuous time diffusion models, where an optimal strategy is obtained as a solution to an appropriate HJB equation, usually reformulated in a variational form. Consequently, the results are based on a sophisticated theory of PDE's and solutions usually do not use directly probabilistic properties of the phenomenon under study. Moreover, due to complexity of studied PDEs the results are of existential form.

In this paper, we approach the optimization problem (1) from a probabilistic point of view. We prove that there exists a self-financing portfolio strategy maximizing the growth-rate (1). We show that this trading strategy has a Markovian form, i.e. an investment decision at time t is based only on values of asset prices and economic factors at t. Main highlights of this paper are: transaction costs with a constant term and a general form of dependence of asset prices on economic factors. As far as we know there is no paper that treats this type of problems in such generality.

The functional (2) is an example of the so-called long-run average cost functional. For a survey of studies on long-run average cost functionals see [4]. A paper by Schäl [29] was a breakthrough and initiated use of Bellman inequalities, which led to significantly more general results. His ideas thrive in [16] (weighted norms), [14] (stochastic games) and recently in [19]. Those results strongly depend on continuity properties of the controlled transition operator of the Markov chain under consideration. In this paper we show that the above ideas can also be used in the study of the problem which significantly violates the continuity assumptions. Moreover, following [19] we are able to remove a requirement for E to be locally compact as is needed in the seminal paper [29]. This significantly generalized the applicability of this framework to incomplete information case (for details on the incomplete information model see [26]).

The paper is organized as follows. In Section 2 we introduce the framework of the problem. We specify the dynamics of asset price process and the form of transaction cost structure. We introduce a process representing proportions of wealth invested in each of the assets at every moment and we reformulate the initial problem in terms of proportions. This reformulation plays a major role in the paper.

Section 3 sees main assumptions presented. We prove ergodicity results and large deviation estimates of the price process. They give a new insight into the dynamics of the price process.

The study of value functions of discounted functionals related to (1) is done in Section 4. A few important technical results on the consequences on the transaction costs are proved. They are used to build a relation between value functions for the discounted problems with and without the constant term in the transaction costs structure. It is a starting point for derivation of the Bellman inequality, which is performed in Section 5. At this stage we also deal with the lack of continuity of the controlled transition operator. We prove the existence of a growth-optimal strategy and show its form. We also relate the results to the case without constant term in the transaction costs structure (see [31]).

2. Preliminaries

In this section we shall specify the model in full detail and introduce notation used in the sequel. The dynamics of the price process is governed by the equation

$$\frac{S^i(t+1)}{S^i(t)} = \zeta^i\Big(Z(t+1), \xi(t+1)\Big), \qquad S^i(0) = s^i > 0, \qquad i = 1, \dots, d, \tag{3}$$

where $\big(\xi(t)\big)_{t=1,2,\dots}$ is a sequence of i.i.d. random variables with values in a Polish space $(E^\xi, \mathcal{E}^\xi)$ and functions $\zeta^i : (E, \mathcal{E}) \times (E^\xi, \mathcal{E}^\xi) \to (0, \infty)$ are Borel measurable, $i = 1, \dots, d$. The process $Z(t)$ is a time-homogeneous Markov process. We assume that $\big(S(t), Z(t)\big)_{t=0,1,\dots}$ forms a weak Feller process, i.e. its transition operator transforms the space of bounded continuous functions into itself. In the sequel we shall write $\zeta^i(t)$ for $\zeta^i\Big(Z(t+1), \xi(t+1)\Big)$, and $\zeta(t)$ for the vector $\big(\zeta^1(t), \dots, \zeta^d(t)\big)$, whenever it does not lead to ambiguity. Filtration generated by $\big(S(t), Z(t)\big)_{t=0,1,\dots}$ represents the knowledge of an investor observing the market. It depends only on the initial value z of the process $\big(Z(t)\big)_{t=0,1,\dots}$, and will be denoted by $\big(\mathcal{F}^z_t\big)_{t=0,1,\dots}$.

Fix $s \in (0, \infty)^d$ and $z \in E$, the initial values of processes $S(t)$ and $Z(t)$. A trading strategy is a sequence of pairs $\big((N_k, \tau_k)\big)_{k=0,1,\dots}$, where $\tau_0 = 0$, $(\tau_k)_{k=1,2,\dots}$ are $\big(\mathcal{F}^z_t\big)$ stopping times, and

$\tau_{k+1} > \tau_k$, $k = 1, 2, \ldots$. Stopping times (τ_k), $k \geq 1$, represent moments of transactions, whereas $\tau_0 = 0$ is only introduced for convenience of notation. The number of shares held in the portfolio in the time interval $[\tau_k, \tau_{k+1})$ is denoted by N_k, which is an $\mathcal{F}^z_{\tau_k}$-measurable random variable with values in $[0, \infty)^d$. Hence, N_0 is a deterministic initial portfolio.

The share holding process at time t is given by

$$N(t) = \sum_{k=1}^{\infty} 1_{t \in [\tau_k, \tau_{k+1})} N_k.$$

In what follows we shall consider transaction costs of one of the forms

$$\tilde{c}(\eta_1, \eta_2, S) = \sum_{i=1}^{d} \Big(c_i^1 S^i (\eta_1^i - \eta_2^i)^+ + c_i^2 S^i (\eta_1^i - \eta_2^i)^- \Big) + C, \tag{4}$$

$$\tilde{c}(\eta_1, \eta_2, S) = \max \left(C, \sum_{i=1}^{d} \Big(c_i^1 S^i (\eta_1^i - \eta_2^i)^+ + c_i^2 S^i (\eta_1^i - \eta_2^i)^- \Big) \right) \tag{5}$$

where $c_i^1, c_i^2 \in [0, 1)$ are proportional costs, $C \geq 0$, S stands for the asset prices at the moment of transaction, η_1 denotes the portfolio contents before transaction, and η_2 – after transaction. We impose a self-financing condition on portfolios, i.e.

$$N_k \cdot S(\tau_k) = N_{k-1} \cdot S(\tau_k) + \tilde{c}\big(N_{k-1}, N_k, S(\tau_k)\big), \quad k = 1, 2, \ldots. \tag{6}$$

Notice that due to the lower bound C on the transaction costs function, transactions cannot be executed if the wealth of the portfolio is smaller than C. It is also clear that (6) depends on the initial value (s, z) of the process $\big(S(t), Z(t)\big)$.

For the clarity of presentation, we shall restrict our attention to the costs of the form (4). However, all the results are easily modified to fit (5). In fact, the results extend to a more general case, see Section 6.

In the case of no transaction costs or proportional transactions costs it is natural to reformulate the problem in terms of proportions as it transforms the set of controls (possible portfolios) to a compact set, which faciliates matematical analysis. In our more general framework, we shall also benefit from this reformulation.

Denote by $X_-(t)$ the wealth of the portfolio before a possible transaction at t and by $X(t)$ the wealth just after the transaction:

$$\begin{aligned} X(t) &= N(t) \cdot S(t), \\ X_-(t) &= N(t-1) \cdot S(t). \end{aligned} \tag{7}$$

If there is no transaction at t both values are identical. In a similar way, for $i = 1, \ldots, d$, we construct two processes representing proportions of our capital invested in the asset i:

$$\begin{aligned} \pi^i(t) &= \frac{N^i(t) S^i(t)}{X(t)}, \\ \pi_-^i(t) &= \frac{N^i(t-1) S^i(t)}{X_-(t)}. \end{aligned} \tag{8}$$

Since short sales are prohibited we have $\pi(t), \pi_-(t) \in \mathcal{S}$, where $\mathcal{S}$ is the unit simplex in $\mathbb{R}^d$:

$$\mathcal{S} = \{(\pi^1, \ldots, \pi^d) : \quad \pi^i \geq 0, \quad \sum_{i=1}^{d} \pi^i = 1\}.$$

Denote by $\mathcal{S}^0$ the simplex $\mathcal{S}$ with its interior

$$\mathcal{S}^0 = \{(\pi^1, \ldots, \pi^d) : \quad \pi^i \geq 0, \quad \sum_{i=1}^{d} \pi^i \leq 1\}$$

and let $g : \mathcal{S}^0 \to \mathcal{S}$ be a projection from $\mathcal{S}^0$ to its boundary $\mathcal{S}$

$$g(\pi^1, \ldots, \pi^d) = \Big(\frac{\pi^1}{\sum \pi^i}, \ldots, \frac{\pi^d}{\sum \pi^i}\Big).$$

Let

$$c(\pi_-, \tilde{\pi}) = \sum_{i=1}^{d} \Big(c_i^1(\tilde{\pi}^i - \pi_-^i)^+ + c_i^2(\tilde{\pi}^i - \pi_-^i)^-\Big).$$

The self-financing condition (6) can be written as

$$X_-(\tau_k) = X(\tau_k) + X_-(\tau_k)\Big(c\big(\pi_-(\tau_k), \tilde{\pi}_k\big) + \frac{C}{X_-(\tau_k)}\Big), \quad k = 1, 2, \ldots \tag{9}$$

for some $\tilde{\pi}_k \in \mathcal{S}^0$ such that $\pi(\tau_k) = g\big(\tilde{\pi}_k\big)$. From (6) one can deduce that $\tilde{\pi}_k = \frac{X(\tau_k)}{X_-(\tau_k)}\pi(\tau_k)$ satisfies (9).

Given $\pi_-, \pi \in \mathcal{S}$, $x_- > 0$ define a function

$$F^{\pi_-, \pi, x_-}(\delta) = c\big(\pi_-, \delta\pi\big) + \frac{C}{x_-} + \delta.$$

Equation (9) can be written equivalently as

$$F^{\pi_-(\tau_k), \pi(\tau_k), X_-(\tau_k)}\Big(\sum_{i=1}^{d} \tilde{\pi}_k^i\Big) = 1.$$

The following lemma states a crucial property of F that will be used to reformulate the self-financing condition.

LEMMA 2.1. There exists a unique function $e : \mathcal{S} \times \mathcal{S} \times (0, \infty) \to [0, 1]$, such that

(1) if $e(\pi_-, \pi, x) > 0$, then $F^{\pi_-, \pi, x_-}\big(e(\pi_-, \pi, x_-)\big) = 1$,

(2) $e(\pi_-, \pi, x_-) = 0$ if and only if the equation $F^{\pi_-, \pi, x_-}(\cdot) = 1$ has no solution in $(0, 1]$.

Moreover, e is continuous.

