Indifference pricing of a life insurance portfolio with systematic mortality risk in a market with an asset driven by a Lévy process

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Probability space

\[(\Omega, \mathcal{F}, \mathbb{P}) \quad \mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T}\]

- the filtration \(\mathbb{F}\) satisfies the usual hypotheses of completeness and right continuity,
- \(\mathbb{F} = \mathbb{F}^F \vee \mathbb{F}^M \vee \mathbb{F}^N\),
- the filtrations \(\mathbb{F}^F\) and \((\mathbb{F}^M, \mathbb{F}^N)\) are independent.
Financial market

\[
\frac{dB(t)}{B(t)} = r \, dt, \quad B(0) = 1,
\]

\[
\frac{dS(t)}{S(t-)} = \mu dt + \xi dL(t), \quad S(0) = 1,
\]

where \((L(t), 0 \leq t \leq T)\) denotes zero-mean Lévy process, \(\mathbb{F}^F\)-adapted with càdlàg sample paths, which satisfies the Lévy-Itô decomposition

\[
L(t) = \sigma W(t) + \int_{(0,t]} \int_{\mathbb{R}} z \left( M(ds \times dz) - \nu(dz)ds \right).
\]
\( (W(t), 0 \leq t \leq T) \) Brownian motion,

\[
M((s, t] \times A) = \#\{s < u \leq t : (L(u) - L(u-)) \in A\}
\]
Poisson random measure,

\[
0 \leq s < t \leq T, A \in \mathcal{B}\{\mathbb{R} - \{0\}\},
\]

\[
\tilde{M}((s, t] \times A) = M((s, t] \times A) - \nu(A)(t - s)
\]
martingale-valued measure,

\[
0 \leq s < t \leq T, A \in \mathcal{B}\{\mathbb{R} - \{0\}\}.
\]
$r, \mu, \sigma$ are non-negative constants, $\mu > r$,

$\xi = 1$, $\xi L(t)$ is a Lévy process,

$\nu$ is a Lévy measure on $(-1, \infty)$, $\nu\{0\} = 0$, $\int_{z \geq 1} z^2 \nu(dz) < \infty$.

$S(t) = \exp\{\mu_E t + \sigma W(t) + \int_{(0,t]} \int_{\mathbb{R}} z \tilde{M}_E(ds \times dz)\}$

$\nu_E$ is a Lévy measure on $\mathbb{R}$,

$\nu\{0\} = 0$, $\int_{z \geq 1} e^{2z} \nu_E(dz) < \infty$. 
Mortality intensity process

\[ d\lambda(t) = \theta(t, \lambda(t))dt + \eta(t, \lambda(t))d\bar{W}(t), \quad \lambda(0) = \lambda, \]

where \((\bar{W}(t), 0 \leq t \leq T)\) denotes \(\mathbb{F}^M\)-adapted Brownian motion.

- \(\theta : [0, T] \times (0, \infty) \to \mathbb{R}, \eta : [0, T] \times (0, \infty) \to (0, \infty)\) are continuous functions, locally Lipschitz continuous in \(\lambda\), uniformly in \(t\),

- there exists a sequence \((E_n)_{n \in \mathbb{N}}\) of bounded sets with \(\bar{E}_n \subseteq (0, \infty)\) and \(\bigcup_{n \geq 1} E_n = (0, \infty)\), such that the functions \(\theta\) and \(\eta\) are uniformly Lipschitz continuous on \([0, T] \times \bar{E}_n\),

- \(\mathbb{P}(\forall s \in [t, T] \lambda(s) \in (0, \infty) | \lambda(t) = \lambda) = 1\) and \(\sup_{s \in [t, T]} \mathbb{E}[|\lambda(s)|^2 | \lambda(t) = \lambda] < \infty\) for all starting points \((t, \lambda) \in [0, T] \times (0, \infty)\).
Life insurance portfolio

