# Convexity Propagation in Interest rate models

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Consider a one-dimensional short rate model

$$dr_t = \beta(r_t)dt + \sigma(r_t)dW_t$$

on a filtered stochastic basis $(\Omega, \mathcal{F}, P)$ carrying a one-dimensional Brownian motion $W$.
Under suitable assumptions on $\beta, \sigma$ we can guarantee that

$$G(r,t) := E(\exp(-\int_0^t r_s ds) g(r_t))$$

makes sense for sufficiently regular pay-off functions $g$.

# The Theorem

Assume sufficient regularity on $\beta, \sigma, g$ and assume that $g$ is a **non-negative, convex, decreasing pay-off**. If

$$\beta_{rr}(r) \leq 2$$

for all $r$, then $r \mapsto G(r,t)$ is convex in $r$.
J. Ekström and J. Tysk also prove assertions on monotonicity (with respect to changes in volatility and drift), log-convexity and log-convity. Properties of option prices in models with jumps can be found in Ekström/Tysk (to appear in Math. Finance).

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- How does a multi-dimensional version of the previous theorem look like?
- Does it hold for jump-diffusions or Lévy-driven processes?
- Does it hold for affine processes?

Fix $N, d \geq 1$. We consider a stochastic differential equation

$$dr_s = \beta(r_s)ds + \sum_{i=1}^{d} \sigma_i(r_s)dW_s^i$$

in $\mathbb{R}^N$ and a linear functional $l$ on $\mathbb{R}^N$. We assume the coefficients to satisfy $C^\infty$-boundedness conditions and the initial value $r$ to be deterministic. We **do not assume** ellipticity.
Let $g : \mathbb{R}^N \to \mathbb{R}$ be a **convex, non-negative function** with sufficient regularity such that the following calculations make sense. We consider

$$G(r,t) = E(exp(-\int_0^t l(r_s)ds)g(r_t))$$

for $t \geq 0$.

# First and Second Variation

We denote the derivatives with respect to $r$ by $T$ (tangent map). We shall apply the first and second variation of the stochastic process $r$ with respect to its initial value in a deterministic direction $v$, which is a stochastic process again,

$$\partial_v r_s = T\beta(r_s)\partial_v r_s ds + \sum_{i=1}^{d} T\sigma_i(r_s)\partial_v r_s dW_s^i, \quad \partial_v r_0 = v$$

and

$$\begin{aligned}\partial_v^2 r_s = &\left(T\beta(r_s)\partial_v^2 r_s + T^2\beta(r_s)(\partial_v r_s)^2\right)ds + \\ &+ \sum_{i=1}^{d}\left(T\sigma_i(r_s)\partial_v^2 r_s + T^2\sigma_i(r_s)(\partial_v r_s)^2\right)dW_s^i\end{aligned}$$

with $\partial_v^2 r_0 = 0$.

# Taking derivatives

The first derivative of $G$ with respect to $r$,

$$TG(r,t)v = -E(exp(-\int_0^t l(r_s)ds)\int_0^t l(\partial_v r_s)ds\, g(r_t))+ \\ + E(exp(-\int_0^t l(r_s)ds)Tg(r_t)\partial_v r_t)$$

and...

# Taking derivatives

...the second one

$$
\begin{aligned}
T^2G(r,t)(v)^2 = & E(exp(-\int_0^t l(r_s)ds)(\int_0^t l(\partial_v r_s)ds)^2 g(r_t))- \\
& - E(exp(-\int_0^t l(r_s)ds) \int_0^t l(\partial_v^2 r_s)ds \; g(r_t))- \\
& - E(exp(-\int_0^t l(r_s)ds) \int_0^t l(\partial_v r_s)ds \; (Tg(r_t)\partial_v r_t))- \\
& - E(exp(-\int_0^t l(r_s)ds) \int_0^t l(\partial_v r_s)ds \; (Tg(r_t)\partial_v r_t))+ \\
& + E(exp(-\int_0^t l(r_s)ds)(T^2g(r_t)(\partial_v r_t)^2))+ \\
& + E(exp(-\int_0^t l(r_s)ds)(Tg(r_t)\partial_v^2 r_t)).
\end{aligned}
$$

There is a symmetric structure, which is best revealed by the following process

$$C_t = 2\int_0^t l(\partial_v r_s)ds\, \partial_v r_t - \partial_v^2 r_t.$$

Hence we can write that

$$\begin{aligned} T^2 G(r,t)(v)^2 =& E(exp(-\int_0^t l(r_s)ds)g(r_t)\int_0^t l(C_s)ds) - \\ & - E(exp(-\int_0^t l(r_s)ds)Tg(r_t)C_t) + \\ & + E(exp(-\int_0^t l(r_s)ds)(T^2 g(r_t)(\partial_v r_t)^2)) \end{aligned}$$

by gathering the respective terms and applying that

$$\int_0^t l(C_s)ds = (\int_0^t l(\partial_v r_s)ds)^2 - \int_0^t l(\partial_v^2 r_s)ds.$$

# The Theorem

Assume first that $l(\partial_v r_t) \geq 0$ and $Tg(r_t)\partial_v r_t \leq 0$ almost surely for all $t \geq 0$, then we obtain that

$$TG(r,t)v \leq 0$$

for $r \in \mathbb{R}^N$ and $t \geq 0$. Next we assume (without granting the first assumption) that

$$2l(v)(TG(r,t)v) - TG(r,t)T^2\beta(r)(v)^2) \leq 0$$

for all $v, r \in \mathbb{R}^N$ and $t \geq 0$. Then we conclude that $T^2 G(t,r)(v)^2 \geq 0$.

