Indifference pricing of a life insurance portfolio with systematic mortality risk in a market with an asset driven by a Lévy process

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Probability space

$$(\Omega, \mathcal{F}, \mathbb{P}) \quad \mathbb{F} = (\mathcal{F}_t)_{0\leq t\leq T}$$

- the filtration $\mathbb{F}$ satisfies the usual hypotheses of completeness and right continuity,
- $\mathbb{F} = \mathbb{F}^F \vee \mathbb{F}^M \vee \mathbb{F}^N$,
- the filtrations $\mathbb{F}^F$ and $(\mathbb{F}^M, \mathbb{F}^N)$ are independent.

Financial market

$$\frac{dB(t)}{B(t)} = r dt, \quad B(0) = 1,$$

$$\frac{dS(t)}{S(t-)} = \mu dt + \xi dL(t), \quad S(0) = 1,$$

where $(L(t), 0 \leq t \leq T)$ denotes zero-mean Lévy process, $\mathbb{F}^F$-adapted with càdlàg sample paths, which satisfies the Lévy-Itô decomposition

$$L(t) = \sigma W(t) + \int_{(0,t]} \int_{\mathbb{R}} z\big(M(ds \times dz) - \nu(dz)ds\big).$$

Financial market

- $(W(t), 0 \leq t \leq T)$ Brownian motion,
- $M((s,t] \times A) = \#\{s < u \leq t : (L(u) - L(u-)) \in A\}$
  Poisson random measure,
  $0 \leq s < t \leq T, A \in \mathcal{B}\{\mathbb{R} - \{0\}\}$,
- $\tilde{M}((s,t] \times A) = M((s,t] \times A) - \nu(A)(t-s)$
  martingale-valued measure,
  $0 \leq s < t \leq T, A \in \mathcal{B}\{\mathbb{R} - \{0\}\}$.

Financial market

- r, μ, σ are non-negative constants, $\mu > r$,
- $\xi = 1, \qquad \xi L(t)$ is a Lévy process,
- ν is a Lévy measure on $(-1, \infty)$,
  $\nu(\{0\}) = 0, \qquad \int_{z \geq 1} z^2 \nu(dz) < \infty.$

$$S(t) = \exp\left\{\mu_E t + \sigma W(t) + \int_{(0,t]} \int_{\mathbb{R}} z \tilde{M}_E(ds \times dz)\right\}$$

- ν_E is a Lévy measure on $\mathbb{R}$,
  $\nu(\{0\}) = 0, \qquad \int_{z \geq 1} e^{2z} \nu_E(dz) < \infty.$

Mortality intensity process

$$d\lambda(t) = \theta(t, \lambda(t))dt + \eta(t, \lambda(t))d\bar{W}(t), \quad \lambda(0) = \lambda,$$

where $(\bar{W}(t), 0 \le t \le T)$ denotes $\mathbb{F}^M$-adapted Brownian motion.

- $\theta : [0, T] \times (0, \infty) \to \mathbb{R}, \eta : [0, T] \times (0, \infty) \to (0, \infty)$ are continuous functions, locally Lipschitz continuous in λ, uniformly in t,
- there exists a sequence $(E_n)_{n \in \mathbb{N}}$ of bounded sets with $\bar{E}_n \subseteq (0, \infty)$ and $\bigcup_{n \ge 1} E_n = (0, \infty)$, such that the functions θ and η are uniformly Lipschitz continuous on $[0, T] \times \bar{E}_n$,
- $\mathbb{P}(\forall_{s \in [t,T]} \lambda(s) \in (0, \infty) | \lambda(t) = \lambda) = 1$ and $\sup_{s \in [t,T]} \mathbb{E}[|\lambda(s)|^2 | \lambda(t) = \lambda] < \infty$ for all starting points $(t, \lambda) \in [0, T] \times (0, \infty)$.

Life insurance portfolio

- the portfolio consists of the same life insurance policies, issued at time 0 to a group of n_0 persons,
- each policyholder is entitled to three types of payments,
- there are amounts payable continuously at the rate c, as long as an insured person is alive, but no longer than T years,
- there is a benefit payable immediately at the moment of the death of an insured person, in the amount of b, within T years,
- there are two types of endowments: a certain, initial premium $B(0)$ and a terminal, survival benefit in the amount of $B(T)$, payable provided that an insured person is still alive at time T.