Proof. The proof is rather straightforward and resembles the proof of Lemma 1 in [31]. ■

Let $\big((N_k, \tau_k)\big)$ be a self-financing trading strategy and let $\pi_-(\tau_k)$, $\pi(\tau_k)$ be defined as above. By virtue of Lemma 2.1 for any $k \in \mathbb{N}$ we have

$$F^{\pi_-(\tau_k),\pi(\tau_k),X_-(\tau_k)}\Big(e\big(\pi_-(\tau_k), \pi(\tau_k), X_-(\tau_k)\big)\Big) = 1$$

and $\frac{X(\tau_k)}{X_-(\tau_k)} = e\big(\pi_-(\tau_k), \pi(\tau_k), X_-(\tau_k)\big)$. Second assertion is a consequence of the uniqueness of e and equation (9). Therefore, any transaction can be described solely by means of proportions $\pi_-(\tau_k)$ and $\pi(\tau_k)$, and the portfolio wealth $X_-(\tau_k)$. Consequently, any trading strategy has a unique representation in the following form: the initial wealth $x_- = N_0 \cdot S(0)$, the initial proportion

$$\pi_- = \left(\frac{N_0^1 S^1(0)}{N_0 \cdot S(0)}, \dots, \frac{N_0^d S^d(0)}{N_0 \cdot S(0)}\right),$$

and $\big((\pi_k, \tau_k)\big)_{k=1,2,\dots}$, where π_k is the post-transaction proportion represented by $\mathcal{S}$-valued $\mathcal{F}_{\tau_k}$-measurable random variable. Indeed, define the corresponding pre-transaction proportion process $\pi_-(t)$ by

$$\begin{aligned} &\pi_-(0) = \pi_-, \\ &\pi_-(t) = \pi_k \diamond \zeta(\tau_k + 1) \diamond \dots \diamond \zeta(t), \qquad \tau_k < t \le \tau_{k+1}, \end{aligned} \tag{10}$$

where for simplicity of the notation we introduce $\tau_0 = 0$ and

$$\pi \diamond \zeta = g(\pi^1 \zeta^1, \dots, \pi^d \zeta^d), \qquad \pi \in \mathcal{S}, \quad \zeta \in (0, \infty)^d. \tag{11}$$

The corresponding post-transaction proportion process is given by

$$\begin{aligned} &\pi(0) = \pi_-, \qquad \tau_1 > 0, \\ &\pi(t) = \pi_k \diamond \zeta(\tau_k + 1) \diamond \dots \diamond \zeta(t), \qquad \tau_k \le t < \tau_{k+1}. \end{aligned} \tag{12}$$

At the moment τ_k the pre-transaction wealth $X_-(\tau_k)$ is diminished to

$$X(\tau_k) = X_-(\tau_k) e\big(\pi_-(\tau_k), \pi(\tau_k), X_-(\tau_k)\big).$$

Furthermore,

$$X_-(t+1) = \sum_{i=1}^{d} \frac{\pi^i(t) X^i(t)}{S^i(t)} S^i(t+1) = X(t)\Big(\pi(t) \cdot \zeta(t)\Big).$$

Consequently,

$$X_-(t) = X_-(0) \prod_{s=0}^{t-1} \Big(\pi(s) \cdot \zeta(s)\Big) \prod_{k=1}^{\infty} \Big(1_{\tau_k < t} e\big(\pi_-(\tau_k), \pi(\tau_k), X_-(\tau_k)\big) + 1_{\tau_k \ge t}\Big), \tag{13}$$

which finishes the construction of the correspondence between the primal definition of a trading strategy with share holding process $N(t)$ and the equivalent form with proportions. Notice that due to our reformulation, the self-financing condition no longer depends on the initial value of the asset price process $\big(S(t)\big)$.

Let $\mathcal{A}^z$ be a set of sequences $\big((\pi_k, \tau_k)\big)_{k=1,2,\ldots}$, where (τ_k) are $(\mathcal{F}_t^z)$ stopping times, $\tau_{k+1} > \tau_k$, $k = 1, 2, \ldots$ and π_k is a $\mathcal{F}_{\tau_k}^z$-measurable random variables with values in $\mathcal{S}$. Elements of $\mathcal{A}^z$ will be called admissible trading strategies or admissible portfolios. Notice that for a fixed initial wealth x_- and an initial proportion π_- not every admissible trading strategy $\big((\pi_k, \tau_k)\big)$ is related to some self-financing strategy $\big((N_k, \tau_k)\big)$. Indeed, if $X_-(\tau_k)$ is small for some k, not all proportions $\pi(\tau_k)$ are attainable from $\pi_-(\tau_k)$. If $\pi(\tau_k)$ is attainable, we have

$$F^{\pi_-(\tau_k),\pi(\tau_k),X_-(\tau_k)}\Big(e\big(\pi_-(\tau_k), \pi(\tau_k), X_-(\tau_k)\big)\Big) = 1$$

and

$$X(\tau_k) = X_-(\tau_k)e\big(\pi_-(\tau_k), \pi(\tau_k), X_-(\tau_k)\big).$$

If $\pi(\tau_k)$ is not attainable, we have, by the above construction,

$$e\big(\pi_-(\tau_k), \pi(\tau_k), X_-(\tau_k)\big) = 0$$

and

$$X(\tau_k) = 0.$$

Therefore, in what follows we may assume that all proportions are attainable from $\pi_-(\tau_k)$ irrespective of the value of $X_-(\tau_k)$, but they may lead to zero wealth process if we cannot afford to pay transaction costs. Since the strategy allowing anihilation of wealth is not optimal (the functional in (14) evaluates as $-\infty$), the extension of the set of trading strategies does not have any impact on optimal strategies.

As we noticed before, the set of admissible strategies and the wealth of the portfolio are independent of the initial prices of the assets. Therefore, instead of writing $\mathbb{P}^{(s,z)}$ and $\mathbb{E}^{(s,z)}$ to stress dependence of the probability measure on the initial condition of the Markov process $\big(S(t), Z(t)\big)$ we will write $\mathbb{P}^z$ and $\mathbb{E}^z$.

The goal of this paper is to maximize the functional

$$J^{\pi_-,x_-,z}(\Pi) = \liminf_{T\to\infty} \frac{1}{T}\mathbb{E}^z \ln X_-(T) \tag{14}$$

over all portfolios $\Pi \in \mathcal{A}^z$, where π_- is the initial proportion, x_- denotes the initial wealth and z is the initial state of the economic factor process. Observe that using (13) we obtain

$$\begin{aligned} J^{\pi_-,x_-,z}(\Pi) = \liminf_{T\to\infty} \frac{1}{T}\bigg\{ &\sum_{t=0}^{T-1} \mathbb{E}^z \ln \pi(t)\cdot\zeta(t) \\ &+ \sum_{k=1}^{\infty} \mathbb{E}^z \Big\{ 1_{\tau_k<T} \ln e\big(\pi_-(\tau_k), \pi_k, X_-(\tau_k)\big)\Big\}\bigg\}. \end{aligned} \tag{15}$$

This transformes our problem to the form suitable for further analysis.

3. Assumptions and basic properties of the price process

Denote by $P(z,dy)$ the transition operator of the process $Z(t)$. Let $\hat{E} = E \times E^{\xi}$, and let ν be the law of $\xi(1)$ on E^{ξ}. For $x = (z,\xi)$ and a bounded measurable function w on $\hat{E}$ define

$$\hat{P}w(x) = \int_E \int_{E^{\xi}} w(z',\xi')\nu(d\xi')P(z,dz').$$

Consider the following assumptions:

(A1) The process $\big(S(t), Z(t)\big)$ satisfies the Feller property i.e. its transition operator maps the space of continuous bounded functions into itself.

(A2) $\mathcal{S} \times E \ni (\pi, z) \mapsto h(\pi,z) = \mathbb{E}^{\,z}\big\{\ln \pi \cdot \zeta\big(z(1),\xi(1)\big)\big\}$ is a bounded, continuous function.

(A3) $\sup_{z,z' \in E} \sup_{B \in \mathcal{E}} \big(P^n(z,B) - P^n(z',B)\big) = \kappa < 1$ for some $n \geq 1$.

(A4) There is a continuous function $\hat{u}_0$ defined on $\hat{E}$ such that $\hat{u}_0(x) \geq 1$ for $x \in \hat{E}$, the function $x \mapsto \hat{P}\hat{u}_0(x)$ is bounded on compact subsets of $\hat{E}$ and for any positive real number l the set $\Big\{x : \frac{\hat{u}_0(x)}{\hat{P}\hat{u}_0(x)} \leq l\Big\}$ is compact.

(A5) The function $\zeta(z,\xi)$ is continuous and separated from 0, i.e. $\inf_{z,\xi} \zeta^i(z,\xi) > 0$ for $i = 1,\ldots,d$.

Due to assumption (A3) the process $Z(t)$ has very strong ergodic properties that we will use frequently. Together with (A4)-(A5) it gives important estimates on the behaviour of the asset prices, as can be seen in the following theorem:

THEOREM 3.1. Under (A1)-(A5):

i) the process $Z(t)$ has a unique invariant probability measure ϑ.

ii) for each non-negative measurable function f such that $\int_E f(z)\vartheta(dz) < \infty$

$$\lim_{t\to\infty} \mathbb{E}^{\,z} f\big(Z(t)\big) \to \int_E f(z)\vartheta(dz).$$

iii) The following large deviations estimate holds: for each $\epsilon > 0$ there exists $T^* > 0$, $\gamma > 0$, $K > 0$ such that for all $T \geq T^*$

$$\mathbb{P}^z\Big\{\frac{1}{T}\ln\Big(\prod_{t=0}^{T-1}\hat{\zeta}\big(Z(t+1),\xi(t+1)\big)\Big) \leq \hat{p} - \epsilon\Big\} \leq Ke^{-\gamma T},$$

where

$$\hat{\zeta}(z,\xi) = \min\big(\zeta^1(z,\xi),\ldots,\zeta^d(z,\xi)\big)$$

and

$$\hat{p} = \int_{E\times E^{\xi}} \ln\hat{\zeta}(z,\xi)\,\vartheta(dz)\nu(d\xi).$$

Proof. Notice that (A3) implies that for arbitrary $z, z' \in E$ and $B \in \mathcal{E}$

$$P^n(z, B) \le \kappa + P^n(z', B)\big).$$

Therefore, Condition (D) in [9], Section 5.2, holds with $\phi(B) = P^n(z', B)$ for some $z' \in E$ and $\epsilon = \frac{1-\kappa}{2}$ (this is a version of Doeblin's hypothesis). Clearly, due to (A3) there is exactly one ergodic set. Indeed, for any bounded measurable function f we have

$$\mathbb{E}^{z_1} f\big(Z(n)\big) - \mathbb{E}^{z_2} f\big(Z(n)\big) = \int_E f(z)\big(P^n(z_1, dz) - P^n(z_2, dz)\big) \le \kappa \|f\|_\infty.$$

Therefore, there is at most one invariant probability measure, which proves uniqueness of the ergodic set. Consequently, Theorem 6.1 in [9] implies (i) and (ii).

Statement (iii) results from application of the Large Deviations Theory to the Markov process $\big(Z(t), \xi(t)\big)$. Recall that the transition operator of this process is denoted by $\hat{P}$. A measure $\vartheta \otimes \nu$ is a unique probabilistic invariant measure of $\hat{P}$. Let

$$L_n = \frac{1}{n} \sum_{t=1}^{n} \delta_{(Z(t), \xi(t))}, \qquad n = 1, 2, \ldots,$$

denote the empirical distribution of the process $\big(Z(t), \xi(t)\big)$. Notice that L_n takes values in the space $\mathcal{P} = \mathcal{P}(E \times E^\xi)$ of probability measures on $E \times E^\xi$ with the weak convergence topology. Due to Section 4 of [8] (see also [11] and [22]) there exists a convex lower semicontinuous function $J : \mathcal{P} \to \mathbb{R}$ (called a good rate function) such that for any compact set $\Gamma \in \mathcal{B}(\mathcal{P})$ we have

$$\limsup_{n\to\infty} \frac{1}{n} \log \left(\sup_{z\in E} \mathbb{P}^z\Big(\{\omega : L_n(\omega) \in \Gamma\}\Big)\right) \le -\inf_{\mu\in\Gamma} J(\mu).$$

Under assumption (A4) the above inequality holds for any closed set Γ (not necessarily compact). By Lemma 4.2 of [8], the set of measures $\mu \in \mathcal{P}$ such that $J(\mu) \le l$ is compact for each $l \in \mathbb{R}$. Consequently, for a closed set $\Gamma \subset \mathcal{P}$ such that $\vartheta \otimes \nu \notin \Gamma$ we have (see Proposition 1 of [11]) $\inf_{\mu\in\Gamma} J(\mu) > 0$.