# Corollaries and Discussion

- Fix $N, d \geq 1$. Assume that the drift $\beta$ is linear and that $g = 1$. If $l(\partial_v r_t) \geq 0$ for all $v \in \mathbb{R}^N$ with $l(v) \geq 0$, then

$$r \mapsto E(\exp(\int_0^t l(r_s)ds))$$

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  $$r \mapsto E(\exp(\int_0^t l(r_s) ds))$$

  is convex with respect to the initial values.
- Assume that $N = 1$, $d \geq 1$ and that $g$ is decreasing. We choose $l(x) = x$ and $v = 1$. Since $\partial_v r_0 = v$ we have that $l(\partial_v r_t) \geq 0$ by continuity of the trajectories and the fact that $\partial_v r_t \neq 0$ almost surely for $t \geq 0$. Consequently we obtain in this case $TG(r,t)v \leq 0$. Supposing finally that

  $$2 - \beta_{rr}(r) \geq 0$$

  yields convexity such as in Ekström/Tysk.

# Corollaries and Discussion

- The meaning of the condition

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- The result does actually not only apply for Ito-diffusions, but is rather true in the world of Markov processes. Indeed we learn from the proof that the sign of the action of a certain generator $\mathcal{A}$ on a certain function $F$ determines the property.

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- The result does actually not only apply for Ito-diffusions, but is rather true in the world of Markov processes. Indeed we learn from the proof that the sign of the action of a certain generator $\mathcal{A}$ on a certain function $F$ determines the property.
- Time-inhomogenous Markov processes can be considered by the same methods. The results remain the same.

# The Proof

We consider the $5N$-dimensional Markov process $X$ composed of $(r_t, \partial_v r_t, \partial_v^2 r_t, \int_0^t r_s ds, \int_0^t \partial_v r_s ds)$ for the approriate initial value $(r, v, 0, 0, 0)$. We interpret the previous expectations as solutions of the respective heat equation for a particular initial function $F$. We know from Dynkin's formula that

$$E(F(X_t)) = F(X_0) + \int_0^t E(\mathcal{A}F(X_s))ds,$$

where $\mathcal{A}$ denotes the generator of $X$.

# The Proof

We choose

$$F(x_1,x_2,x_3,x_4,x_5)=\exp(-I(x_4))\,Tg(x_1)(2I(x_5)x_2)-x_3),$$

which yields

$$\begin{aligned}\mathcal{A}F(x_1,x_2,x_3,x_4,x_5) &= -I(x_1)\exp(-I(x_4))\,Tg(x_1)(2I(x_5)x_2)-x_3)+\\ &+\exp(-I(x_4))\,Tg(x_1)(2I(x_5)x_2)-x_3)+\\ &+\exp(-I(x_4))\,Tg(x_1)\times\\ &\times(2I(x_2)x_2+2I(x_5)\,T\beta(x_1)x_2-T^2\beta(x_1)(x_2)^2-T\beta(x_1)x_3).\end{aligned}$$

# The Proof

We assume that $T^2 G(r,t)(v)^2 = 0$ and want to prove that the derivative at $t$ of this quantity has to be positive. We obtain by the previous consideration

$$\frac{\partial}{\partial t}|_{t=0} T^2 G(r,t)(v)^2 \geq -\mathcal{A}F(r,v,0,0,0)$$
$$= -(2l(v)Tg(r)v - Tg(r)T^2\beta(r)(v)^2) \geq 0.$$

By slightly deforming $\beta$ we can assume that the inequality is strict and therefore we obtain that $T^2 G(r,\delta)(v)^2 > 0$ for $0 \leq \delta \leq \delta_0$, where $\delta_0 > 0$ is some number. The general result follows by approximation arguments.

# Discussion

For the proof we need the following structures:

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- A Markov process admitting first and second variations (see for instance Ph. Protter's book for conditions when semi-martingale-driven SDEs admit stochastic flows).

# Discussion

For the proof we need the following structures:

- A Markov process admitting sufficiently regular solutions of its associated heat equation.
- A Markov process admitting first and second variations (see for instance Ph. Protter's book for conditions when semi-martingale-driven SDEs admit stochastic flows).
- The sign of the synthesized generator $\mathcal{A}$ on

$$F(x_1, x_2, x_3, x_4, x_5) = \exp(-I(x_4))\, Tg(x_1)(2I(x_5)x_2) - x_3)$$

at $(r, v, 0, 0, 0)$ is decisive.

# Applications

- Greeks play a major role in hedging contingent claims.

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- Convexity of the matrix of "Gammas" tells about the distance of tangent processes to the actual claim.

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- Further informations on this research @ *Publications* soon on my webpage (google Josef Teichmann).