
Information, Inflation, and Interest

Lane P. Hughston

Professor of Financial Mathematics
Department of Mathematics, King's College London,
The Strand, London WC2R 2LS, UK
(lane.hughston@kcl.ac.uk, www.mth.kcl.ac.uk)

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(Co-author: Andrea Macrina, King's College London)

Asset pricing in discrete time

Many authors have pursued the development of asset pricing in discrete time.

In the context of interest rate modelling, it is worth recalling that the first example of a fully developed term-structure model where the initial discount function is freely specifiable is that of Ho & Lee 1986, in a discrete-time setting.

Here we develop a general discrete-time scheme, taking an axiomatic approach.

Let $\{t_i\}_{i=0,1,2,\dots}$ denote a sequence of discrete times, not necessarily equally spaced, where t_0 is the present and $t_{i+1} > t_i$ for all $i \in \mathbb{N}_0$.

We assume the sequence $\{t_i\}$ is unbounded in the sense that for any given time T there exists a value of i such that $t_i > T$.

The economy will be represented with the specification of a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $\{\mathcal{F}_i\}_{i \geq 0}$ which we call the “market filtration”.

Each asset is characterised by a pair of processes $\{S_{t_i}\}_{i \geq 0}$ and $\{D_{t_i}\}_{i \geq 0}$ which we refer to as the “value process” and the “dividend process”.

We interpret D_{t_i} as a random cash flow or dividend paid to the owner of the asset at time t_i . Then S_{t_i} denotes the “ex-dividend” value of the asset at t_i .

For simplicity, we frequently write $S_i = S_{t_i}$ and $D_i = D_{t_i}$.

To ensure the absence of arbitrage in the financial markets and to establish intertemporal pricing relations, we assume the existence of a strictly positive pricing kernel $\{\pi_i\}_{i \geq 0}$, and make the following assumptions:

Axiom A. For any asset with associated value process $\{S_i\}_{i \geq 0}$ and dividend process $\{D_i\}_{i \geq 0}$, the process $\{M_i\}_{i \geq 0}$ defined by

$$M_i = \pi_i S_i + \sum_{n=0}^i \pi_n D_n \quad (1)$$

is a martingale.

Axiom B. There exists a strictly positive non-dividend-paying asset with value process $\{\bar{B}_i\}_{i \geq 0}$ such that $\bar{B}_{i+1} > \bar{B}_i$ for all $i \in \mathbb{N}_0$. We assume that for any $b \in \mathbb{R}$ there exists a time t_i such that $\bar{B}_i > b$.

Given this axiomatic scheme, we proceed to explore its consequences.

The notation $\{\bar{B}_i\}$ is used in Axiom B to distinguish the positive return asset from the previsible money-market account asset $\{B_i\}$ introduced later.

Since $\{\bar{B}_i\}$ is non-dividend paying, Axiom A shows that $\{\pi_i \bar{B}_i\}$ is a martingale.

Writing $\bar{\rho}_i = \pi_i \bar{B}_i$, we have $\pi_i = \bar{\rho}_i / \bar{B}_i$. Since $\{\bar{B}_i\}$ is strictly increasing, we see that $\{\pi_i\}$ is a supermartingale. It then follows from Axiom B that

$$\lim_{i \rightarrow \infty} \mathbb{E}[\pi_i] = 0. \quad (2)$$

With these facts in hand, we shall establish a useful result concerning the pricing of assets such that $S_i \geq 0$ and $D_i \geq 0$ for all $i \in \mathbb{N}$.

Proposition 1. *Let $\{S_i\}_{i \geq 0}$ and $\{D_i\}_{i \geq 0}$ be the value and dividend processes associated with a limited-liability asset. Then $\{S_i\}$ is of the form*

$$S_i = \frac{m_i}{\pi_i} + \frac{1}{\pi_i} \mathbb{E}_i \left[\sum_{n=i+1}^{\infty} \pi_n D_n \right], \quad (3)$$

where $\{m_i\}$ is a non-negative martingale that vanishes if and only if the following transversality condition holds:

$$\lim_{j \rightarrow \infty} \mathbb{E}[\pi_j S_j] = 0. \quad (4)$$

In the case of an asset for which the transversality condition is satisfied, the price is directly related to the future dividend flow:

$$S_i = \frac{1}{\pi_i} \mathbb{E}_i \left[\sum_{n=i+1}^{\infty} \pi_n D_n \right]. \quad (5)$$

This is the so-called “fundamental equation” often used directly as a basis for asset pricing theory (e.g. as in Cochrane 2005).