Define

$$\hat{\Gamma} = \Big\{\mu \in \mathcal{P} : \int_{E\times E^\xi} \ln \hat{\zeta}(z, \xi)\, \mu(dz \times d\xi) \le \hat{p} - \epsilon\Big\}.$$

To complete the proof it is enough to show that $\inf_{\mu\in\hat{\Gamma}} J(\mu) > 0$. Due to unboundedness of $\hat{\zeta}$ the set $\hat{\Gamma}$ may not be closed in $\mathcal{P}$. However, under (A5) for every $N > 0$

$$\Gamma_N = \Big\{\mu \in \mathcal{P} : \int_{E\times E^\xi} \min\big(\ln \hat{\zeta}(z, \xi), N\big)\, \mu(dz \times d\xi) \le \hat{p} - \epsilon\Big\}$$

is closed and $\hat{\Gamma} \subseteq \Gamma_N$. Moreover, due to monotone convergence theorem there exists N such that $\vartheta \otimes \nu \notin \Gamma_N$, and consequently $\inf_{\mu\in\hat{\Gamma}_N} J(\mu) > 0$. ∎

Statement (iii) of the above lemma reads that whenever the average one-step growth rate of the asset prices is positive then the prices grow exponentially fast on a large subset of Ω, i.e. for $T > T^*$

$$\mathbb{P}\Big\{S^i(T) \geq S^i(0)e^{T(\hat{p}-\epsilon)} \ \ \forall\, i = 1, \ldots d\Big\} \geq 1 - Ke^{-\gamma T}.$$

Although this property seams to be a very restrictive result, it is satisfied by a large class of market models and give an indepth understanding of the dynamics of the price process. The following remarks shall explain the assumptions (A1)-(A3):

(1) Assume that $Z(t)$ is a Feller process, which is clearly required for (A1) to hold. If $\zeta^i(z,\xi)$, $i = 1, \ldots, d$, are continuous in z then (A1) is satisfied. Indeed, let $\phi : (0,\infty)^d \times E \to \mathbb{R}$ be continuous bounded. Define

$$g(s,z,\xi) = \int_E \phi\big(s^1\zeta^1(\tilde{z},\xi), \ldots, s^d\zeta^d(\tilde{z},\xi), \tilde{z}\big)P(z,d\tilde{z}).$$

It is continuous by the Feller property of $Z(t)$. Consequently, the mapping

$$(s,z) \mapsto \mathbb{E}^{\,(s,z)}\phi\big(S(1), Z(1)\big) = \int_{E^\xi} g(s,z,\xi)\nu(d\xi),$$

where ν is a distribution of $\xi(1)$ on E^ξ, is continuous by dominated convergence theorem and (A1) holds. In particular, if $Z(t)$ is a Markov chain with a finite state space (A1) is always satisfied.

(2) Assumption (A2) reads that the expected one period growth rate is finite.

(3) Assume that $\zeta^i(z,\xi)$, $i = 1, \ldots, d$, are bounded functions separated from 0 and continuous in z. Clearly, $h(\pi, z)$ is bounded. By (A1) $Z(t)$ is a Feller process, hence $h(\pi, z)$ is continuous by the same argument as above and (A2) holds.

(4) By Jensen's inequality

$$\inf_{\pi \in \mathcal{S}} h(z,\pi) = \min_{i=1,\ldots,d} \mathbb{E}^{\,z}\big\{\ln \zeta^i\big(Z(1), \xi(1)\big)\big\}.$$

Therefore, $h(\pi, z)$ is bounded from below if and only if

$$\inf_{z\in E} \mathbb{E}^{\,z}\big\{\ln \zeta^i\big(Z(1), \xi(1)\big)\big\} > -\infty, \qquad i = 1, \ldots, d.$$

(5) Condition (A2) does not imply boundedness of ζ^i. Consider a generalized Black-Scholes model with economic factors (see [5], [6], [25]), i.e.

$$S^i(t+1) = S^i(t)\exp\Big(\sigma^i\big(Z(t+1)\big)\cdot\big(W(t+1) - W(t)\big) + \mu^i\big(Z(t+1)\big)\Big), \quad i = 1, \ldots, d,$$

where $Z(t)$ is a Feller process, $W(t)$ is an m-dimensional Wiener process and $\sigma^i : E \to \mathbb{R}^m$, $\mu^i : E \to \mathbb{R}$, $i = 1, \ldots, d$, are continuous bounded functions. Clearly, (A1) is satisfied by (1). To show (A2) we recall the definition

$$h(\pi, z) = \mathbb{E}^{\,z} \ln \sum_{i=1}^d \pi^i \exp\Big(\sigma^i\big(Z(1)\big)\cdot\xi(1) + \mu^i\big(Z(1)\big)\Big)$$

with $\xi(1) = W(1) - W(0)$. Consequently,

$$\mathbb{E}^z\big\{ - D_1\big(Z(1)\big)\|\xi(1)\|_2 - D_2\big(Z(1)\big)\big\} \leq h(\pi, z) \leq \mathbb{E}^z\big\{D_1\big(Z(1)\big)\|\xi(1)\|_2 + D_2\big(Z(1)\big)\big\},$$

where ξ has a standard normal distribution ν on $\mathbb{R}^m$, $D_1(z) = \max_{i=1,\ldots,d} \|\sigma^i(z)\|_2$, $D_2(z) = \max_{i=1,\ldots,d} |\mu^i(z)|$ and $\|\cdot\|_2$ stands for the L^2 norm in $\mathbb{R}^m$. Therefore, we have boundedness of $h(\pi, z)$. Continuity with respect to π is easy by dominated convergence theorem. Similarly,

$$g(\tilde{z}) = \int_{\mathbb{R}^m} \ln \sum_{i=1}^d \pi^i \exp\big(\sigma^i(\tilde{z}) \cdot \xi + \mu^i(\tilde{z})\big)\nu(d\xi)$$

is continuous. Hence, by the Feller property of $Z(t)$ the function $h(\pi, z)$ is continuous with respect to z and (A2) is satisfied. In particular, (A2) is satisfied if $Z(t)$ is a Markov chain with a finite state space.

(6) In the stochastic control literature a one-step uniform ergodicity is usually assumed, i.e. (A3) with $n = 1$ (see e.g. condition (UE) in [31]). Allowing for $n > 1$ opened a new class of applications. Indeed, (A3) is satisfied if $Z(t)$ is a recursive Markov chain on a finite state space.

(7) Assumption (H*): there is a continuous function u_0 defined on E such that $u_0(x) \geq 1$ for $x \in E$, $Pu_0(x)$ is bounded on compact subsets of E and for any l the set $\left\{z : \frac{u_0(z)}{Pu_0(z)} \leq l\right\}$ is compact.

LEMMA 3.2. If E^ξ is locally compact and (H*) is satisfied then (A4) is also satisfied.

Proof. Without loss of generality we may assume that the support of ν is not compact (otherwise we can replace E^ξ by a compact set). Let (K_n) be an increasing sequence of compact sets such that $\nu(K_{n+1} \setminus K_n) \leq \frac{1}{n^2}$, and $\overline{K_{n+1} \setminus K_n} \cap K_{n-1} = \emptyset$, for $n = 1, 2, \ldots$, and $\bigcup_n K_n = E^\xi$. Define a function g on E^ξ to be equal to 1 on K_1 and $\sqrt{n}$ on $\overline{K_{n+1} \setminus K_n}$ for odd n, and extend g using Tietze theorem to a continuous function on the whole E^ξ. The construction in Tieze theorem implies that $g(\xi) \geq 1$ and $\nu(g) := \int_{E^\xi} g(\xi)\nu(d\xi) < \infty$. Let $\hat{u}_0(z, \xi) = u_0(z)g(\xi)$. We shall prove that the set

$$\Gamma_l = \left\{x \in \hat{E} : \frac{\hat{u}_0(x)}{\hat{P}\hat{u}_0(x)} \leq l\right\} = \left\{(z, \xi) \in \hat{E} : \frac{u_0(z)g(\xi)}{Pu_0(z)\,\nu(g)} \leq l\right\}$$

is compact for any l. Let $(z_n, \xi_n) \subset \Gamma_l$. If (ξ_n) leaves all compact sets K_m then $g(\xi_n) \to \infty$. Consequently $\frac{u_0(z_n)}{Pu_0(z_n)} \to 0$, which contradicts $\inf_{z\in E} \frac{u_0(z)}{Pu_0(z)} > 0$. Therefore, there exists m such that (ξ_n) is contained in K_m. Compactness of K_m implies that $\xi_{n_k} \to \xi \in K_m$ for some subsequence n_k. Since

$$\frac{u_0(z_{n_k})}{Pu_0(z_{n_k})} \leq \frac{l\nu(g)}{g(\xi_{n_k})} \leq l\nu(g),$$

then by (H*) there is a subsequence of z_{n_k} convergent to z. Due to continuity of $\frac{\hat{u}_0}{\hat{P}\hat{u}_0}$, we have $(z, \xi) \in \Gamma_l$, which completes the proof of compactness of Γ_l. ∎

4. Discounted functionals and estimates

This section is devoted to an in-depth study of the discounted functional connected to our problem. They play a major role in the derivation of the Bellman equation for the average cost functional (15).

Given π_-, x_-, z consider a discounted functional

$$J_\beta^{\pi_-,x_-,z}(\Pi) = \mathbb{E}^z\Big\{\sum_{t=0}^{\infty}\beta^t h\big(\pi(t), Z(t)\big) + \sum_{k=1}^{\infty}\beta^{\tau_k}\ln e\big(\pi_-(\tau_k), \pi_k, X_-(\tau_k)\big)\Big\}, \qquad \beta \in (0,1), \tag{16}$$

and its value function

$$v_\beta(\pi_-, x_-, z) = \sup_{\Pi\in\mathcal{A}^z} J_\beta^{\pi_-,x_-,z}(\Pi).$$

Denote by M an impulse operator acting on measurable functions

$$Mw(\pi_-, x_-, z) = \sup_{\pi\in\mathcal{S}}\Big\{\ln e(\pi_-, \pi, x_-) + w\big(\pi, x_-\, e(\pi_-, \pi, x_-), z\big)\Big\}. \tag{17}$$

LEMMA 4.1. The impulse operator maps the space of continuous bounded functions into itself. Moreover, given any bounded continuous function w there exists a measurable selector for Mw.

Proof. The proof is standard (see [7] Corollary 1 or [15]). ■

THEOREM 4.2. Under (A1)-(A2) the function v_β is continuous and bounded, and satisfies Bellman equation

$$v_\beta(\pi_-, x_-, z) = \sup_\tau\ \mathbb{E}^z\Big\{\sum_{t=0}^{\tau-1}\beta^t h(\pi(t), Z(t)) + \beta^\tau M v_\beta\Big(\pi_-(\tau), X_-(\tau), Z(\tau)\Big)\Big\}, \tag{18}$$

where

$$\begin{aligned} \pi_-(0) &= \pi_-, \qquad & \pi_-(t+1) &= \pi_-(t)\diamond\zeta(t), \\ X_-(0) &= x_-, \qquad & X_-(t+1) &= X_-(t)\,\big(\pi_-(t)\cdot\zeta(t)\big) \end{aligned}$$

are counterparts of (10), (12), (13).