Future-life times

- $T_1, T_2, ..., T_{n_0}$ future life-times of insured persons,
- the random variables $T_1, T_2, ..., T_{n_0}$ are identically distributed with the survival function

$$\mathbb{P}(T_i > t | \mathcal{F}_t^M) = e^{-\int_0^t \lambda(s)ds}, \quad i = 1, 2, ..., n_0,$$

- the censored life-times $\big((T_1 \wedge T, \mathbf{1}\{T_1 \leq T\}), ..., (T_{n_0} \wedge T, \mathbf{1}\{T_{n_0} \leq T\})\big)$ are $\mathbb{F}^N$-measurable,
- τ_i the moment of $n_0 - i$th death, $i = 0, 1, .., n_0 - 1, \tau_{n_0} = 0$.

Liability

- the counting process $(N(t), 0 \leq t \leq T)$

$$N(t) = n_0 - \sum_{i=1}^{n_0} \mathbf{1}\{T_i \leq t\},$$

- the payment process $(P(t), 0 \leq t \leq T)$

$$\begin{aligned} dP(t) &= N(t-)cdt - bdN(t) + N(T)B(T)d\mathbf{1}\{t \geq T\}, \\ P(0) &= -n_0 B(0). \end{aligned}$$

Literature

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Wealth process

$$
\begin{aligned}
dX^\pi(t) &= \pi(t)\Big(\mu dt + \sigma dW(t) + \int_{z>-1} z\tilde{M}(dt\times dz)\Big) \\
&\quad + \Big(X^\pi(t-) - \pi(t)\Big) r dt - dP(t), \\
X(0) &= x - P(0).
\end{aligned}
$$

- premiums and benefits are specified at the issue of the contract,
- π is a control variable.

Wealth process

$$
\begin{aligned}
dX_n^{\pi_n}(t) &= \pi_n(t)\Big(\mu dt + \sigma dW(t) + \int_{z>-1} z\tilde{M}(dt \times dz)\Big) \\
&\quad + \Big(X_n^{\pi_n}(t-) - \pi_n(t)\Big) r dt - ncdt,
\end{aligned}
$$

$$
X^\pi(t) = \begin{cases}
x + n_0 B(0), & t = 0 \\
X_n^{\tau_n, X^\pi(\tau_n), \pi_n}(t), & \tau_n < t < \tau_{n-1} \wedge T, \tau_n < T, \\
X_n^{\tau_n, X^\pi(\tau_n), \pi_n}(\tau_{n-1}) - b, & t = \tau_{n-1}, \tau_{n-1} < T, \\
X_n^{\tau_n, X^\pi(\tau_n), \pi_n}(T) - b - (n-1)B(T), & t = T, \tau_{n-1} = T, \\
X_n^{\tau_n, X^\pi(\tau_n), \pi_n}(T) - nB(T), & t = T, \tau_{n-1} > T, \\
X_0^{\tau_0, X^\pi(\tau_0), \pi_0}(t), & \tau_0 < t \le T, \tau_0 < T.
\end{cases}
$$

Discounted wealth process

$$
\begin{aligned}
dY_n^{\pi_n}(t) \;&=\; e^{-\rho t}\pi_n(t)\Big((\mu - r)ds + \sigma dW(t) + \int_{z>-1} z\tilde{M}(dt \times dz)\Big) \\
&\quad + Y_n^{\pi_n}(t-)(r-\rho)dt - nce^{-\rho t}dt,
\end{aligned}
$$

$$
Y^{\pi}(t) = \begin{cases}
y + n_0 B(0), & t = 0 \\
Y_n^{\tau_n, Y^{\pi}(\tau_n), \pi_n}(t), & \tau_n < t < \tau_{n-1} \wedge T, \\
Y_n^{\tau_n, Y^{\pi}(\tau_n), \pi_n}(\tau_{n-1}) - be^{-\rho \tau_{n-1}}, & t = \tau_{n-1}, \tau_{n-1} < T \\
Y_n^{\tau_n, Y^{\pi}(\tau_n), \pi_n}(T) - be^{-\rho T} - (n-1)B(T)e^{-\rho T}, & t = T, \tau_{n-1} = T, \\
Y_n^{\tau_n, Y^{\pi}(\tau_n), \pi_n}(T) - nB(T)e^{-\rho T}, & t = T, \tau_{n-1} > T \\
Y_0^{\tau_0, Y^{\pi}(\tau_0), \pi_0}(t), & \tau_0 < t \leq T, \tau_0 < T
\end{cases}
$$