- the portfolio consists of the same life insurance policies, issued at time 0 to a group of $n_0$ persons,
- each policyholder is entitled to three types of payments,
- there are amounts payable continuously at the rate $c$, as long as an insured person is alive, but no longer than $T$ years,
- there is a benefit payable immediately at the moment of the death of an insured person, in the amount of $b$, within $T$ years,
- there are two types of endowments: a certain, initial premium $B(0)$ and a terminal, survival benefit in the amount of $B(T)$, payable provided that an insured person is still alive at time $T$. 
Future-life times

- $T_1, T_2, \ldots, T_{n_0}$ future life-times of insured persons,
- the random variables $T_1, T_2, \ldots, T_{n_0}$ are identically distributed with the survival function
  \[ P(T_i > t | \mathcal{F}_t^M) = e^{-\int_0^t \lambda(s)ds}, \quad i = 1, 2, \ldots, n_0, \]
- the censored life-times
  \( ((T_1 \wedge T, 1\{T_1 \leq T\}), \ldots, (T_{n_0} \wedge T, 1\{T_{n_0} \leq T\})) \)
  are $\mathcal{F}^N$-measurable,
- $\tau_i$ the moment of $n_0 - i$th death, $i = 0, 1, \ldots, n_0 - 1, \tau_{n_0} = 0.$
the counting process \((N(t), 0 \leq t \leq T)\)

\[
N(t) = n_0 - \sum_{i=1}^{n_0} 1\{T_i \leq t\},
\]

the payment process \((P(t), 0 \leq t \leq T)\)

\[
dP(t) = N(t-)cdt - bdN(t) + N(T)B(T)d1\{t \geq T\},
\]

\[P(0) = -n_0B(0).\]
Literature


Wealth process

\[
    dX^\pi(t) = \pi(t)(\mu dt + \sigma dW(t) + \int_{z>-1} z \tilde{M}(dt \times dz))
    + (X^\pi(t^+) - \pi(t))rdt - dP(t),
\]

\[
    X(0) = x - P(0).
\]

- premiums and benefits are specified at the issue of the contract,
- \(\pi\) is a control variable.
Wealth process

\[ dX_n^\pi(t) = \pi_n(t)(\mu dt + \sigma dW(t) + \int_{z > -1} z \tilde{M}(dt \times dz)) \]
\[ + (X_n^\pi(t-) - \pi_n(t)) r dt - n c dt, \]

\[ X^\pi(t) = \begin{cases} 
  x + n_0 B(0), & t = 0 \\
  X_{\tau_n}^\pi(\tau_n), \pi_n(t), & \tau_n < t < \tau_{n-1} \wedge T, \tau_n < T, \\
  X_{\tau_n}^\pi(\tau_n), \pi_n(\tau_{n-1}) - b, & t = \tau_{n-1}, \tau_{n-1} < T, \\
  X_{\tau_n}^\pi(\tau_n), \pi_n(T) - b - (n - 1) B(T), & t = T, \tau_{n-1} = T, \\
  X_{\tau_n}^\pi(\tau_n), \pi_n(T) - n B(T), & t = T, \tau_{n-1} > T, \\
  X_0^\pi(\tau_0), \pi_0(t), & \tau_0 < t \leq T, \tau_0 < T. 
\end{cases} \]
Discounted wealth process

\[ dY_{\pi n}(t) = e^{-\rho t}\pi_n(t)((\mu - r)ds + \sigma dW(t) + \int_{z > -1} z\tilde{M}(dt \times dz)) + Y_{\pi n}(t-) (r - \rho)dt - nce^{-\rho t} dt, \]