Proof. By Lemma 2.1 the function $\ln e(\pi_-, \pi, x_-)$ is bounded, by (A2) $h(\pi, z)$ is bounded. Therefore, $v_\beta(\pi_-, x_-, z)$ is bounded. For a continuous bounded function $v : \mathcal{S}\times(0,\infty)\times E \mapsto \mathbb{R}$ let

$$\mathcal{T}_\beta v(\pi, x, z) = \sup_\tau\ \mathbb{E}^z\Big\{\sum_{t=0}^{\tau-1}\beta^t h(\pi(t), Z(t)) + \beta^\tau M v\Big(\pi_-(\tau), X_-(\tau), Z(\tau)\Big)\Big\}.$$

The operator $\mathcal{T}_\beta$ maps the space $C^b = C^b(\mathcal{S}\times(0,\infty)\times E; \mathbb{R})$ of bounded continuous functions into itself. It results from the Feller property (A1) of the transition operator of the process

$\big(S(t), Z(t)\big)$ by a general result on the continuity of the value function of optimal stopping problems. Let

$$v_\beta^0(\pi_-, x_-, z) = \sum_{t=0}^{\infty} \beta^t \mathbb{E}^{\,z} h\big(\pi_-(t)\big), X_-(t)\big).$$

Put $v_\beta^{k+1} = \mathcal{T}_\beta v_\beta^k$. Thanks to continuity of v_β^k and Mv_β^k it can be shown that v_β^k is a value function for the maximization of J_β over the portfolios with at most k transactions. Observe that it is never optimal to have two transactions at the same time ($\mathbb{P}(\tau_k = \tau_{k+1}) > 0$) by the subadditivity of the transaction cost structure. Therefore, we have the estimate

$$\|v_\beta - v_\beta^k\|_\infty \le \sum_{l=k}^{\infty} \beta^l \|h\|_\infty = \beta^k \frac{\|h\|_\infty}{1-\beta},$$

which implies that v_β^k tends uniformly to v_β. Consequently, v_β is a continuous bounded function and satisfies $v_\beta = \mathcal{T}_\beta v_\beta$, which is equivalent to the Bellman equation (18). ■

There are two distinct cases: $C > 0$ (constant plus proportional transaction costs) and $C = 0$ (proportional costs only). Theorem 4.2 applies to both of them. The rest of this section is devoted to estimation of the difference between value functions of a problem with and without constant transaction costs. In the beginning let us examine the equation for $e(\pi_-, \pi, x_-)$:

$$c\big(\pi_-, e(\pi_-, \pi, x_-)\pi\big) + \frac{C}{x_-} + e(\pi_-, \pi, x_-) = 1.$$

Clearly, if $C = 0$, the solution is independent of x_- and we shall denote it by $\tilde{e}(\pi_-, \pi)$, i.e. it is a unique solution to

$$c\big(\pi_-, \tilde{e}(\pi_-, \pi)\pi\big) + \tilde{e}(\pi_-, \pi) = 1.$$

Similarly, while $C = 0$, the impulse operator M does not depend on x_-, hence the value function v_β is independent of x_-. We shall therefore refer to the case without a constant term in transaction costs by skipping x_- in the list of arguments and writing $\tilde{J}_\beta^{\pi_-, z}(\Pi)$, $\tilde{v}_\beta(\pi, z)$ and $\tilde{e}(\pi_-, \pi)$.

4.1. Technical estimates

This subsection presents auxiliary results. They are analogous to those obtained in [26]. For completeness, their proofs are included in Appendix.

Due to self-financing of portfolios, transaction costs decrease portfolio wealth. It is therefore important to derive estimates on the diminution factor $e(\pi_-, \pi, x_-)$ and to study the relationship between $e(\pi_-, \pi, x_-)$ and $\tilde{e}(\pi_-, \pi)$. First of the following lemmas gives lower bounds for e and $\tilde{e}$:

LEMMA 4.3. We have

$$1 - \tilde{e}(\pi_-, \pi) \le \frac{2\max_i(c_i^1, c_i^2)}{1 - \max_i(c_i^1, c_i^2)},$$
$$1 - e(\pi_-, \pi, x_-) \le \frac{2\max_i(c_i^1, c_i^2) + \frac{C}{x_-}}{1 - \max_i(c_i^1, c_i^2)}.$$

Let $x^* = \inf\{x_- : \ e(\pi_-, \pi, x_-) > 0 \text{ for all } \pi_-, \pi \in \mathcal{S}\}$. If the wealth of the portfolio is greater than x^*, any transaction can be executed. We shall use this rough treshold in the following lemma:

LEMMA 4.4. For $\pi_-, \pi \in \mathcal{S}$

i) $\tilde{e}(\pi_-, \pi) \geq e(\pi_-, \pi, x_-) \geq e(\pi_-, \pi, \tilde{x}_-), \qquad x_- \geq \tilde{x}_- > 0.$

ii) $\tilde{e}(\pi_-, \pi) - e(\pi_-, \pi, x_-) \leq \dfrac{C}{\big(1 - \max_i c_i^1\big) x_-} \qquad \text{for } x_- > x^*.$

iii) For all $M > x^*$ and $x_- \geq M$

$$\ln \frac{\tilde{e}(\pi_-, \pi)}{e(\pi_-, \pi, x_-)} \leq \frac{1}{\inf_{\tilde{\pi}_-, \tilde{\pi}} e(\tilde{\pi}_-, \tilde{\pi}, M)} \, \frac{C}{\big(1 - \max_i c_i^1\big) x_-}.$$

COROLLARY 4.5. The value function $v_\beta(\pi_-, x_-, z)$ is non-decreasing in x_-.

Due to Theorem 4.2 the value function $\tilde{v}_\beta(x_-, z)$ is bounded and continuous for each β. However, it does not imply that it is uniformly bounded in β. Conversely, it increases to infinity as β grows to 1 in market models of interest. To account for this fact, we shall study the span seminorm of $\tilde{v}_\beta$, which is defined as $\|\tilde{v}_\beta\|_{sp} = \sup \tilde{v}_\beta(\cdot) - \inf \tilde{v}_\beta(\cdot)$.

LEMMA 4.6. Under (A3) there exists $M < \infty$ such that

$$\|\tilde{v}_\beta\|_{sp} \leq M,$$

for all $\beta \in (0, 1)$.

4.2. Large deviations and proportional transaction costs

Theorem 3.1 provides important insight into the dynamics of asset prices. In this section we shall apply this result to describe the dynamics of the portfolio wealth under proportional transaction costs. Let us introduce a general assumption:

(A6) $\eta < \hat{p}$,

where $\hat{p}$ is a constant from Theorem 3.1 and

$$\eta = -\ln\left(1 - \frac{2\max_i(c_i^1, c_i^2)}{1 - \max_i(c_i^1, c_i^2)}\right).$$

Since η is a unique solution to the equation

$$e^{-\eta} = 1 - \frac{2\max_i(c_i^1, c_i^2)}{1 - \max_i(c_i^1, c_i^2)},$$

by virtue of Lemma 4.3, $e^{-\eta}$ is a lower bound on $\tilde{e}(\pi_-, \pi)$. Formula (13) gives us the following estimate

$$X_-(t) \geq X_-(0) \prod_{s=0}^{t-1} \Big(e^{-\eta}\, \pi(s) \cdot \zeta(s) \Big) = X_-(0) e^{-\eta t} \prod_{s=0}^{t-1} \Big(\pi(s) \cdot \zeta(s) \Big).$$

Consequently, denoting $\hat{\zeta}(t) = \min\big(\zeta^1(t), \ldots, \zeta^d(t)\big)$, we obtain

$$X_-(t) \geq X_-(0) e^{-\eta t} \prod_{s=0}^{t-1} \hat{\zeta}(s). \tag{19}$$

In view of Theorem 3.1 for any $\epsilon > 0$ there exists $K > 0$, $\gamma > 0$ and T^* such that

$$\mathbb{P}\Big\{ e^{T(\hat{p}-\epsilon)} \leq \prod_{s=0}^{t-1} \hat{\zeta}(s) \Big\} \geq 1 - K e^{-\gamma T}, \quad T > T^*.$$

Due to (19), the wealth of the portfolio can be estimated by

$$\mathbb{P}\Big\{ X_-(0) e^{T(\hat{p}-\eta-\epsilon)} \leq X_-(T) \Big\} \geq 1 - K e^{-\gamma T}, \quad T > T^*.$$

Due to (A6) there exists $0 < \epsilon < \hat{p} - \eta$, which implies that $X_-(t)$ increases exponentially fast irrespective of trading strategy.

4.3. Bounds on $\tilde{v}_\beta(\pi_-, z) - v_\beta(\pi_-, x_-, z)$

For the rest of this section assume that (A1)-(A6) are satisfied. Since

$$\lim_{m\to\infty} -\ln \left(1 - \frac{2 \max_i(c_i^1, c_i^2) + \frac{C}{m}}{1 - \max_i(c_i^1, c_i^2)} \right) = \eta < \hat{p},$$

there exists a constant $M > 0$ such that

$$\hat{p} > \eta_M := -\ln \left(1 - \frac{2 \max_i(c_i^1, c_i^2) + \frac{C}{M}}{1 - \max_i(c_i^1, c_i^2)} \right). \tag{20}$$

THEOREM 4.7. For any $z \in E$ and any admissible strategy $\tilde{\Pi} \in \mathcal{A}^z$, $\pi_- \in \mathcal{S}$, $x_- \in (0, \infty)$ there exists an admissible trading strategy $\Pi \in \mathcal{A}^z$ such that

$$\tilde{J}_\beta^{\pi_-, z}(\tilde{\Pi}) - J_\beta^{\pi_-, x_-, z}(\Pi) \leq L(x_-), \quad \beta \in (0, 1),$$

where

$$L(x_-) = K_1 + K_2 \max(K_3, -\ln x_-)$$

for some strictly positive constants K_1, K_2, K_3 independent of the choice of z, $\tilde{\Pi}$, π_- and x_-.

Before we proceed to the proof we formulate an important consequence of the theorem. Notice that the bound is uniform in β, π_- and z.

COROLLARY 4.8. We have

$$0 \le \tilde{v}_\beta(\pi_-, z) - v_\beta(\pi_-, x_-, z) \le L(x_-), \quad \beta \in (0,1),\ \pi_- \in \mathcal{S},\ z \in E.$$

where $L(x_-)$ is a function from Theorem 4.7.

Proof. The inequality $0 \le \tilde{v}_\beta(\pi_-, z) - v_\beta(\pi_-, x_-, z)$ is obvious. For the second inequality it is enough to notice that

$$\tilde{v}_\beta(\pi_-, z) - v_\beta(\pi_-, x_-, z) \le \sup_{\tilde{\Pi} \in \mathcal{A}} \Big\{ \tilde{J}_\beta^{\pi_-, z}(\tilde{\Pi}) - \tilde{J}_\beta^{\pi_-, x_-, z}(\Pi) \Big\},$$

where by Π we denote a strategy related to $\tilde{\Pi}$ as in Theorem 4.7. ■

Proof of Theorem 4.7. Fix π_-, x_-, z and $\tilde{\Pi} \in \mathcal{A}^z$. Denote by $\tilde{\pi}_-(t)$ and $\tilde{\pi}(t)$ the pre-transaction and the post-transaction process linked to the strategy $\tilde{\Pi}$. We construct Π in the following way: if $X_-(t)$ is smaller than M we do nothing waiting for the wealth to raise over $M^* = Me^\eta$. At that moment we perform a transaction to make the proportions equal to those defined by $\tilde{\Pi}$. This decreases the wealth at most by $e^{-\eta}$, so the resulting portfolio wealth is not less than M. On the other hand, if $X_-(t) \ge M$ we mimic the strategy $\tilde{\Pi}$, i.e. we keep the same proportions of stocks. Denote by $\pi_-(t)$, $\pi(t)$ and $X_-(t)$ the processes corresponding to the strategy Π. By the definition of Π we know that $\pi(t) = \tilde{\pi}(t)$ if $X_-(t) \ge M^*$. However, the wealths between M and M^* can be attained either in the process of recovering from the shortage of wealth (being below M) or in normal investing process, thus we cannot easily determine whether $\pi(t) = \tilde{\pi}(t)$.