Indifference arguments

$$\sup_\pi \mathbb{E}\big[u(X^\pi(T))|X(0) = x + n_0 B(0), \lambda(0) = \lambda, N(0) = n_0\big]$$
$$= \sup_{\pi_0} \mathbb{E}\big[u(X_0^{\pi_0}(T))|X_0(0) = x\big]$$

$$\sup_\pi \mathbb{E}\big[u(Y^\pi(\tau_0 \wedge T))|Y(0) = y + n_0 B(0), \lambda(0) = \lambda, N(0) = n_0\big]$$
$$= u(y)$$

$$\sup_\pi \mathbb{E}\big[u(Y^\pi(\tau_0 \wedge T))|Y(0) = y + n_0 B(0), \lambda(0) = \lambda, N(0) = n_0\big]$$
$$\sup_{\pi_0} \mathbb{E}\big[u(Y_0^{\pi_0}(\tau_0 \wedge T))|Y(0) = y, \lambda(0) = \lambda\big]$$

Value function

$$V_n(t,x,\lambda) = \sup_{\pi \in \mathcal{A}} \mathbb{E}\big[u(X^\pi(T)) | X(t) = x, \lambda(t) = \lambda, N(t) = n\big]$$

Lemma 1. *For* $n = 1, 2, ..., n_0$ *and all* $(t, x, \lambda) \in [0, T] \times \mathbb{R} \times (0, \infty)$ *the value function* $V_n(t, x, \lambda)$ *has the representation:*

$$\begin{aligned} V_n(t,x,\lambda) &= \sup_{\pi_n \in \mathcal{A}} \mathbb{E}^{t,x,\lambda,n}\big[u(X_n^{\pi_n}(T) - nB(T))\mathbf{1}\{\tau_{n-1} > T\} \\ &\quad + V_{n-1}(\tau_{n-1}, X_n^{\pi_n}(\tau_{n-1}) - b)\mathbf{1}\{\tau_{n-1} \le T\}\big], \end{aligned}$$

provided that the value function V_{n-1} *is well-defined.*

Optimal investment and consumption in the presence of default

$$\sup_{(c,\pi)} \mathbb{E}\Big[\int_0^T \mathbf{1}\{\tau \geq s\} u(c(s))ds \quad + \quad \alpha u(X^{c,\pi}(\tau))\mathbf{1}\{\tau \leq T\}$$
$$\quad + \quad \beta u(X^{c,\pi}(T))\mathbf{1}\{\tau > T\}\Big]$$

- Blanchet-Scalliet, C., El Karoui, N., Jeanblanc, M., Martellini, L., 2003, Optimal investment and consumption decisions when time-horizon is uncertain, *preprint*,
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Admissible strategies

Definition 1. *The sequence of controls* $(\pi_n(t), 0 < t \leq T)$ *is admissible,* $\pi_n \in \mathcal{A}$*, if it satisfies the following conditions:*

1. π_n *is a Markov control,*
2. *the mapping* $t \mapsto \pi_n(t, \omega)$ *is a.s. left continuous and has right limits,*
3. *the stochastic differential equation has a unique solution* X^{π_n} *on* $[0, T]$*,*

for all $n = 0, 1, ..., n_0$.