Alternatively we can write (5) in the form

$$S_i = \frac{1}{\pi_i} (\mathbb{E}_i[F_\infty] - F_i), \quad (6)$$

where

$$F_i = \sum_{n=0}^i \pi_n D_n, \quad \text{and} \quad F_\infty = \lim_{i \rightarrow \infty} F_i. \quad (7)$$

Hence the price of a pure “income-generating” asset can be expressed as a ratio of potentials, thus giving us a discrete-time analogue of a result of Rogers 1997.

Pricing kernel and positive return assets

To proceed further we need to say more about the relation between the pricing kernel $\{\pi_i\}$ and the positive-return asset $\{\bar{B}_i\}$. To this end let us write

$$\bar{r}_i = \frac{\bar{B}_i - \bar{B}_{i-1}}{\bar{B}_{i-1}} \quad (8)$$

for the rate of return on the positive-return asset realised at time t_i on an investment made at time t_{i-1} .

The notation \bar{r}_i is used here to distinguish the rate of return on $\{\bar{B}_i\}$ from the rate of return r_i on the money market account, which will be introduced later.

Proposition 2. *There exists an asset with constant nominal value $S_i = 1$ for all $i \in \mathbb{N}_0$, for which the associated cash flows are given by $\{\bar{r}_i\}_{i \geq 1}$.*

Proposition 3. *Let $\{\bar{B}_i\}$ be a positive-return asset satisfying the conditions of Axiom B, and let $\{\bar{r}_i\}$ be its rate-of-return process. Then the pricing kernel can be expressed in the form $\pi_i = \mathbb{E}_i[G_\infty] - G_i$, where $G_i = \sum_{n=1}^i \pi_n \bar{r}_n$ and $G_\infty = \lim_{i \rightarrow \infty} G_i$.*

We shall establish a converse to this result, which allows one to construct a system satisfying Axioms A and B from any strictly-increasing non-negative adapted process that converges, and satisfies a certain integrability condition.

Proposition 4. *Let $\{G_i\}_{i \geq 0}$ be a strictly increasing adapted process satisfying $G_0 = 0$, and $\mathbb{E}[G_\infty] < \infty$, where $G_\infty = \lim_{i \rightarrow \infty} G_i$.*

Let the processes $\{\pi_i\}$, $\{\bar{r}_i\}$, and $\{\bar{B}_i\}$, be defined by $\pi_i = \mathbb{E}_i[G_\infty] - G_i$ for $i \geq 0$, $\bar{r}_i = (G_i - G_{i-1})/\pi_i$ for $i \geq 1$, and $\bar{B}_i = \prod_{n=1}^i (1 + \bar{r}_n)$ for $i \geq 1$, with $\bar{B}_0 = 1$.

Let the process $\{\bar{\rho}_i\}$ be defined by $\bar{\rho}_i = \pi_i \bar{B}_i$ for $i \geq 0$. Then $\{\bar{\rho}_i\}$ is a martingale, and $\lim_{j \rightarrow \infty} \bar{B}_j = \infty$.

Thus $\{\pi_i\}$ and $\{\bar{B}_i\}$, as constructed, satisfy Axioms A and B.

Discount bonds

Now we proceed to consider the properties of nominal discount bonds.

By such an instrument we mean an asset that pays a single dividend consisting of one unit of currency at some designated time t_j .

The price P_{ij} at time t_i ($i < j$) of a discount bond that matures at time t_j is:

$$P_{ij} = \frac{1}{\pi_i} \mathbb{E}_i[\pi_j]. \quad (9)$$

Since $\pi_i > 0$ for $i \in \mathbb{N}$, and $\mathbb{E}_i[\pi_j] < \pi_i$ for $i < j$, it follows that $0 < P_{ij} < 1$.

We observe that the associated interest rate R_{ij} defined by

$$P_{ij} = \frac{1}{1 + R_{ij}} \quad (10)$$

is strictly positive.

Since $\{\pi_i\}$ is given, there is no need in the present framework to model the volatility structure of the discount bonds.

Indeed, from the present point of view this would be a little artificial. The important issue, rather, is how to model the pricing kernel.

Thus, our scheme differs somewhat in spirit from the discrete-time models discussed, e.g., in Heath *et al.* 1990, and Filipović & Zabczyk 2002.

As an example of a tractable family of discrete-time interest rate models set

$$\pi_i = \alpha_i + \beta_i N_i, \quad (11)$$

where $\{\alpha_i\}$ and $\{\beta_i\}$ are strictly positive, strictly decreasing deterministic processes, satisfying $\lim_{i \rightarrow \infty} \alpha_i = 0$ and $\lim_{i \rightarrow \infty} \beta_i = 0$, and where $\{N_i\}$ is a strictly positive martingale.

Then by (9) we have

$$P_{ij} = \frac{\alpha_j + \beta_j N_i}{\alpha_i + \beta_i N_i}, \quad (12)$$

thus giving a family of “rational” interest rate models.