By the definition of $J_\beta^{\pi_-, x_-, z}$ and $\tilde{J}_\beta^{\pi_-, z}$ we have

$$\begin{aligned}
&\tilde{J}_\beta^{\pi_-, z}(\tilde{\Pi}) - J_\beta^{\pi_-, x_-, z}(\Pi) \\
&\quad = \mathbb{E}^z \bigg\{ \sum_{t=0}^{\infty} \beta^t \Big(h\big(\tilde{\pi}(t), Z(t)\big) - h\big(\pi(t), Z(t)\big) \\
&\qquad\qquad + 1_{\tilde{\pi}_-(t) \ne \tilde{\pi}(t)} \ln \tilde{e}\big(\tilde{\pi}_-(t), \tilde{\pi}(t)\big) - 1_{\pi_-(t) \ne \pi(t)} \ln e\big(\pi_-(t), \pi(t), X_-(t)\big) \Big) \bigg\}.
\end{aligned}$$

Above difference can be bounded from above by the sum of the following two expressions:

$$\mathbb{E}^z \bigg\{ \sum_{t=0}^{\infty} \beta^t 1_{X_-(t) < M^*} \Big(h\big(\tilde{\pi}(t), Z(t)\big) - h\big(\pi(t), Z(t)\big) - 1_{\pi_-(t) \ne \pi(t)} \ln e\big(\pi_-(t), \pi(t), X_-(t)\big) \Big) \bigg\}, \tag{21}$$

$$\mathbb{E}^z \bigg\{ \sum_{t=0}^{\infty} \beta^t 1_{X_-(t) \ge M^*} \Big(h\big(\tilde{\pi}(t), Z(t)\big) - h\big(\pi(t), Z(t)\big) + 1_{\pi_-(t) \ne \pi(t)} \ln \frac{\tilde{e}\big(\pi_-(t), \pi(t)\big)}{e\big(\pi_-(t), \pi(t), X_-(t)\big)} \Big) \bigg\}. \tag{22}$$

By construction of the strategy Π no transaction is performed if the wealth $X_-(t)$ is below M, so we have

$$-1_{\pi_-(t)\neq\pi(t)} \ln e\big(\pi_-(t), \pi(t), X_-(t)\big) \leq \eta_M.$$

This yields

$$(21) \leq L_1 \, \mathbb{E}^{\,z} \sum_{t=0}^{\infty} 1_{X_-(t)<M^*},$$

where $L_1 = \sup h(\cdot) - \inf h(\cdot) + \eta_M$. On the other hand, if $X_-(t) \geq M^*$, we have $\pi(t) = \tilde{\pi}(t)$, so

$$(22) \leq \mathbb{E}^{\,z}\bigg\{ \sum_{t=0}^{\infty} \bigg(1_{X_-(t)\geq M^*} \ln \frac{\tilde{e}\big(\pi_-(t), \pi(t)\big)}{e\big(\pi_-(t), \pi(t), X_-(t)\big)} \bigg) \bigg\}.$$

By virtue of Lemma 4.4 (iii) we obtain

$$(22) \leq \mathbb{E}^{\,z}\bigg\{ \sum_{t=0}^{\infty} \bigg(1_{X_-(t)\geq M^*} \frac{L_2}{X_-(t)} \bigg) \bigg\},$$

where

$$L_2 = \frac{C}{\inf_{\hat{\pi}_-, \hat{\pi}} e(\hat{\pi}_-, \hat{\pi}, M^*)}.$$

Consequently, we have the estimate

$$\tilde{J}^{\pi_-,z}_{\beta}(\tilde{\Pi}) - J^{\pi_-,x_-,z}_{\beta}(\Pi) \leq L_1 \, \mathbb{E}^{\,z} \sum_{t=0}^{\infty} 1_{X_-(t)<M^*} + L_2 \, \mathbb{E}^{\,z} \sum_{t=0}^{\infty} \bigg(1_{X_-(t)\geq M^*} \frac{1}{X_-(t)} \bigg). \tag{23}$$

To complete the proof we use the large deviations estimate. Fix $\epsilon > 0$ small enough so that $\hat{p} - \eta_M - \epsilon > 0$. Denote by A_t the event

$$A_t = \bigg\{ \frac{1}{t} \ln \bigg(\prod_{j=0}^{t-1} \hat{\zeta}\big(Z(j+1), \xi(j+1)\big) \bigg) - \hat{p} \geq -\epsilon \bigg\}.$$

Strategy Π is constructed in such a way that trade takes place only if $X_-(t) \geq M$. Thus on the set A_t we have

$$X_-(t) \geq x_- e^{-t\eta_M} e^{t(\hat{p}-\epsilon)} = x_- e^{t(\hat{p}-\eta_M-\epsilon)}. \tag{24}$$

This reads as an exponentially fast growth of the wealth due to $\hat{p} - \eta_M - \epsilon > 0$.

Let $K > 0$, $\gamma > 0$, and T^* be the constants from Theorem 3.1 (iii) for the given ϵ. By the large deviations estimate we have

$$\mathbb{P}^z(A_t^c) \leq K e^{-\gamma t} \quad \text{for } t \geq T^*,$$

where by A_t^c we denote the complement of A_t. Let t_0 be the smallest integer such that $t_0 \geq T^*$ and

$$e^{t_0(\hat{p}-\eta_M-\epsilon)} \geq \frac{M^*}{x_-}.$$

Clearly, $X_-(t) \geq M^*$ on A_t for all $t \geq t_0$. Hence

$$\mathbb{E}^z \sum_{t=0}^{\infty} 1_{X_-(t)<M^*} \leq t_0 + \sum_{t=t_0}^{\infty} \mathbb{P}(A_t^c) \leq t_0 + \sum_{t=t_0}^{\infty} Ke^{-\gamma t} =: L_3.$$

Computation of a bound for the second term of (23) has to be split into two parts depending on A_t:

$$\mathbb{E}^z \sum_{t=0}^{\infty} \left(1_{X_-(t)\geq M^*} \frac{1}{X_-(t)} \right) = \mathbb{E}^z \sum_{t=0}^{\infty} \left(1_{X_-(t)\geq M^*} \frac{1_{A_t}}{X_-(t)} \right) + \mathbb{E}^z \sum_{t=0}^{\infty} \left(1_{X_-(t)\geq M^*} \frac{1_{A_t^c}}{X_-(t)} \right).$$

Easily,

$$\mathbb{E}^z \sum_{t=0}^{\infty} \left(1_{X_-(t)\geq M^*} \frac{1_{A_t^c}}{X_-(t)} \right) \leq \frac{1}{M^*} \mathbb{E}^z \sum_{t=0}^{\infty} \mathbb{P}(A_t^c) \leq \frac{1}{M^*} \mathbb{E}^z \sum_{t=0}^{\infty} Ke^{-\gamma t} =: L_4.$$

Due to (24)

$$\begin{aligned} \mathbb{E}^z \sum_{t=0}^{\infty} \left(1_{X_-(t)\geq M^*} \frac{1_{A_t}}{X_-(t)} \right) &\leq \mathbb{E}^z \sum_{t=0}^{t_0-1} \left(1_{X_-(t)\geq M^*} \frac{1_{A_t}}{X_-(t)} \right) + \mathbb{E}^z \sum_{t=t_0}^{\infty} \left(1_{X_-(t)\geq M^*} \frac{1_{A_t}}{X_-(t)} \right) \\ &\leq \frac{t_0}{M^*} + \frac{1}{M^*} \sum_{t=t_0}^{\infty} e^{-(t-t_0)(\hat{p}-\eta_M-\epsilon)} \\ &=: L_5. \end{aligned}$$

Consequently,

$$\tilde{J}_\beta^{\pi_-,z}(\tilde{\Pi}) - J_\beta^{\pi_-,x_-,z}(\Pi) \leq L_1 L_3 + L_2(L_4 + L_5).$$

It can be easily noticed that these constants does not depend on π_-, $\tilde{\Pi}$ and z. However, they depend on x_- through t_0. Combining the estimates for $L_1, \ldots, L_5$ we obtain the formula for $L(x_-)$. ■

5. Growth optimal portfolios

Now we are in a position to state and prove the main result of this paper: existence and form of an optimal strategy maximizing the expected average rate of return of a portfolio of financial assets.

THEOREM 5.1. Under assumptions (A1)-(A6) there exists a measurable function $p : \mathcal{S} \times (0, \infty) \times E \to \mathcal{S}$, a constant λ and a measurable set $I \subseteq \mathcal{S} \times (0, \infty) \times E$ such that

$$\lambda = J^{\pi_-,x_-,z}(\Pi^*) = \sup_{\Pi \in \mathcal{A}^z} J^{\pi_-,x_-,z}(\Pi), \tag{25}$$

where the optimal portfolio $\Pi^* = \big((\pi_1^*, \tau_1^*), (\pi_2^*, \tau_2^*), \dots\big)$ is given by the formulas

$$\begin{aligned}
\tau_1^* &= \inf\{t \geq 0 : \quad \big(\pi_-(t), X_-(t), Z(t)\big) \in I\},\\
\tau_{k+1}^* &= \inf\{t > \tau_k^* : \quad \big(\pi_-(t), X_-(t), Z(t)\big) \in I\},\\
\pi_k^* &= p\big(\pi_-(\tau_k^*), X_-(\tau_k^*), Z(\tau_k^*)\big).
\end{aligned}$$

The statement of the theorem is not surprising. What is surprising is its generality. Assumptions (A1)-(A6) proved to be nonrestrictive (see Section 3). We are not aware of papers dealing with the maximisation of the average rate of return in such a general setting with constant and proportional transaction costs. Notice also that the statement of the theorem holds with different transaction costs structures, as mentioned in the introduction. This result also extends the area of applicability of the vanishing discount approach to models with non-weakly continuous controlled transition probabilities. Existing results require either strongly or weakly continuous (Feller) controlled transition probabilities (see [15], [19], [29], [30]).

Surprisingly, the following corollary holds:

COROLLARY 5.2. The optimal value for the problem with only proportional transaction costs ($C = 0$) is equal to λ from Theorem 5.1. Moreover, it is reached by the portfolio constructed in the statement of this theorem.

The proof of the above corollary is a direct consequence of the proof of Theorem 5.1. It is presented in details later.

Proof of Theorem 5.1. We shall use a generalization of the vanishing discount method ([4], [15], [19], [29], [31]). We will obtain a Bellman inequality for our optimization problem as a limit of Bellman equations for discounted problems (16). However, we cannot directly apply known results since they require continuity of the controlled transition function q defined below. Instead, we shall follow the approach pioneered by [29] and later used by many authors (see eg. [19]) filling parts where the continuity of q is needed by considerations based on specific properties of our control problem. Moreover, we will ease the requirement of local compactness of the state space in the spirit of [19].