\[
Y_{\pi n}(t) = \begin{cases} 
y + n_0 B(0), & t = 0 \\
y_{\tau_n, Y_{\pi n}(\tau_n), \pi_n}(t), & \tau_n < t < \tau_{n-1} \land T, \\
y_{\tau_n, Y_{\pi n}(\tau_n), \pi_n}(\tau_{n-1}) - be^{-\rho \tau_{n-1}}, & t = \tau_{n-1}, \tau_{n-1} < T, \\
y_{\tau_n, Y_{\pi n}(\tau_n), \pi_n}(T) - be^{-\rho T} - (n - 1)B(T)e^{-\rho T}, & t = T, \tau_{n-1} = T, \\
y_{\tau_n, Y_{\pi n}(\tau_n), \pi_n}(T) - nB(T)e^{-\rho T}, & t = T, \tau_{n-1} > T, \\
y_{\tau_0, Y_{\pi n}(\tau_0), \pi_0}(t), & \tau_0 < t \leq T, \tau_0 < T. \end{cases}
\]
Indifference arguments

\[
\begin{align*}
\sup_{\pi} \mathbb{E}\left[ u(X^\pi(T)) \middle| X(0) = x + n_0 B(0), \lambda(0) = \lambda, N(0) = n_0 \right] &= \sup_{\pi_0} \mathbb{E}\left[ u(X_0^{\pi_0}(T)) \middle| X_0(0) = x \right] \\
\sup_{\pi} \mathbb{E}\left[ u(Y^\pi(\tau_0 \wedge T)) \middle| Y(0) = y + n_0 B(0), \lambda(0) = \lambda, N(0) = n_0 \right] &= u(y) \\\n\sup_{\pi} \mathbb{E}\left[ u(Y^\pi(\tau_0 \wedge T)) \middle| Y(0) = y + n_0 B(0), \lambda(0) = \lambda, N(0) = n_0 \right] &= \sup_{\pi_0} \mathbb{E}\left[ u(Y_0^{\pi_0}(\tau_0 \wedge T)) \middle| Y(0) = y, \lambda(0) = \lambda \right]
\end{align*}
\]
Value function

\[ V_n(t, x, \lambda) = \sup_{\pi \in \mathcal{A}} \mathbb{E}[u(X^\pi(T))|X(t) = x, \lambda(t) = \lambda, N(t) = n] \]

**Lemma 1.** For \( n = 1, 2, ..., n_0 \) and all \((t, x, \lambda) \in [0, T] \times \mathbb{R} \times (0, \infty)\) the value function \( V_n(t, x, \lambda) \) has the representation:

\[
V_n(t, x, \lambda) = \sup_{\pi_n \in \mathcal{A}} \mathbb{E}^{t,x,\lambda,n}[u(X_n^{\pi_n}(T) - nB(T))1\{\tau_{n-1} > T}\]
\[
+ V_{n-1}(\tau_{n-1}, X_n^{\pi_n}(\tau_{n-1}) - b)1\{\tau_{n-1} \leq T}\],

provided that the value function \( V_{n-1} \) is well-defined.
Optimal investment and consumption in the presence of default

\[
\sup_{(c,\pi)} \mathbb{E} \left[ \int_0^T 1\{\tau \geq s\} u(c(s)) \, ds + \alpha u(X^{c,\pi}(\tau)) 1\{\tau \leq T\} + \beta u(X^{c,\pi}(T)) 1\{\tau > T\} \right]
\]


Definition 1. The sequence of controls \((\pi_n(t), 0 < t \leq T)\) is admissible, \(\pi_n \in \mathcal{A}\), if it satisfies the following conditions:

1. \(\pi_n\) is a Markov control,

2. the mapping \(t \mapsto \pi_n(t, \omega)\) is a.s. left continuous and has right limits,