Note that in a discrete-time setting we can produce classes of models that have no immediate analogues in continuous time—for example, we can let $\{N_i\}$ be the natural martingale associated with a branching process.

It turns out that any discount bond system consistent with our general scheme admits a representation of the Flesaker-Hughston type.

Proposition 5. *Let $\{\pi_i\}$, $\{\bar{B}_i\}$, $\{P_{ij}\}$ satisfy the conditions of Axioms A and B. Then there exists a family of positive martingales $\{m_{in}\}_{0 \leq i \leq n}$ indexed by $n \in \mathbb{N}$ such that*

$$P_{ij} = \frac{\sum_{n=j+1}^{\infty} m_{in}}{\sum_{n=i+1}^{\infty} m_{in}}. \quad (13)$$

Nominal money-market account

Let us consider now the situation where the positive-return asset is previsible.

Thus we assume that B_i is \mathcal{F}_{i-1} -measurable and we drop the “bar” over B_i to signify that we are considering a money-market account. In that case we have:

$$\begin{aligned}
 P_{i-1,i} &= \frac{1}{\pi_{i-1}} \mathbb{E}_{i-1}[\pi_i] \\
 &= \frac{B_{i-1}}{\rho_{i-1}} \mathbb{E}_{i-1} \left[\frac{\rho_i}{B_i} \right] \\
 &= \frac{B_{i-1}}{B_i},
 \end{aligned} \tag{14}$$

by virtue of the martingale property of $\{\rho_i\}$.

Thus, in the case of a money-market account we see that

$$P_{i-1,i} = \frac{1}{1 + r_i}. \tag{15}$$

where $r_i = R_{i-1,i}$.

In other words, the rate of return on the money-market account is previsible, and is given by the one-period simple discount factor associated with the discount bond that matures at time t_i .

Reverting now to the general situation, it follows that if we are given a pricing kernel $\{\pi_i\}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with filtration $\{\mathcal{F}_i\}$, and a system of assets satisfying Axioms A and B, then we can construct a candidate for an associated previsible money market account by setting $B_0 = 1$ and defining

$$B_i = (1 + r_i)(1 + r_{i-1}) \cdots (1 + r_1), \quad (16)$$

for $i \geq 1$, where

$$r_i = \frac{\pi_{i-1}}{\mathbb{E}_{i-1}[\pi_i]} - 1. \quad (17)$$

We shall refer to the process $\{B_i\}$ thus constructed as the “natural” money market account associated with the pricing kernel $\{\pi_i\}$.

To justify this nomenclature, we need to verify that $\{B_i\}$, so constructed, satisfies the conditions of Axioms A and B.

To this end, we make note of the following decomposition. Let $\{\pi_i\}$ be a positive supermartingale satisfying $\mathbb{E}_i[\pi_j] < \pi_i$ for all $i < j$ and $\lim_{j \rightarrow \infty} \pi_j = 0$.

Then as an identity we can write

$$\pi_i = \frac{\rho_i}{B_i}, \quad (18)$$

where

$$\rho_i = \frac{\pi_i}{\mathbb{E}_{i-1}[\pi_i]} \frac{\pi_{i-1}}{\mathbb{E}_{i-2}[\pi_{i-1}]} \cdots \frac{\pi_1}{\mathbb{E}_0[\pi_1]} \pi_0 \quad (19)$$

for $i \geq 0$, and

$$B_i = \frac{\pi_{i-1}}{\mathbb{E}_{i-1}[\pi_i]} \frac{\pi_{i-2}}{\mathbb{E}_{i-2}[\pi_{i-1}]} \cdots \frac{\pi_1}{\mathbb{E}_1[\pi_2]} \frac{\pi_0}{\mathbb{E}_0[\pi_1]} \quad (20)$$

for $i \geq 1$, with $B_0 = 1$. Thus, in this scheme we have

$$\rho_i = \frac{\pi_i}{\mathbb{E}_{i-1}[\pi_i]} \rho_{i-1}, \quad (21)$$

with the initial condition $\rho_0 = \pi_0$; and

$$B_i = \frac{\pi_{i-1}}{\mathbb{E}_{i-1}[\pi_i]} B_{i-1}, \quad (22)$$

with the initial condition $B_0 = 1$. It is evident that $\{\rho_i\}$ as thus defined is $\{\mathcal{F}_i\}$ -adapted, and that $\{B_i\}$ is previsible and strictly increasing.

Making use of the identity (22) we are then able to establish the following result:

Proposition 6. *Let $\{\pi_i\}$ be a non-negative supermartingale satisfying $\mathbb{E}_i[\pi_j] < \pi_i$ for all $i < j \in \mathbb{N}_0$, and $\lim_{i \rightarrow \infty} \mathbb{E}[\pi_i] = 0$.*

Let $\{B_i\}$ be defined by $B_0 = 1$ and $B_i = \prod_{n=1}^i (1 + r_n)$ for $i \geq 1$, where $1 + r_i = \pi_{i-1} / \mathbb{E}_{i-1}[\pi_i]$, and set $\rho_i = \pi_i B_i$ for $i \geq 0$.