Denote by $\mathcal{H} = \mathcal{S} \times (0, \infty) \times E$ the state space of our Markovian control model. Clearly, it is complete and separable, which will be needed for the existence of measurable selectors. Denote by q the controlled transition operator, i.e. a function $q : \mathcal{H} \times \mathcal{S} \to \mathcal{P}(\mathcal{H})$, where $\mathcal{P}(\mathcal{H})$ is the space of Borel probability measures on $\mathcal{H}$, uniquely determined by the formula

$$\int_{\mathcal{H}} f(\tilde{\pi}_-, \tilde{x}_-, \tilde{z})\, q(\pi_-, x_-, z, \pi)(d\tilde{\pi}_-, d\tilde{x}_-, d\tilde{z}) = \mathbb{E}^{\,z} f\Big(\pi \diamond \zeta\big(z, \xi(1)\big), X_-^{\pi}(1), z(1)\Big) \tag{26}$$

for bounded measurable $f : \mathcal{H} \to \mathbb{R}$, where

$$X_-^{\pi}(1) = \begin{cases} x_-\, e(\pi_-, \pi, x_-)\, \Big(\pi \cdot \zeta\big(z, \xi(1)\big)\Big), & \text{when } \pi_- \neq \pi,\\ x_-\, \pi \cdot \zeta\big(z, \xi(1)\big), & \text{when } \pi_- = \pi. \end{cases}$$

Obviously, q is not weakly continuous as long as the constant term in transaction costs is non-null. Indeed, $X_-^{\pi_-}(1) - X_-^{\tilde{\pi}}(1) \geq C$ for any $\tilde{\pi} \neq \pi_-$. Consider

$$\eta(\pi_-, \pi, x_-, z) = \begin{cases} h(\pi, z), & \pi_- = \pi, \\ h(\pi, z) + \ln e(\pi_-, \pi, x_-), & \pi_- \neq \pi. \end{cases}$$

Bellman equation (18) writes in an equivalent form

$$v_\beta(\pi_-, x_-, z) = \sup_{\pi \in \mathcal{S}} \left\{ \eta(\pi_-, \pi, x_-, z) + \beta \int v_\beta \, dq(\pi_-, x_-, z, \pi) \right\}. \tag{27}$$

Let $a_\beta : \mathcal{H} \to \mathcal{S}$ be a measurable selector for Mv_β (see Lemma 4.1) and I_β be the impulse region

$$I_\beta = \{(\pi_-, x_-, z) \in \mathcal{H} : \quad v_\beta(\pi_-, x_-, z) = Mv_\beta(\pi_-, x_-, z)\}.$$

The optimal strategy in this formulation is given by a measurable function $f_\beta : \mathcal{H} \to \mathcal{S}$

$$f_\beta(\pi_-, x_-, z) = \begin{cases} \pi_-, & (\pi_-, x_-, z) \notin I_\beta, \\ a_\beta(\pi_-, x_-, z), & (\pi_-, x_-, z) \in I_\beta. \end{cases}$$

Since v_β is unbounded as β grows to ∞ we introduce the relative discounted value function

$$w_\beta(\pi_-, x_-, z) = m_\beta - v_\beta(\pi_-, x_-, z),$$

where

$$m_\beta = \sup_{\pi_- \in \mathcal{S}} \sup_{z \in E} \tilde{v}_\beta(\pi_-, z)$$

is well-defined due to Lemma 4.6. Moreover, we have

LEMMA 5.3.

i) $0 \leq w_\beta(\pi_-, x_-, z) \leq M_1 + M_2 \max(M_3, -\ln x_-)$ with $M_1, M_2, M_3 > 0$ independent of β, π_-, x_-, z.

ii) $\{(1-\beta)m_\beta : \beta \in (0,1)\}$ is a compact set.

Proof. By Lemma 4.6, and Corollary 4.8 we have

$$w_\beta(\pi_-, x_-, z) \leq m_\beta - v_\beta(\pi_-, z) + v_\beta(\pi_-, z) - v_\beta(\pi_-, x_-, z) \leq M + L(x_-),$$

where $L(x_-)$ is a function defined in Theorem 4.7. We conclude by using the form of $L(x_-)$. Part ii) follows from boundedness of $h(\cdot)$ and $\ln \tilde{e}(\cdot)$. ■

Put $\overline{\lambda} = \limsup_{\beta \uparrow 1} (1-\beta) m_\beta$, which is finite by Lemma 5.3 (ii). Denote by β_k the sequence of discount factors converging to 1 such that

$$\overline{\lambda} = \limsup_{k \to \infty} (1 - \beta_k) m_{\beta_k}.$$

Let

$$\underline{w}(\vartheta) = \liminf_{k\to\infty,\, \vartheta'\to\vartheta} w_{\beta_k}(\vartheta'), \quad \vartheta \in \mathcal{H}.$$

It can be written equivalently as

$$\underline{w}(\vartheta) = \inf\big\{ \liminf_{k\to\infty} w_{\beta_k}(\vartheta_k) : \vartheta_k \to \vartheta \big\}, \quad \vartheta \in \mathcal{H}.$$

LEMMA 5.4. ([20] Lemma 3.1) The function $\underline{w}$ is lower semi continuous.

The proof of this lemma is straightforward and is based on the following reformulation of the definition of $\underline{w}$:

$$\underline{w}(\vartheta) = \sup_n \inf_{k\geq n} \Big(\inf_{\vartheta'\in B(\vartheta,1/n)} w_{\beta_k}(\vartheta') \Big),$$

where $B(\vartheta, 1/n)$ is a ball in $\mathcal{H}$ of radius $1/n$.

In the sequel we shall need two transition operators related to q. Let $\underline{q}$ be given by the formula (26) with

$$X_-^\pi(1) = x_-\ e(\pi_-, \pi, x_-)\ \big(\pi \cdot \zeta\big(z, \xi(1)\big)\big)$$

and $\overline{q}$ with

$$X_-^\pi(1) = x_-\ \big(\pi \cdot \zeta\big(z, \xi(1)\big)\big).$$

They are weakly continuous. Indeed, it is straightforward by (A1) and the continuity of $e(\pi_-, \pi, x_-)$ (see Lemma 2.1) that the mapping $(\pi_-, x_-, z) \mapsto \big(\int_{\mathcal{H}} f\, d\underline{q}(\pi_-, x_-, z), \int_{\mathcal{H}} f\, d\overline{q}(\pi_-, x_-, z)\big)$ is continuous for any continuous bounded function $f : \mathcal{H} \to \mathbb{R}$.

LEMMA 5.5. ([28] Lemma 3.2) Let $\{\mu_n\}$ be a sequence of probability measures on a separable metric space $\mathcal{X}$ converging weakly to μ and $\{g_n\}$ be a sequence of measurable nonnegative functions on $\mathcal{X}$. Then

$$\int \underline{g}\, d\mu \leq \liminf_{n\to\infty} \int g_n\, d\mu_n, \qquad \text{where} \qquad \underline{g}(x) = \liminf_{n\to\infty,\ y\to x} g_n(y), \qquad x \in \mathcal{X}.$$

THEOREM 5.6. Under assumptions (A1)-(A5) there exists a measurable function $f_1 : \mathcal{H} \to \mathcal{S}$ and a measurable function $w : \mathcal{H} \to (-\infty, 0]$ such that

$$w(\vartheta) + \overline{\lambda} \leq \eta\big(\vartheta, f_1(\vartheta)\big) + \int w(\vartheta') q\big(\vartheta, f_1(\vartheta)\big)(d\vartheta'), \qquad \vartheta \in \mathcal{H}. \tag{28}$$

Proof. From equation (27) we derive

$$w_\beta(\vartheta) + (\beta - 1) m_\beta = -\eta\big(\vartheta, f_\beta(\vartheta)\big) + \beta \int w_\beta(\vartheta') q\big(\vartheta, f_\beta(\vartheta)\big)(d\vartheta'), \qquad \vartheta \in \mathcal{H}, \quad \beta \in (0,1),$$

where f_β defines an optimal strategy for v_β. Fix $\vartheta \in \mathcal{H}$ and a sequence (ϑ_k) converging to ϑ. Above equation can be rewritten as

$$w_{\beta_k}(\vartheta_k) + (\beta_k - 1) m_{\beta_k} = -\eta\big(\vartheta_k, f_{\beta_k}(\vartheta_k)\big) + \beta_k \int w_{\beta_k}(\vartheta') q\big(\vartheta_k, f_{\beta_k}(\vartheta_k)\big)(d\vartheta').$$

Applying $\liminf_{k\to\infty}$ on both sides yields

$$\liminf_{k\to\infty} w_{\beta_k}(\vartheta_k) - \overline{\lambda} = -\limsup_{k\to\infty} \eta\big(\vartheta, f_{\beta_k}(\vartheta_k)\big) + \liminf_{k\to\infty} \int \beta_k w_{\beta_k}(\vartheta') q\big(\vartheta_k, f_{\beta_k}(\vartheta_k)\big)(d\vartheta'). \quad (29)$$

Since $\mathcal{S}$ is compact there exists a sequence (n_k) such that $f_{\beta_{n_k}}(\vartheta) \to \pi^*$ and either (a) $\vartheta_{n_k} \in I_{\beta_{n_k}}$ for every k, or (b) $\vartheta_{n_k} \notin I_{\beta_{n_k}}$ for every k. Assume first that (a) holds. By virtue of Lemma 5.5 we have

$$\liminf_{k\to\infty} \int \beta_k w_{\beta_k}(\vartheta') q\big(\vartheta_k, f_{\beta_k}(\vartheta_k)\big)(d\vartheta') \geq \int \underline{w}(\vartheta') \underline{q}\big(\vartheta, \pi^*\big)(d\vartheta').$$

By Corollary 4.5 the functions $v_\beta(\pi_-, x_-, z)$ are nondecreasing in x_-. This implies that $\underline{w}(\pi_-, x_-, z)$ is non-increasing in x_-. Hence $\int \underline{w}(\vartheta')\underline{q}\big(\vartheta, \pi^*\big)(d\vartheta') \geq \int \underline{w}(\vartheta') q\big(\vartheta, \pi^*\big)(d\vartheta')$ and

$$\liminf_{k\to\infty} \int \beta_k w_{\beta_k}(\vartheta') q\big(\vartheta_k, f_{\beta_k}(\vartheta_k)\big)(d\vartheta') \geq \int \underline{w}(\vartheta') q\big(\vartheta, \pi^*\big)(d\vartheta'). \quad (30)$$

In the case (b) we have $f_{\beta_{n_k}}(\vartheta_k) = \pi_-^k$, where $\vartheta_k = (\pi_-^k, x_-^k, z^k)$. Obviously, $\pi^* = \pi_-$, where $\vartheta = (\pi_-, x_-, z)$. From equalities $q\big(\vartheta_{n_k}, f_{\beta_{n_k}}(\vartheta_k)\big) = \overline{q}\big(\vartheta_{n_k}, f_{\beta_{n_k}}(\vartheta_k)\big)$ and $q\big(\vartheta, \pi^*\big) = \overline{q}\big(\vartheta, \pi^*\big)$ and Lemma 5.5 we obtain (30). Since η is upper semicontinuous we conclude that

$$\liminf_{k\to\infty} w_{\beta_k}(\vartheta_k) - \overline{\lambda} \geq -\eta\big(\vartheta, \pi^*\big) + \int \underline{w}(\vartheta') q\big(\vartheta, \pi^*\big)(d\vartheta').$$

Consequently,

$$\liminf_{k\to\infty} w_{\beta_k}(\vartheta_k) - \overline{\lambda} \geq \inf_{\pi\in\mathcal{S}} \Big\{ -\eta\big(\vartheta, \pi\big) + \int \underline{w}(\vartheta') q\big(\vartheta, \pi\big)(d\vartheta')\Big\}.$$

Taking infimum over all sequences ϑ_n converging to ϑ we finally obtain

$$\underline{w}(\vartheta) - \overline{\lambda} \geq \inf_{\pi\in\mathcal{S}} \Big\{ -\eta\big(\vartheta, \pi\big) + \int \underline{w}(\vartheta') q\big(\vartheta, \pi\big)(d\vartheta')\Big\}.$$

We only need to prove that there exists a measurable selector for the infimum on the right-hand side. First notice that due to Lemma 5.4 the function $\underline{w}$ is lower semicontinuous. By weak continuity of the transition probability q the mapping

$$(\pi_-, x_-, z) \mapsto \int_{\mathcal{H}} \underline{w}(\vartheta')\, q(\pi_-, x_-, z)(d\vartheta')$$

is lower semicontinuous (see [14] Lemma 3.3 (a)). Corollary 1 in [7] implies that there exists a measurable mapping $f_1 : \mathcal{H} \to \mathcal{S}$ such that

$$\underline{w}(\vartheta) - \overline{\lambda} \geq \Big\{ -\eta\big(\vartheta, f_1(\vartheta)\big) + \int \underline{w}(\vartheta') q\big(\vartheta, f_1(\vartheta)\big)(d\vartheta')\Big\},$$

which yields (28) with $w = -\underline{w}$.

