Growth rate optimization under transaction costs

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Joint work with Łukasz Stettner

Portfolio problem

Find a trading strategy Π solving

$$\liminf_{T\to\infty} \frac{1}{T}\mathbb{E}\,\ln\left(\frac{X_-^{\Pi}(T)}{X_-^{\Pi}(0)}\right) \longrightarrow \max$$

- economic factors
- transaction costs separated from zero
- partial information

Why returns?

- intuitive
- widely accepted in finance theory & practice
- there are results stating that return-conscious investment is the best long-run strategy

Thorp (1975), Aase and Øksendal (1988), Algoet and Cover (1988), Akian et al. (2001), Le and Platen (2006), Gerencsér et al. (2005), . . .

Our financial universe

$$S(t) = \big(S^1(t), S^2(t), \ldots, S^d(t)\big), \qquad t = 0, 1, 2, \ldots$$

economic factors (Markov process on a Polish space $(E, \mathcal{E})$)

$$\frac{S^i(t+1)}{S^i(t)} = \zeta^i\big(Z(t+1), \xi(t+1)\big), \qquad i = 1, \ldots, d$$

sequence of i.i.d. random variables independent of $\big(Z(t)\big)_{t \geq 0}$

For example...

- Binomial model with factor-dependent probabilities
- discretized Black-Scholes model with economic factors

Wiener process in $\mathbb{R}^p$

$$\frac{S^i(t+1)}{S^i(t)} = \exp\Big(\sigma^i\big(Z(t+1)\big)\cdot\big(W(t+1)-W(t)\big)+\mu^i\big(Z(t+1)\big)\Big),$$

$$i = 1, \ldots, d$$

σ^i, μ^i – functions with values in $\mathbb{R}^p, \mathbb{R}$, resp.

$Z(t)$ – a recurrent Markov chain with finite state space

Large Deviations

(A1) Process $(S(t), Z(t))$ satisfies the Feller property.

(A2) $\sup_{z,z' \in E} \sup_{B \in \mathcal{E}} \big(P^n(z, B) - P^n(z', B)\big) = \kappa < 1$ for some $n \geq 1$.

(A3) Doeblin's type condition for $(S(t), Z(t))$.

Then:

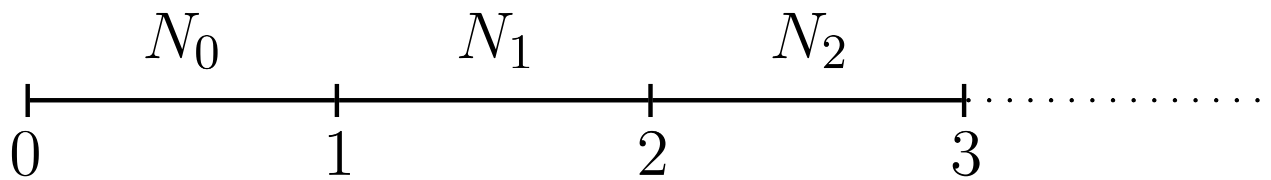
average “market” growth rate > 0

small number

$$\mathbb{P}^z\Big\{X_-(0)e^{T(\hat{p}-\epsilon)} \leq X_-(T)\Big\} \geq 1 - e^{-\gamma T}, \qquad T \geq T^*.$$

portfolio wealth

Admissible trading strategies $\mathcal{A}(n_0, s_0, z_0)$



Trading strategy $(N_0, N_1, \ldots)$ is admissible for (n_0, s_0, z_0) if

- $N_t \in [0, \infty)^d$ – number of shares
- N_t is $\mathcal{F}_t$-measurable
- if $N_{t-1} \neq N_t$ (transaction)

$$N_{t-1} \cdot S(t) = N_t \cdot S(t) + c\big(N_{t-1}, N_t, S(t)\big) \quad \mathbb{P}^{z_0} - a.s.$$