Operators

Definition 2. *Let $\mathcal{L}_{F,\rho}$ denote the integro-differential operator given by*

$$\begin{aligned}
\mathcal{L}^{\pi}_{F,\rho} h(t,x) &= \left(e^{-\rho t}\pi(\mu - r) + x(r-\rho) - nce^{-\rho t}\right)\frac{\partial h}{\partial x}(t,x) \\
&\quad + \frac{1}{2} e^{-2\rho t}\pi^2\sigma^2 \frac{\partial^2 h}{\partial x^2}(t,x) \\
&\quad + \int_{z>-1} \left(h(t, x + e^{-\rho t}\pi z) - \phi(t,x) - e^{-\rho t}\pi z \frac{\partial h}{\partial x}(t,x)\right)\nu(dz),
\end{aligned}$$

and let $\mathcal{L}_M$ denote the differential operator given by

$$\mathcal{L}_M h(t,\lambda) = \theta(t,\lambda)\frac{\partial h}{\partial \lambda}(t,\lambda) + \frac{1}{2}\eta^2(t,\lambda)\frac{\partial^2 h}{\partial \lambda^2}(t,\lambda).$$

Verification theorem

Theorem 1. *Assume that v_{n-1} is a candidate function which coincides with the optimal value function V_{n-1}, such that*

$$\mathbb{E}\left[\int_0^T \left|v_{n-1}(t, X_n^{\pi_n}(t), \lambda(t))\right|^2 dt\right] < \infty,$$

for all $\pi_n \in \mathcal{A}$. Let $v_n \in \mathcal{C}^{1,2,2}([0,T) \times \mathbb{R} \times (0,\infty)) \cap \mathcal{C}([0,T] \times \mathbb{R} \times (0,\infty))$ satisfies for all $\pi_n \in \mathcal{A}$

$$\begin{aligned} 0 &\geq \frac{\partial v_n}{\partial t}(t,x,\lambda) + \mathcal{L}_F^{\pi_n} v_n(t,x,\lambda) + \mathcal{L}_M v_n(t,x,\lambda) \\ &\quad + n\lambda\big(v_{n-1}(t,x-b,\lambda) - v_n(t,x,\lambda)\big), \quad (t,x,\lambda) \in [0,T) \times \mathbb{R} \times (0,\infty), \\ v_n(T,x,\lambda) &= u(x - nb(T)), \quad (x,\lambda) \in \mathbb{R} \times (0,\infty). \end{aligned}$$

Assume also that for all $\pi_n \in \mathcal{A}$

$$\begin{aligned} \int_0^T \int_{z \geq 1} \left|v_n(t, X_n^{\pi_n}(t) + \pi_n(t)z, \lambda(t))\right|^2 \nu(dz)dt &< \infty, \quad \mathbb{P}-\textit{a.s.}, \\ \mathbb{E}\left[\int_0^T \left|v_n(t, X_n^{\pi_n}(t), \lambda(t))\right|^2 dt\right] &< \infty, \end{aligned}$$

Verification theorem

$\{v_n^-(\tau, X_n^{\pi_n}(\tau), \lambda(\tau))\}_{0<\tau\leq T}$ *is uniformly integrable for all* $\mathbb{F}$*-stopping times* τ.

Then

$$v_n(t,x,\lambda) \geq V_n(t,x,\lambda), \quad (t,x,\lambda) \in [0,T] \times \mathbb{R} \times (0,\infty).$$

If additionally there exists an admissible control $\hat{\pi}_n \in \mathcal{A}$ *such that*

$$\begin{aligned} 0 \quad &= \quad \frac{\partial v_n}{\partial t}(t,x,\lambda) + \mathcal{L}_F^{\hat{\pi}_n} v_n(t,x,\lambda) + \mathcal{L}_M v_n(t,x,\lambda) \\ &\quad +n\lambda\big(v_{n-1}(t,x-b,\lambda) - v_n(t,x,\lambda)\big), \quad (t,x,\lambda) \in [0,T) \times \mathbb{R} \times (0,\infty), \end{aligned}$$

and

$\{v_n(\tau, X_n^{\hat{\pi}_n}(\tau), \lambda(\tau))\}_{0<\tau\leq T}$ *is uniformly integrable for all* $\mathbb{F}$*-stopping times* τ,

then

$$v_n(t,x,\lambda) = V_n(t,x,\lambda), \quad (t,x,\lambda) \in [0,T] \times \mathbb{R} \times (0,\infty).$$