3. the stochastic differential equation has a unique solution \(X^{\pi_n}\) on \([0, T]\),

for all \(n = 0, 1, \ldots, n_0\).
**Definition 2.** Let $\mathcal{L}_{F,\rho}$ denote the integro-differential operator given by

$$
\mathcal{L}_{F,\rho}^{\pi} h(t, x) = \left( e^{-\rho t} \pi (\mu - r) + x(r - \rho) - nce^{-\rho t} \right) \frac{\partial h}{\partial x}(t, x) + \frac{1}{2} e^{-2\rho t} \pi^2 \sigma^2 \frac{\partial^2 h}{\partial x^2}(t, x) + \int_{z>-1} (h(t, x + e^{-\rho t} \pi z) - \phi(t, x) - e^{-\rho t} \pi z \frac{\partial h}{\partial x}(t, x)) \nu(dz),
$$

and let $\mathcal{L}_M$ denote the differential operator given by

$$
\mathcal{L}_M h(t, \lambda) = \theta(t, \lambda) \frac{\partial h}{\partial \lambda}(t, \lambda) + \frac{1}{2} \eta^2(t, \lambda) \frac{\partial^2 h}{\partial \lambda^2}(t, \lambda).
$$
Verification theorem

**Theorem 1.** Assume that \( v_{n-1} \) is a candidate function which coincides with the optimal value function \( V_{n-1} \), such that

\[
\mathbb{E} \left[ \int_0^T \left| v_{n-1}(t, X_{n}^\pi(t), \lambda(t)) \right|^2 dt \right] < \infty,
\]

for all \( \pi_n \in \mathcal{A} \). Let \( v_n \in C^{1,2,2}([0, T] \times \mathbb{R} \times (0, \infty)) \cap C([0, T] \times \mathbb{R} \times (0, \infty)) \) satisfies for all \( \pi_n \in \mathcal{A} \)

\[
0 \geq \frac{\partial v_n}{\partial t}(t, x, \lambda) + \mathcal{L}_{\pi}^{\pi_n} v_n(t, x, \lambda) + \mathcal{L}_M v_n(t, x, \lambda) \\
+ n\lambda(v_{n-1}(t, x-b, \lambda) - v_n(t, x, \lambda)), \quad (t, x, \lambda) \in [0, T] \times \mathbb{R} \times (0, \infty),
\]

\[
v_n(T, x, \lambda) = u(x - nb(T)), \quad (x, \lambda) \in \mathbb{R} \times (0, \infty).
\]

Assume also that for all \( \pi_n \in \mathcal{A} \)

\[
\int_0^T \int_{z \geq 1} \left| v_n(t, X_{n}^\pi(t) + \pi_n(t)z, \lambda(t)) \right|^2 \nu(dz) dt < \infty, \quad \mathbb{P} - a.s.,
\]

\[
\mathbb{E} \left[ \int_0^T \left| v_n(t, X_{n}^\pi(t), \lambda(t)) \right|^2 dt \right] < \infty,
\]
Verification theorem

\{v_n^-(\tau, X_n^\pi_n(\tau), \lambda(\tau))}\}_{0<\tau\leq T} \text{ is uniformly integrable for all } \mathbb{F}\text{-stopping times } \tau.

Then

\[ v_n(t, x, \lambda) \geq V_n(t, x, \lambda), \quad (t, x, \lambda) \in [0, T] \times \mathbb{R} \times (0, \infty). \]

If additionally there exists an admissible control \( \hat{\pi}_n \in A \) such that

\[ 0 = \frac{\partial v_n}{\partial t}(t, x, \lambda) + \mathcal{L}_F^{\hat{\pi}_n} v_n(t, x, \lambda) + \mathcal{L}_M v_n(t, x, \lambda) \]

\[ + n\lambda \left( v_{n-1}(t, x - b, \lambda) - v_n(t, x, \lambda) \right), \quad (t, x, \lambda) \in [0, T') \times \mathbb{R} \times (0, \infty), \]

and

\{v_n(\tau, X_n^{\hat{\pi}_n}(\tau), \lambda(\tau))\}_{0<\tau\leq T} \text{ is uniformly integrable for all } \mathbb{F}\text{-stopping times } \tau,

then

\[ v_n(t, x, \lambda) = V_n(t, x, \lambda), \quad (t, x, \lambda) \in [0, T] \times \mathbb{R} \times (0, \infty). \]
Optimal value function (problems I and II)