Fix $(\pi_-, x_-, z) \in \mathcal{H}$ and define a portfolio $\Pi = \big((\pi_1, \tau_1), (\pi_2, \tau_2), \dots\big)$ by formulas given in Theorem 5.1 with $I = \{(\pi_-, x_-, z) \in \mathcal{H} : f_1(\pi_-, x_-, z) \neq \pi_-\}$ and $p = f_1$. Iterating (28) T times, dividing by T and passing with T to infinity we obtain

$$\overline{\lambda} \leq J^{\pi_-,x_-,z}(\Pi) + \liminf_{T\to\infty} \mathbb{E}^{\,z} \frac{w\big(\pi_-^\Pi(T), X_-^\Pi(T), Z(T)\big)}{T} \leq J^{\pi_-,x_-,z}(\Pi),$$

since w is nonpositive. On the other hand, by a well-known Tauberian relation

$$\begin{aligned} J^{\pi_-,x_-,z}(\Pi) &\leq \liminf_{\beta\to 1}(1-\beta) J_\beta^{\pi_-,x_-,z}(\Pi) \\ &\leq \liminf_{\beta\to 1}(1-\beta) v_\beta(\pi_-, x_-, z) \leq \liminf_{\beta\to 1}(1-\beta) v_\beta(\pi_-, z) \leq \overline{\lambda}, \end{aligned}$$

which proves the optimality of Π and completes the proof of Theorem 5.1. ■

Proof of Corollary 5.2. First notice that $\overline{\lambda}$ is the optimal value for the problem with proportional transaction costs. Indeed, consider the proof of Theorem 5.6 with $w_\beta(\pi_-, z) = m_\beta - \tilde{v}_\beta(\pi_-, z)$. We obtain an analog of (28) with function w depending on π_-, z and $\overline{\lambda}$ as above. Consequently $\overline{\lambda}$ is the optimal value for the problem with proportional transaction costs.

Let Π be the optimal portfolio for the case with fixed and proportional transaction costs (as defined in Theorem 5.1). Denote by $\tilde{X}_-^\Pi(t)$ the wealth of the portolio governed by Π when the fixed term of the transaction cost function is equal to 0. Obviously $\tilde{X}_-^\Pi(t) \geq X_-^\Pi(t)$ and

$$\lim_{T\to\infty} \frac{1}{T} \mathbb{E}^{\,z} \ln \tilde{X}_-^\Pi(t) \geq \overline{\lambda}.$$

Since $\overline{\lambda}$ is the optimal value for the problem with proportional transaction costs we have the opposite inequality. ■

6. Extensions

The paper can be extended twofolds. First consider a generalization with respect to the cost function $\tilde{c}$. Assume that the cost function $\tilde{c}$ satisfies

$$\tilde{c}(N_1, N_2, S) \geq \sum_{i=1}^d \Big(c_i^1 S^i (N_1^i - N_2^i)^+ + c_i^2 S^i (N_1^i - N_2^i)^-\Big) \tag{31}$$

$$\tilde{c}(N_1, N_2, S) \leq \sum_{i=1}^d \Big(c_i^1 S^i (N_1^i - N_2^i)^+ + c_i^2 S^i (N_1^i - N_2^i)^-\Big) + C \tag{32}$$

for some $C \geq 0$ and $c_i^1, c_2^i \in [0, 1)$, $i = 1, \dots d$. If the cost function in the right-hand side of (32) satisfies (A6) than there exists an optimal portfolio of the form presented in Theorem 5.1. Moreover, the portfolio optimal for the cost

$$\sum_{i=1}^d \Big(c_i^1 S^i (N_1^i - N_2^i)^+ + c_i^2 S^i (N_1^i - N_2^i)^-\Big) + C \tag{33}$$

is optimal for $\tilde{c}$ as well. To see this let us denote by $\hat{J}^{\pi_-,x_-,z}(\Pi)$ the functional (15) for the cost function $\tilde{c}$, by $J^{\pi_-,x_-,z}(\Pi)$ the functional (15) for the cost function (33), and finally by $\tilde{J}^{\pi_-,x_-,z}(\Pi)$ the functional (15) for the cost function

$$\sum_{i=1}^{d} \Big(c_i^1 S^i (N_1^i - N_2^i)^+ + c_i^2 S^i (N_1^i - N_2^i)^- \Big). \tag{34}$$

Easily, for any portfolio $\Pi \in \mathcal{A}^z$ we have

$$\tilde{J}^{\pi_-,x_-,z}(\Pi) \geq \hat{J}^{\pi_-,x_-,z}(\Pi) \geq J^{\pi_-,x_-,z}(\Pi).$$

This implies that

$$\sup_{\Pi\in\mathcal{A}^z} \tilde{J}^{\pi_-,x_-,z}(\Pi) \geq \sup_{\Pi\in\mathcal{A}^z} \hat{J}^{\pi_-,x_-,z}(\Pi) \geq \sup_{\Pi\in\mathcal{A}^z} J^{\pi_-,x_-,z}(\Pi).$$

Since, by virtue of Theorem 5.1 and Corollary 5.2 there exists a constant λ such that

$$\lambda = \sup_{\Pi\in\mathcal{A}^z} \tilde{J}^{\pi_-,x_-,z}(\Pi) = \sup_{\Pi\in\mathcal{A}^z} J^{\pi_-,x_-,z}(\Pi)$$

we conclude that $\lambda = \sup_{\Pi\in\mathcal{A}^z} \hat{J}^{\pi_-,x_-,z}(\Pi)$. Moreover, due to Corollary 5.2 the optimal portfolio for the functional $J^{\pi_-,x_-,z}$ is also optimal for $\tilde{J}^{\pi_-,x_-,z}$. Therefore, it is also optimal for $\hat{J}^{\pi_-,x_-,z}$. Notice now that the cost function (5) satisfies (31) and (32). Therefore, Theorem 5.1 extends to this important case.

The results of this paper can be applied to an incomplete information case and extend [26]. Let us first sketch some motivation for this development. It is well known that investors do not have full information about variables influencing the economy. It is due to the time needed to collect and process statistical data or simply due to inaccesibility of some information. Therefore, it is natural to extend our model to cover the case where a number of economic factors is either observable with delay and noise or not observable at all. This general setting is obtained by considering an observation process whose dynamics depends on the factors. This is well-established in engineering applications, where the observation process usually consists of noisy and possibly biased readings of the variables. However, it was argued that in the financial context it is natural to assume that we have complete observation of a group of factors and the rest is not observable. It does not substantially change the reasoning but simplifies the notation.

Following the above remark assume that the space of economic factors E is a direct sum of metric spaces E^1, E^2 with Borel σ-algebras $\mathcal{E}^1$, $\mathcal{E}^2$. Therefore, $Z(t)$ has a unique decomposition into $\big(Z^1(t), Z^2(t)\big)$. We shall treat E^1 as the observable part of the economic factor space and $Z^1(t)$ as the observable factor process. The process $Z^2(t)$ is the unobservable part of the factor process. We denote by $\mathcal{M}_t, \mathcal{Z}_t^1, \mathcal{Z}_t^2$ filtrations generated, respectively, by $\zeta(t)$, $Z^1(t)$ and $Z^2(t)$. Denote by $\mathcal{Y}_t$ the filtration generated by $\mathcal{M}_t$ and $\mathcal{Z}_t^1$ and by $\tilde{\mathcal{A}}^z$ a space of $\mathcal{Y}_t$-adapted portfolios admissible for z, i.e. $\tilde{\mathcal{A}}^z \subseteq \mathcal{A}^z$. Our aim is to prove existence of optimal strategy maximizing the functional

$$J^{\pi_-,x_-,z^1,\rho}(\Pi) = \liminf_{T\to\infty} \frac{1}{T} \mathbb{E}^{\,z^1,\rho} \ln X_-^{\Pi}(T)$$

over all strategies $\Pi \in \tilde{\mathcal{A}}$. Here $(z^1, \rho) \in E^1 \times \mathcal{P}(Z^2)$ denotes the initial distribution of $\big(Z^1(t), Z^2(t)\big)$ and $\mathcal{P}(Z^2)$ stands for the space of probability measures on $(Z^2, \mathcal{E}^2)$. Now, we can follow a similar reasoning as in [26] to obtain existence of an optimal portfolio. Here, however, we improve two main aspects of the result; firstly, the transaction costs structure is more general and covers important examples (4) and (5). Moreover, in [26] the space E^2 was forced to be compact to guarantee that $\mathcal{P}(E^2)$ is locally compact. Here, due to a different method of proof of Theorem 5.1 we allow E^2 to be a general separable metric space (in this case, $\mathcal{P}(E^2)$ is also a separable metric space). For further details see [26].

7. Appendix

Proof of Lemma 4.3. First inequality is a direct consequence of the second one. Denote by $d = \max_i(c_i^1, c_i^2)$ and $\delta = e(\pi_-, \pi, x_-)$. By Lemma 2.1, if a transaction from π_- to π with the wealth x_- is possible, then $\delta \geq 0$ and it is a unique solution to

$$c(\pi_-, \delta\pi) + \frac{C}{x_-} + \delta = 1.$$

Otherwise, $\delta = 0$ and easily $c(\pi_-, \delta\pi) + \frac{C}{x_-} + \delta \geq 1$. Notice that $c(\pi_-, \delta\pi) \leq d \sum_{i=1}^d |\pi_-^i - \delta\pi^i|$. Consequenly, we have the following sequence of inequalities:

$$\begin{aligned} 1 &\leq d \sum_{i=1}^d |\pi_-^i - \delta\pi^i| + \frac{C}{x_-} + \delta \\ &\leq d \sum_{i=1}^d |\pi_-^i - \delta\pi_-^i| + d \sum_{i=1}^d |\delta\pi_-^i - \delta\pi^i| + \frac{C}{x_-} + \delta \\ &\leq d(1-\delta) + 2d\delta + \frac{C}{x_-} + \delta, \end{aligned}$$

which easily leads to the required inequality. ■

Proof of Lemma 4.4. Noticing $a^+ - b^+ \leq (a-b)^+$ and $a^- - b^- \leq (a-b)^-$ we obtain for $\delta_1, \delta_2 \in [0, 1]$

$$\begin{aligned} c(\pi_-, \delta_2\pi) - c(\pi_-, \delta_1\pi) &= \sum_{i=1}^d \Big(c_i^1 \big((\pi_-)_i - \delta_2\pi_i\big)^+ - c_i^1 \big((\pi_-)_i - \delta_1\pi_i\big)^+ \\ &\qquad + c_i^2 \big((\pi_-)_i - \delta_2\pi_i\big)^- - c_i^2 \big((\pi_-)_i - \delta_1\pi_i\big)^- \Big) \\ &\leq \sum_{i=1}^d \big(c_i^1(\delta_1 - \delta_2)^+\pi_i + c_i^2(\delta_1 - \delta_2)^-\pi_i \big). \end{aligned} \tag{35}$$