Optimal value function (problems I and II)

$$W_n(t,x,\lambda) = \sup_{\pi\in\mathcal{A}} \mathbb{E}\Big[u(Y^\pi(\tau_0 \wedge T))|Y(t)=y, \lambda(t)=\lambda, N(t)=n\Big]$$

$$\begin{cases} 0 = \frac{\partial w_n}{\partial t}(t,y,\lambda) + \sup_{\pi_n\in\mathbb{R}}\{\mathcal{L}_{F,\rho}^{\pi_n} w_n(t,y,\lambda)\} + \mathcal{L}_M w_n(t,y,\lambda) \\ \qquad +n\lambda\Big(w_{n-1}(t,y-be^{-\rho t},\lambda) - w_n(t,y,\lambda)\Big) \\ w_n(T,y,\lambda) = u(y - nB(T)e^{-\rho T}) \end{cases}$$

Exponential utility

Lemma 2. *Consider the Hamilton-Jacobi-Bellman equation:*

$$0 = \frac{\partial v_0}{\partial t}(t,x) + \sup_{\pi_0 \in \mathbb{R}} \{\mathcal{L}_F^{\pi_0} v_0(t,x)\}, \quad (t,x) \in [0,T) \times \mathbb{R},$$
$$v_0(T,x) = -\frac{1}{\alpha} e^{-\alpha x}, \quad x \in \mathbb{R}.$$

The function v_0 *defined as*

$$v_0(t,x) = -\frac{1}{\alpha} e^{-\alpha f(t) x + g(t)}$$

satisfies the above equation in the classical sense, with $f(t) = e^{r(T-t)}$ *and* $g(t) = G(\hat{\kappa})(T-t)$, *where* $\hat{\kappa}$ *is the unique minimizer of the convex function*

$$G(\kappa) = -\kappa(\mu - r) + \frac{1}{2}\kappa^2\sigma^2 + \int_{z>-1} (e^{-\kappa z} - 1 + \kappa z)\nu(dz).$$

The optimal investment strategy is $\hat{\pi}_0(t) = \frac{\hat{\kappa}}{\alpha f(t)}$.

Pricing with respect to terminal time T

$$v_n(t,x,\lambda) = v_0(t,x)\phi_n(t,\lambda), \quad (t,x,\lambda) \in [0,T] \times \mathbb{R} \times (0,\infty)$$

$$\hat{\pi}_n(t) = \frac{\hat{\kappa}}{\alpha} e^{-r(T-t)}, \quad n = 0, 1, ..., n_0$$

$$\begin{cases} 0 = \frac{\partial \phi_n}{\partial t}(t,\lambda) + \mathcal{L}_M \phi_n(t,\lambda) + (\alpha f(t) nc - n\lambda)\phi_n(t,\lambda) \\ \qquad\quad + n\lambda e^{\alpha f(t) b} \phi_{n-1}(t,\lambda) \\ \phi_n(T,\lambda) = e^{\alpha n B(T)} \end{cases}$$

Pricing with respect to terminal time T

Lemma 3. *Assume that* $\phi_{n-1}(t,\lambda) \in \mathcal{C}^{1,2}([0,T) \times (0,\infty))$ $\cap \mathcal{C}_b([0,T] \times (0,\infty))$. *Then the equation has the unique solution* $\phi_n(t,\lambda) \in \mathcal{C}^{1,2}([0,T) \times (0,\infty)) \cap \mathcal{C}_b([0,T] \times (0,\infty))$. *The following probabilistic representation holds:*

$$\begin{aligned}\phi_n(t,\lambda) &= \mathbb{E}^{t,\lambda}\Big[e^{\alpha n B(T)} e^{\int_t^T (\alpha f(s) nc - n\lambda(s))ds} \\ &\quad + \int_t^T e^{\alpha f(s) b} n\lambda(s)\phi_{n-1}(s,\lambda(s)) e^{\int_t^s (\alpha f(u) nc - n\lambda(u))du} ds\Big].\end{aligned}$$

Pricing with respect to terminal time T

Lemma 4. *The function* $\phi_{n_0}(0,\lambda)$ *can be represented as*

$$\phi_{n_0}(0,\lambda) = \mathbb{E}^{0,\lambda}\Big[\exp\Big(\alpha\int_{(0,T]} e^{r(T-t)}dP(t)\Big)\Big].$$

Pricing equation (method I):

$$n_0 B(0) = \frac{1}{\alpha}e^{-rT}\log\mathbb{E}^{0,\lambda}\Big[\exp\Big(\alpha\int_{(0,T]} e^{r(T-t)}dP(t)\Big)\Big]$$