Transaction costs

$n_1 \in [0,\infty)^d$ – number of shares before transaction
$n_2 \in [0,\infty)^d$ – number of shares after transaction
$S \in (0,\infty)^d$ – stock prices
$C \geq 0$, $\quad c \in [0,1)$ – transaction costs

$$c(n_1, n_2, S) = C + c\sum_{i=1}^{d} S^i |n_1^i - n_2^i|$$

$$c(n_1, n_2, S) = \max\left(C, c\sum_{i=1}^{d} S^i |n_1^i - n_2^i|\right)$$

Transaction costs

$n_1 \in [0,\infty)^d$ – number of shares before transaction
$n_2 \in [0,\infty)^d$ – number of shares after transaction
$S \in (0,\infty)^d$ – stock prices
$C \geq 0$, $\quad c^1, c^2 \in [0,1)^d$ – transaction costs

$$c(n_1, n_2, S) = C + \sum_{i=1}^{d} \left(c_i^1 S^i (n_1^i - n_2^i)^+ + c_i^2 S^i (n_1^i - n_2^i)^- \right)$$

$$c(n_1, n_2, S) = \max\left(C, \sum_{i=1}^{d} \left(c_i^1 S^i (n_1^i - n_2^i)^+ + c_i^2 S^i (n_1^i - n_2^i)^- \right) \right)$$

Optimization problem

Find a trading strategy $\Pi = (N_0^*, N_1^*, \ldots) \in \mathcal{A}(n_0, s_0, z_0)$ solving

$$\liminf_{T\to\infty} \frac{1}{T}\mathbb{E}^{z_0} \ln\left(\frac{X_-^{\Pi}(T)}{X_-^{\Pi}(0)}\right) \longrightarrow \max$$

Optimization problem

Find a trading strategy $\Pi = (N_0^*, N_1^*, \ldots) \in \mathcal{A}(n_0, s_0, z_0)$ solving

$$\liminf_{T\to\infty} \frac{1}{T} \mathbb{E}^{\,z_0} \ln\left(\frac{X_-^{\Pi}(T)}{X_-^{\Pi}(0)}\right) \longrightarrow \max$$

It is equivalent to

$$\liminf_{T\to\infty} \frac{1}{T} \mathbb{E}^{\,z_0} \ln\left(X_-^{\Pi}(T)\right) \longrightarrow \max$$

Main result

Theorem. (JP, Ł. Stettner 2007) There exists a constant λ such that for any (n_0, s_0, z_0)

$$\sup_{\Pi \in \mathcal{A}(n_0,s_0,z_0)} \liminf_{T\to\infty} \frac{1}{T} \mathbb{E}^{z_0} \ln\big(X_-^{\Pi}(T)\big) = \lambda.$$

This supremum is attained by a strategy $(N_0^*, N_1^*, \ldots)$ given by

$$N_t^* = D\big(N_{t-1}^*, S(t), Z(t)\big).$$

a measurable function, independent of (n_0, s_0, z_0)

Proof

- proportions
- vanishing discount method [Schäl 1993]
- Tauberian theorem
- large deviations

- significant violation of continuity assumptions required for existence of optimal strategies
- lack of requirement for $(E, \mathcal{E})$ to be a locally compact space – important for incomplete information case

Ergodic functional

$$
\begin{aligned}
&\liminf_{T\to\infty} \frac{1}{T}\mathbb{E}^{\,z_0} \ln\left(\frac{X_-^{\Pi}(T)}{X_-^{\Pi}(0)}\right) \\
&\qquad = \liminf_{T\to\infty} \frac{1}{T}\mathbb{E}^{\,z_0} \sum_{k=0}^{T-1} \ln \frac{X_-^{\Pi}(t+1)}{X_-^{\Pi}(t)} \\
&\qquad = \liminf_{T\to\infty} \frac{1}{T}\mathbb{E}^{\,z_0} \sum_{t=0}^{T-1} \mathbb{E}^{\,z_0}\left\{ \ln \frac{N_t \cdot S(t+1)}{N_{t-1} \cdot S(t)} \middle| \mathcal{F}_t \right\} \\
&\qquad = \liminf_{T\to\infty} \frac{1}{T}\mathbb{E}^{\,z_0} \sum_{t=0}^{T-1} h\big(N_{t-1}, N_t, S(t), Z(t)\big)
\end{aligned}
$$

Bellman equation

$$w(n_0, s_0, z_0) + \lambda = \sup_{n_0^* \in I(n_0, s_0)} \left(h(n_0, n_0^*, s_0, z_0) + \int w(n_0^*, s_1, z_1) q(s_0, z_0)(\cdot) \right)$$

- n_0^* is selected from a set depending on n_0 and s_0.
- There are two unknowns: a constant λ and a function w.
- If in transaction costs $C > 0$, then h is discontinuous. Problems!