\[ W_n(t, x, \lambda) = \sup_{\pi \in \mathcal{A}} \mathbb{E}\left[u(Y^\pi(\tau_0 \wedge T)) | Y(t) = y, \lambda(t) = \lambda, N(t) = n \right] \]

\[
\begin{cases}
0 = \frac{\partial w_n}{\partial t}(t, y, \lambda) + \sup_{\pi_n \in \mathbb{R}} \left\{\mathcal{L}^{\pi_n}_{F, \rho} w_n(t, y, \lambda) \right\} + \mathcal{L}_M w_n(t, y, \lambda) \\
+ n\lambda \left( w_{n-1}(t, y - be^{-\rho t}, \lambda) - w_n(t, y, \lambda) \right) \\
w_n(T, y, \lambda) = u(y - nB(T)e^{-\rho T})
\end{cases}
\]
Lemma 2. Consider the Hamilton-Jacobi-Bellman equation:

\[ 0 = \frac{\partial v_0}{\partial t}(t, x) + \sup_{\pi_0 \in \mathbb{R}} \{ \mathcal{L}_{F_0}^0 v_0(t, x) \}, \quad (t, x) \in [0, T) \times \mathbb{R}, \]

\[ v_0(T, x) = -\frac{1}{\alpha} e^{-\alpha x}, \quad x \in \mathbb{R}. \]

The function \( v_0 \) defined as

\[ v_0(t, x) = -\frac{1}{\alpha} e^{-\alpha f(t)x + g(t)} \]

satisfies the above equation in the classical sense, with \( f(t) = e^{r(T-t)} \) and \( g(t) = G(\hat{\kappa})(T - t) \), where \( \hat{\kappa} \) is the unique minimizer of the convex function

\[ G(\kappa) = -\kappa(\mu - r) + \frac{1}{2}\kappa^2 \sigma^2 + \int_{z > -1} (e^{-\kappa z} - 1 + \kappa z) \nu(dz). \]

The optimal investment strategy is \( \hat{\pi}_0(t) = \frac{\hat{\kappa}}{\alpha f(t)} \).
Pricing with respect to terminal time $T$

$$v_n(t, x, \lambda) = v_0(t, x)\phi_n(t, \lambda), \quad (t, x, \lambda) \in [0, T] \times \mathbb{R} \times (0, \infty)$$

$$\hat{\pi}_n(t) = \frac{\hat{\kappa}}{\alpha} e^{-r(T-t)}, \quad n = 0, 1, \ldots, n_0$$

$$0 = \frac{\partial \phi_n}{\partial t}(t, \lambda) + \mathcal{L}_M \phi_n(t, \lambda) + (\alpha f(t)nc - n\lambda)\phi_n(t, \lambda)$$

$$+ n\lambda e^{\alpha f(t)b} \phi_{n-1}(t, \lambda)$$

$$\phi_n(T, \lambda) = e^{\alpha nB(T)}$$
Lemma 3. Assume that $\phi_{n-1}(t, \lambda) \in C^{1,2}([0, T) \times (0, \infty)) \cap C_b([0, T] \times (0, \infty))$. Then the equation has the unique solution $\phi_n(t, \lambda) \in C^{1,2}([0, T) \times (0, \infty)) \cap C_b([0, T] \times (0, \infty))$. The following probabilistic representation holds:

$$
\phi_n(t, \lambda) = E^{t, \lambda} \left[ e^{\alpha_n B(T)} e^{\int_t^T (\alpha f(s) nc - n\lambda(s)) ds} \right]
+ \int_t^T e^{\alpha f(s)b} n\lambda(s) \phi_{n-1}(s, \lambda(s)) e^{\int_t^s (\alpha f(u) nc - n\lambda(u)) du} ds.
$$
Lemma 4. The function $\phi_{n_0}(0, \lambda)$ can be represented as

$$\phi_{n_0}(0, \lambda) = \mathbb{E}^{0,\lambda}\left[ \exp\left( \alpha \int_{(0,T]} e^{r(T-t)}dP(t) \right) \right].$$