Consequently,

$$|c(\pi_-, \delta_2\pi) - c(\pi_-, \delta_1\pi)| \leq |\delta_2 - \delta_1| \max_i(c_i^1, c_i^2). \tag{36}$$

Recall that

$$\tilde{e}(\pi_-,\pi) = 1 - c\big(\pi_-, \tilde{e}(\pi_-,\pi)\pi\big),$$
$$e(\pi_-,\pi,x_-) = 1 - c\big(\pi_-, e(\pi_-,\pi,x_-)\pi\big) - \frac{C}{x_-}. \tag{37}$$

We shall prove (i) by contradiction. Assume that $\tilde{e}(\pi_-,\pi) < e(\pi_-,\pi,x_-)$. Easily

$$0 \le e(\pi_-,\pi,x_-) - \tilde{e}(\pi_-,\pi) \le c\big(\pi_-, \tilde{e}(\pi_-,\pi)\pi\big) - c\big(\pi_-, e(\pi_-,\pi,x_-)\pi\big) - \frac{C}{x_-}.$$

By (36) we obtain

$$e(\pi_-,\pi,x_-) - \tilde{e}(\pi_-,\pi) \le \big(e(\pi_-,\pi,x_-) - \tilde{e}(\pi_-,\pi)\big) \max_i(c_i^1, c_i^2) - \frac{C}{x_-}.$$

It gives the estimate

$$1 + \frac{C}{x_- \,\big(e(\pi_-,\pi,x_-) - \tilde{e}(\pi_-,\pi)\big)} \le \max_i(c_i^1, c_i^2),$$

which contradicts the assumption that $c_i^1, c_i^2 \in [0,1)$.

The proof of $e(\pi_-,\pi,x_-) \le e(\pi_-,\pi,\tilde{x}_-)$ can be done in an analogous way.

From (i), (37) and (35) we obtain

$$\begin{aligned}\tilde{e}(\pi_-,\pi) - e(\pi_-,\pi,x_-) &= c\big(\pi_-, e(\pi_-,\pi,x_-)\pi\big) - c\big(\pi_-, \tilde{e}(\pi_-,\pi)\pi\big) + \frac{C}{x_-}\\ &\le \big(\tilde{e}(\pi_-,\pi) - e(\pi_-,\pi,x_-)\big) \max_i c_i^1 + \frac{C}{x_-},\end{aligned}$$

which immediately proves (ii). For (iii) we apply the inequality $\ln(1+x) \le x$ for $x > 0$. ∎

Proof of Corollary 4.5. Given $\pi_- \in \mathcal{S}$, $z \in E$ and $\tilde{x}_- \le x_-$

$$v_\beta(\pi_-,\tilde{x}_-,z) - v_\beta(\pi_-,x_-,z) \le \sup_{\Pi\in\mathcal{A}^z} \big\{J_\beta^{\pi,\tilde{x}_-,z}(\Pi) - J_\beta^{\pi,x_-,z}(\Pi)\big\}.$$

Fix $\pi \in \mathcal{A}^z$ and observe that

$$J_\beta^{\pi_-,\tilde{x}_-,z}(\Pi) - J_\beta^{\pi_-,x_-,z}(\Pi) = \sum_{k=1}^{\infty} \beta^{\tau_k}\Big(\ln e\big(\pi_-(\tau_k),\pi_k,\tilde{X}_-(\tau_k)\big) - \ln e\big(\pi_-(\tau_k),\pi_k,X_-(\tau_k)\big)\Big),$$

where $\tau_0 = 0$, $\pi_-(0) = \pi_-$,

$$\pi_-(t) = \pi_k \diamond \zeta(\tau_k+1) \diamond \ldots \diamond \zeta(t), \qquad \tau_k < t \le \tau_{k+1}$$

and $X_-(t)$, $\tilde{X}_-(t)$ are given by (13). By Lemma 4.4 (i) we have $X_-(t) \ge \tilde{X}_-(t)$, $t \ge 0$ and consequently $J_\beta^{\pi,\tilde{x}_-,z}(\Pi) - J_\beta^{\pi,x_-,z}(\Pi) \le 0$. ∎

Proof of Lemma 4.6. Let $\underline{e} = \inf_{\pi_-,\pi\in\mathcal{S}} \tilde{e}(\pi_-,\pi)$. Since $\max_i(c_i^1, c_i^2) < 1$, we have $\underline{e} > 0$. Fix $z, z' \in E$ and $\pi_-, \pi'_- \in \mathcal{S}$. Denote by Π the portfolio optimal for $\tilde{v}_\beta(\pi_-, z)$, and by Π' the portfolio optimal for $\tilde{v}_\beta(\pi'_-, z')$ (they exist due to Theorem 4.2). The corresponding proportion processes $\pi_-^{\Pi,z}(t)$, $\pi_-^{\Pi',z'}(t)$ will be written as $\pi_-(t)$, $\pi'_-(t)$ and the corresponding wealth processes $X_-^{\Pi,z}(t)$, $X_-^{\Pi',z'}(t)$ as $X_-(t), X'_-(t)$. We have then

$$\begin{aligned}\tilde{v}_\beta(\pi_-, z) - \tilde{v}_\beta(\pi'_-, z') = &\sum_{t=0}^{n-1} \beta^t \mathbb{E}^z h\big(\pi_-(t), z(t)\big) + \sum_{k=1}^{\infty} \mathbb{E}^z \Big\{1_{\tau_k<n} \beta^{\tau_k} \ln \tilde{e}\big(\pi_-(\tau_k), \pi_k\big)\Big\} \\ &- \sum_{t=0}^{n-1} \beta^t \mathbb{E}^{z'} h\big(\pi'_-(t), z'(t)\big) - \sum_{k=1}^{\infty} \mathbb{E}^{z'} \Big\{1_{\tau_k<n} \beta^{\tau_k} \ln \tilde{e}\big(\pi'_-(\tau_k), \pi_k\big)\Big\} \\ &+ \beta^n \Big(\mathbb{E}^z \tilde{v}_\beta\big(\pi_-(n), z(n)\big) - \mathbb{E}^{z'} \tilde{v}_\beta\big(\pi'_-(n), z'(n)\big)\Big).\end{aligned}$$

There are at most n transactions between 0 and $n-1$, since it is never optimal to have more than one transaction at a moment (by subadditivity of the cost function). Due to the fact that h is bounded and $-\infty < \ln \underline{e} \le \ln \tilde{e}(\pi_-, \pi) \le 0$ by Lemma 2.1, we have

$$\tilde{v}_\beta(\pi_-, z) - \tilde{v}_\beta(\pi'_-, z') \le n\|h\|_{sp} - n \ln \underline{e} + \beta^n \Big(\mathbb{E}^z \tilde{v}_\beta\big(\pi_-(n), z(n)\big) - \mathbb{E}^{z'} \tilde{v}_\beta\big(\pi'_-(n), z'(n)\big)\Big).$$

Choose arbitrary $\pi^* \in \mathcal{S}$ and observe that

$$\begin{aligned}\mathbb{E}^z \tilde{v}_\beta\big(\pi_-(n), z(n)\big) - \mathbb{E}^{z'} \tilde{v}_\beta\big(\pi'_-(n), z'(n)\big) \le &\ \mathbb{E}^z \big\{\tilde{v}_\beta\big(\pi_-(n), z(n)\big) - \tilde{v}_\beta\big(\pi^*, z(n)\big)\big\} \\ &+ \mathbb{E}^{z'} \big\{\tilde{v}_\beta\big(\pi^*, z'(n)\big) - \tilde{v}_\beta\big(\pi'_-(n), z'(n)\big)\big\} \\ &+ \mathbb{E}^z \tilde{v}_\beta\big(\pi^*, z(n)\big) - \mathbb{E}^{z'} \tilde{v}_\beta\big(\pi^*, z'(n)\big).\end{aligned}$$

Since for arbitrary $\beta \in (0,1)$, $\pi_-, \pi'_- \in \mathcal{S}$, $z \in E$

$$\tilde{v}_\beta(\pi_-, z) - \tilde{v}_\beta(\pi'_-, z) \le -\ln \tilde{e}(\pi, \pi'),$$

we have

$$\begin{aligned}\mathbb{E}^z \big\{\tilde{v}_\beta\big(\pi_-(n), z(n)\big) - \tilde{v}_\beta\big(\pi^*, z(n)\big)\big\} &\le -\ln \underline{e}, \\ \mathbb{E}^{z'} \big\{\tilde{v}_\beta\big(\pi^*, z'(n)\big) - \tilde{v}_\beta\big(\pi'_-(n), z'(n)\big)\big\} &\le -\ln \underline{e}.\end{aligned}$$

Notice that

$$\begin{aligned}\mathbb{E}^z \tilde{v}_\beta\big(\pi^*, z(n)\big) - \mathbb{E}^{z'} \tilde{v}_\beta\big(\pi^*, z'(n)\big) &= \int_E \tilde{v}_\beta(\pi^*, y)\, dP^n(z, dy) - \int_E \tilde{v}_\beta(\pi^*, y)\, dP^n(z', dy) \\ &= \int_E \tilde{v}_\beta(\pi^*, y)\, q(dy),\end{aligned}$$

with $q = P^n(z,\cdot) - P^n(z',\cdot)$. Let $\Gamma \in \mathcal{E}$ be the set coming from the Hahn-Jordan decomposition of the signed measure q, i.e. q is non-negative on Γ and non-positive on Γ^c. By (A3)

$$\begin{aligned}\int_E \tilde{v}_\beta(\pi^*,y)\, q(dy) &= \int_E \Big(\tilde{v}_\beta(\pi^*,y) - \inf_{y'\in E} \tilde{v}_\beta(\pi^*,y')\Big)\, q(dy)\\ &\le \int_\Gamma \Big(\tilde{v}_\beta(\pi^*,y) - \inf_{y'\in E} \tilde{v}_\beta(\pi^*,y')\Big)\, q(dy)\\ &\quad + \int_{\Gamma^c} \Big(\tilde{v}_\beta(\pi^*,y) - \inf_{y'\in E} \tilde{v}_\beta(\pi^*,y')\Big)\, q(dy)\\ &\le \|\tilde{v}_\beta(\pi^*,\cdot)\|_{sp}\, q(\Gamma) \le \kappa\, \|\tilde{v}_\beta(\pi^*,\cdot)\|_{sp}.\end{aligned}$$

Consequently,

$$\tilde{v}_\beta(\pi_-,z) - \tilde{v}_\beta(\pi'_-,z') \le n\|h\|_{sp} - (n+2)\ln \underline{e} + \kappa \|\tilde{v}_\beta(\pi^*,\cdot)\|_{sp}.$$

Since $\pi_-, \pi'_- \in \mathcal{S}$ and $z, z' \in E$ were arbitrary we obtain

$$\|\tilde{v}_\beta(\pi^*,\cdot)\|_{sp} \le n\|h\|_{sp} - (n+2)\ln \underline{e} + \kappa \|\tilde{v}_\beta(\pi^*,\cdot)\|_{sp},$$

which yields the required result with

$$M = \frac{n\|h\|_{sp} - (n+2)\ln \underline{e}}{1-\kappa}.$$

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