Then $\{\rho_i\}$ is a martingale, and the interest rate system defined by $\{\pi_i\}$, $\{B_i\}$, and $\{P_{ij}\}$ satisfies Axioms A and B.

The martingale $\{\rho_i\}$ is the likelihood ratio process appropriate for a change of measure from the objective measure \mathbb{P} to the equivalent martingale measure \mathbb{Q} characterised by the property that non-dividend-paying assets are martingales when expressed in units of the money-market account.

An interesting feature of Proposition 6 is that no integrability condition is required on $\{\rho_i\}$.

In other words, the natural previsible money market account defined by (20) “automatically” satisfies the conditions of Axiom A.

For some purposes it may therefore be advantageous to incorporate the existence of the natural money market account directly into the axioms.

Then in place of Axiom B we can assume:

Axiom B*. *There exists a strictly-positive non-dividend paying asset, the money-market account $\{B_i\}_{i \geq 0}$, having the properties that $B_{i+1} > B_i$ for all $i \in \mathbb{N}_0$ and that B_i is \mathcal{F}_{i-1} -measurable for all $i \in \mathbb{N}$. We assume that for any $b \in \mathbb{R}$ there exists a time t_i such that $B_i > b$.*

The content of Proposition 6 is that Axioms A and B together imply Axiom B*.

As an exercise we shall establish that the class of interest rate models satisfying Axioms A and B* is non-vacuous.

In particular, suppose we consider the “rational” models defined by equations (11) and (12) for some choice of the martingale $\{N_i\}$.

It is straightforward to see that the unique previsible money market account in this model is given for $i = 0$ by $B_0 = 1$ and for $i \geq 1$ by

$$B_i = \prod_{n=1}^i \frac{\alpha_{n-1} + \beta_{n-1}N_{n-1}}{\alpha_n + \beta_n N_{n-1}}. \quad (23)$$

For $\{\rho_i\}$ we then have

$$\rho_i = \rho_0 \prod_{n=1}^i \frac{\alpha_n + \beta_n N_n}{\alpha_n + \beta_n N_{n-1}}, \quad (24)$$

where $\rho_0 = \alpha_0 + \beta_0 N_0$.

But it is easy to check that for each $i \geq 0$ the random variable ρ_i is bounded; therefore $\{\rho_i\}$ is a martingale, and the money market account process $\{B_i\}$ satisfies the conditions of Axioms A and B*.

Foreign exchange

Consider now the Doob decomposition for $\{\pi_i\}$. Evidently, we have

$\pi_i = \mathbb{E}_i[A_\infty] - A_i$, with

$$\begin{aligned}
 A_i &= \sum_{n=0}^{i-1} (\pi_n - \mathbb{E}_n[\pi_{n+1}]) \\
 &= \sum_{n=0}^{i-1} \pi_n \left(1 - \frac{\mathbb{E}_n[\pi_{n+1}]}{\pi_n} \right) \\
 &= \sum_{n=0}^{i-1} \pi_n (1 - P_{n,n+1}) \\
 &= \sum_{n=0}^{i-1} \pi_n r_{n+1} P_{n,n+1},
 \end{aligned} \tag{25}$$

where $\{r_i\}$ is the previsible short rate process.

The pricing kernel can therefore be put in the form

$$\pi_i = \mathbb{E}_i \left[\sum_{n=i}^{\infty} \pi_n r_{n+1} P_{n,n+1} \right]. \quad (26)$$

Comparing the Doob decomposition with the decomposition $\pi_i = \mathbb{E}_i[G_\infty] - G_i$, where $G_i = \sum_{n=1}^i \pi_n \bar{r}_n$ given by Proposition 3, we deduce that by setting

$$\bar{r}_i = \frac{r_i \pi_{i-1} P_{i-1,i}}{\pi_i} \quad (27)$$

we obtain a positive-return asset based on the Doob decomposition.

On the other hand, since the money-market account is a positive-return asset, by Proposition 3 we can also write

$$\pi_i = \mathbb{E}_i \left[\sum_{n=i+1}^{\infty} \pi_n r_n \right]. \quad (28)$$

As a consequence, we see that the price process of a pure income-generating asset can be written in the following symmetrical form:

$$S_i = \frac{\mathbb{E}_i \left[\sum_{n=i+1}^{\infty} \pi_n D_n \right]}{\mathbb{E}_i \left[\sum_{n=i+1}^{\infty} \pi_n r_n \right]}. \quad (29)$$

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