Pricing equation (method I):

$$n_0B(0) = \frac{1}{\alpha} e^{-rT} \log \mathbb{E}^{0,\lambda}\left[ \exp\left( \alpha \int_{(0,T]} e^{r(T-t)}dP(t) \right) \right].$$
Pricing with respect to random time

\[ w_n(t, y, \lambda) = u(y)\varphi_n(t, \lambda), \quad (t, y, \lambda) \in [0, T] \times \mathbb{R} \times (0, \infty) \]

\[ \tilde{\pi}_n(t) = \frac{\hat{\kappa}}{\alpha} e^{rt}, \quad n = 1, 2, ..., n_0 \]

\[
\begin{cases}
0 = \frac{\partial \varphi_n}{\partial t}(t, \lambda) + \mathcal{L}_M \varphi_n(t, \lambda) + (G(\hat{\kappa}) + \alpha n e^{-rt} - n\lambda)\varphi_n(t, \lambda) \\
+ n\lambda e^{\alpha be^{-rt}} \varphi_{n-1}(t, \lambda) \\
\varphi_n(T, \lambda) = e^{\alpha n B(T)e^{-rT}}
\end{cases}
\]
Pricing with respect to random date

\[
\frac{1}{\alpha} \log \mathbb{E}^{0,\lambda} \left[ \exp \left( G(\hat{\kappa})(\tau_0 \wedge T) + \alpha \int_{(0,T]} e^{-rt} dP(t) \right) \right]
\]

\[
\frac{d\mathbb{P}^G}{d\mathbb{P}} = \frac{e^{G(\hat{\kappa})(\tau_0 \wedge T)}}{\mathbb{E}^{0,\lambda} \left[ e^{G(\hat{\kappa})(\tau_0 \wedge T)} \right]}
\]
Lemma 5. Assume that the individual premium $B(0)$ is set according to (I). Then

1. $B^\alpha(0)$ is (strictly) increasing in $\alpha$,

2. $\lim_{\alpha \to 0} B^\alpha(0) = \mathbb{E} \left[ \int_0^{T_i \wedge T} ce^{-rt} dt + be^{-rT_i} \mathbf{1}\{T_i \leq T\} + B(T)e^{-rT} \mathbf{1}\{T_i > T\} \right]$,

3. $\lim_{\alpha \to \infty} B^\alpha(0) = \mathbb{P}\text{-ess sup} \ \left\{ \int_0^{T_i \wedge T} ce^{-rt} dt + be^{-rT_i} \mathbf{1}\{T_i \leq T\} + B(T)e^{-rT} \mathbf{1}\{T_i > T\} \right\}$,

4. if $c^1 \leq c^2$, $b^1 \leq b^2$, $B^1(T) \leq B^2(T)$ then $B^1(0) \leq B^2(0)$.
Properties of the premiums

If the premium is set according to (II) or (III), then the points 1, 3, 4 hold as well. For the premium (II) we have

2'. \( \lim_{\alpha \to 0} B^\alpha(0) = -\infty \),

while for the premium (III) we have

2''. \( \lim_{\alpha \to 0} B^\alpha(0) = \mathbb{E}^{\text{PG}} \left[ \int_0^{T_i \wedge T} ce^{-rt} \, dt + be^{-rT_i} 1\{T_i \leq T\} + B(T)e^{-rT} 1\{T_i > T\} \right] \).
Lemma 6. Assume that $r = 0$ and that the premium is calculated according to (I). The ruin probability decreases to zero exponentially and the following inequality holds

$$
\mathbb{P}(\inf_{t \in [0,T]} X^\hat{\pi}(t) < 0 | X(0) = x + n_0 B(0)) \leq e^{-\beta x + \beta n_0 (B^\beta(0) - B^\alpha(0))},
$$

where $\beta > \alpha$ is the unique solution of the equation

$$
G(\beta \frac{\hat{\kappa}}{\alpha}) = 0.
$$
Thank you for your attention.

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