Pricing with respect to random time

$$w_n(t,y,\lambda) = u(y)\varphi_n(t,\lambda), \quad (t,y,\lambda) \in [0,T] \times \mathbb{R} \times (0,\infty)$$

$$\tilde{\pi}_n(t) = \frac{\hat{\kappa}}{\alpha} e^{rt}, \quad n = 1, 2, ..., n_0$$

$$\begin{cases} 0 = \frac{\partial \varphi_n}{\partial t}(t,\lambda) + \mathcal{L}_M \varphi_n(t,\lambda) + (G(\hat{\kappa}) + \alpha n c e^{-rt} - n\lambda)\varphi_n(t,\lambda) \\ \qquad +n\lambda e^{\alpha b e^{-rt}} \varphi_{n-1}(t,\lambda) \\ \varphi_n(T,\lambda) = e^{\alpha n B(T) e^{-rT}} \end{cases}$$

Pricing with respect to random date

Pricing equation (method II):

$$n_0 B(0) = \frac{1}{\alpha} \log \mathbb{E}^{0,\lambda}\Big[\exp\Big(G(\hat{\kappa})(\tau_0 \wedge T) + \alpha \int_{(0,T]} e^{-rt} dP(t) \Big) \Big]$$

Pricing equation (method III):

$$n_0 B(0) \;\; = \;\; \frac{1}{\alpha} \log \mathbb{E}^{\mathbb{P}^G}\Big[\exp\Big(\alpha \int_{(0,T]} e^{-rt} dP(t) \Big) \Big]$$

$$\frac{d\mathbb{P}^G}{d\mathbb{P}} = \frac{e^{G(\hat{\kappa})(\tau_0 \wedge T)}}{\mathbb{E}^{0,\lambda}\left[e^{G(\hat{\kappa})(\tau_0 \wedge T)} \right]}$$

Properties of the premiums

Lemma 5. *Assume that the individual premium $B(0)$ is set according to (I). Then*

1. $B^\alpha(0)$ *is (strictly) increasing in* α,
2. $\lim_{\alpha\to 0} B^\alpha(0) = \mathbb{E}\big[\int_0^{T_i\wedge T} ce^{-rt}dt + be^{-rT_i}\mathbf{1}\{T_i \leq T\} + B(T)e^{-rT}\mathbf{1}\{T_i > T\}\big]$,
3. $\lim_{\alpha\to\infty} B^\alpha(0) = \mathbb{P}$*-ess sup* $\{\int_0^{T_i\wedge T} ce^{-rt}dt + be^{-rT_i}\mathbf{1}\{T_i \leq T\} + B(T)e^{-rT}\mathbf{1}\{T_i > T\}]\}$,
4. *if* $c^1 \leq c^2, b^1 \leq b^2, B^1(T) \leq B^2(T)$ *then* $B^1(0) \leq B^2(0)$.

Properties of the premiums

If the premium is set according to (II) or (II), then the points 1,3,4 hold as well. For the premium (II) we have

2'. $\lim_{\alpha \to 0} B^\alpha(0) = -\infty,$

while for the premium (III) we have

2". $\lim_{\alpha \to 0} B^\alpha(0) = \mathbb{E}^{\mathbb{P}^G}\Big[\int_0^{T_i \wedge T} ce^{-rt}dt + be^{-rT_i}\mathbf{1}\{T_i \leq T\} + B(T)e^{-rT}\mathbf{1}\{T_i > T\}\Big].$

Ruin probability

Lemma 6. *Assume that $r = 0$ and that the premium is calculated according to (I). The ruin probability decreases to zero exponentially and the following inequality holds*

$$\mathbb{P}\Big(\inf_{t\in[0,T]} X^{\hat{\pi}}(t) < 0 | X(0) = x + n_0 B(0)\Big) \leq e^{-\beta x + \beta n_0 \left(B^{\beta}(0) - B^{\alpha}(0)\right)},$$

where $\beta > \alpha$ is the unique solution of the equation

$$G(\beta\frac{\hat{\kappa}}{\alpha}) = 0.$$

Thank you for your attention.

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