Why Bellman equation?

$$w(n_0, s_0, z_0)+\lambda = \sup_{n_0^*} \Big(h(n_0, n_0^*, s_0, z_0)+\int w(n_0^*, s_1, z_1) q(s_0, z_0)(\cdot) \Big)$$

By iteration we get

$$\frac{w(\cdot)}{T} + \lambda = \frac{1}{T}\mathbb{E}^{\,z_0} \sum_{t=0}^{T-1} h\big(N_{t-1}^*, N_t^*, S(t), Z(t)\big) + \frac{1}{T}\mathbb{E}^{\,z_0} \int w(N_T^*, s_1, z_1) q\big(S(T), Z(T)\big)(\cdot)$$

Deriving Bellman equation

Theorem. (Ł. Stettner 2006) $C = 0 \implies$ there exist a continuous, bounded function w and a constant λ such that

$$w(n_0, s_0, z_0)+\lambda = \sup_{n_0^*} \Big(h(n_0, n_0^*, s_0, z_0)+\int w(n_0^*, s_1, z_1) q(s_0, z_0)(\cdot)\Big).$$

Theorem. (JP, Ł. Stettner 2007) $C > 0 \implies$ there exist a l.s.c. non-positive function w, a constant λ and a measureable function f such that

$$w(n_0, s_0, z_0) + \lambda \leq h(n_0, n_0^*, s_0, z_0) + \int w(n_0^*, s_1, z_1) q(s_0, z_0)(\cdot).$$

where

$$n_0^* = f(n_0, s_0, z_0).$$

Incomplete observation

$$Z(t) = \big(Z^1(t), Z^2(t)\big)$$

where

- $Z^1(t) \in E^1$ denotes observable factors
- $Z^2(t) \in E^2$ denotes unobservable factors

Let

- $\mathcal{M}_t = \sigma\{S(k) : k \leq t\}$
- $\mathcal{Z}_t^1 = \sigma\{Z^1(k) : k \leq t\}$
- $\mathcal{Z}_t^2 = \sigma\{Z^2(k) : k \leq t\}$

Complete observation: $\mathcal{F}_t = \mathcal{M}_t \vee \mathcal{Z}_t^1 \vee \mathcal{Z}_t^2$.
Incomplete observation: $\mathcal{Y}_t = \mathcal{M}_t \vee \mathcal{Z}_t^1$.

Optimization problem

- Initial point:

$$(n_0, s_0, z_0^1, \rho),$$

where ρ a prior for z_0^2 (it is a probability measure).

- A set of admissible trading strategies measurable with respect to $(\mathcal{Y}_t)$:

$$\tilde{\mathcal{A}}(n_0, s_0, z_0^1, \rho).$$

- Optimization problem: Find a trading strategy $\Pi = (N_0^*, N_1^*, \ldots) \in \tilde{\mathcal{A}}(n_0, s_0, z_0^1, \rho)$ solving

$$\liminf_{T\to\infty} \frac{1}{T} \mathbb{E}^{z_0} \ln\left(\frac{X_-^{\Pi}(T)}{X_-^{\Pi}(0)}\right) \longrightarrow \max$$

Filtering

Theorem.

1. The process $\big(S(t), Z(t)\big)$ conditioned on $(\mathcal{Y}_t)$ has a form

$$\big(S(t), Z^1(t), \rho(t)\big) \in (0,\infty)^d \times E^1 \times \mathcal{P}(E^2).$$

   Moreover, it is Markovian and there exists an explicit formula for the transition operator for $\rho(t)$.

2. If E^2 is a Polish space then $\mathcal{P}(E^2)$ is also a Polish space.

$\big(Z^1(t), \rho(t)\big)$ – new factor process

The result

Theorem. (JP, Ł. Stettner 2006) There exists a constant λ such that for any (n_0, s_0, z_0^1, ρ)

$$\sup_{\Pi \in \tilde{\mathcal{A}}(n_0, s_0, z_0^1, \rho)} \liminf_{T \to \infty} \frac{1}{T} \mathbb{E}^{\,z_0^1,\rho} \ln\big(X_-^{\Pi}(T)\big) = \lambda.$$

This supremum is attained by a strategy $(N_0^*, N_1^*, \dots)$ given by

$$N_t^* = D\big(N_{t-1}^*, S(t), Z^1(t), \rho(t)\big).$$

a measurable function, independent of (n_0, s_0, z_0^1, ρ)

Summary

- general discrete-time model with economic factors
- large deviations for portfolio wealth
- transaction costs with constant term
- optimal strategy for average growth rate
- partial observation