Weak and Strong Taylor methods for approximative solutions of SDEs

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Outline of the Talk

- Description of the problem and the method;
- A motivating example;
- The general setting: basic Definitions and Theorems;
- Applications and examples to Interest Rate Theory: the LIBOR market model, the frozen drift and models with stochastic volatility.

Motivation and Basic Ideas

- European-option prices with underlying assets driven by parameter-dependent equations;
- Analytically tractable formulas available *only* for particular parameter values;
- **Idea**: Taylor expansions of expectation functionals around these known values;
- **Tools**: Malliavin calculus and *integration-by-parts* on the Wiener space when distribution is unknown;
- **Results**: Strong and weak Taylor expansions of option prices; Tractable pricing formulas on parameter intervals.

The Model and the Problem

Design SDEs smoothly dependent on a parameter $a \in \mathbb{R}$:

$$dX_t^{x,a} = V(X_t^{x,a})dt + \sum_{i=1}^{d} V_i(X_t^{x,a})dW_t^i, \; X_0^{x,a} = x \in \mathbb{R}^N. \quad (1)$$

For $a = 0$, model reduces to a well-known one, e.g. Black and Scholes.

For $a \neq 0$ there might be no analytically tractable formulas.

Efficiently approximate the expectation functional, f expresses payoff-profile:

$$u^a(T,x) := \mathbb{E}(f(X_T^{x,a})).$$

The Remedy - Strong Taylor

- Appropriate conditions on the vector fields to ensure existence of densities for the law of $X_t^{x,a}$ around $a = 0$;

- Approximate drift or volatility terms by strong Taylor expansions of $X_t^{x,a}$;

  For instance, the first-order strong Taylor of $X_T^{x,a}$ is given by:

$$X_T^{x,a} = X_T^{x,0} + \frac{\partial}{\partial a}\Big|_{a=0} X_T^{x,a}.$$

The Remedy - Weak Taylor

- Approximate $u^a(T,x)$ by a finite-order weak Taylor expansion around $u^0(T,x)$;
- Use integration-by-parts on the Wiener space to calculate:

$$\frac{\partial^n}{\partial a^n}\Big|_{a=0} u^a(T,x) = \mathbb{E}(f(X_T^{x,0})\pi_n);$$

$\Rightarrow$ tractable formulas for $\mathbb{E}(f(X_T^{x,a}))$ including Malliavin weights π_n.

An application of a first-order Taylor expansion around $u^0(T,x)$ yields:

$$u^a(T,x) = u^0(T,x) + a\frac{\partial}{\partial a}\Big|_{a=0} u^a(T,x) + o(a). \tag{2}$$

By the integration-by-parts, we shall prove existence of weights:

$$\frac{\partial}{\partial a}\Big|_{a=0} u^a(T,x) = \mathbb{E}(f(X_T^{x,0})\pi).$$

Thus, the first-order weak Taylor expansion looks like:

$$\mathbb{E}(f(X_T^{x,a})) = \mathbb{E}(f(X_T^{x,0})) + a\mathbb{E}(f(X_T^{x,0})\pi).$$

Advantages of the Method

- Explicit knowledge of the distribution of $X_T^{x,a}$ is not necessary;
- There is no dependence on higher derivatives of f;
- The weights are *global*, i.e. *independent* of the payoff function f;
- The process $X_t^{x,0}$ and the weight π are easier to simulate than $X_t^{x,a}$ for $a \neq 0$; the simulation of the weight can be done together with that of the underlying;

$\Rightarrow$ especially suitable for high-dimensional systems, where full numerical schemes can be very time consuming.

A Simplified Motivating Example

Toy-example of a one-factor stochastic volatility model:

$$dS_t^a = (\sigma + aV_t^a)S_t^a dW_t,\ S_0^a = s,$$
$$dV_t^a = (k - bV_t^a)a dt + a\varsigma dW_t,\ V_0^a = v.$$

For $a = 0$ the model is reduced to the Black and Scholes (BS) model with zero interest rates:

$$dS_t^0 = \sigma S_t^0 dW_t \Rightarrow S_t^0 = s\exp\left(\sigma W_t - \frac{1}{2}\sigma^2 t\right) \text{ with } V_t^0 = v.$$

Consider pricing a call with payoff $C_T^a = (S_T^a - K)_+$, thus we want to calculate $\mathbb{E}(C_T^a)$. By (2), we need to calculate the a-sensitivity at

$a = 0$, in the market referred to as volga(0):

$$\frac{\partial}{\partial a}\Big|_{a=0}\mathbb{E}(C_T^a) = \mathbb{E}\Big((C_T^a)'\big|_{a=0}\frac{\gamma(S_T^0)}{\gamma(S_T^0)}\frac{\partial}{\partial a}\Big|_{a=0}S_T^a\Big) =$$

$$= \mathbb{E}\Big(\int_0^T \Big[(C_T^0)' D_t S_T^0\Big]\Big[\frac{D_t S_T^0}{\gamma(S_T^0)}\frac{\partial}{\partial a}\Big|_{a=0}S_T^a\Big]dt\Big) =$$

$$= \mathbb{E}\Big(C_T^0 \delta\Big(\frac{D_t S_T^0 \frac{\partial}{\partial a}|_{a=0} S_T^a}{\gamma(S_T^0)}\Big)\Big) = \mathbb{E}(C_T^0 \pi).$$

For the calculation of the Skorohod integral π we need $Z_T := \frac{\partial}{\partial a}|_a S_T^a$, $D_t S_T^0$ and $\gamma(S_T^0)$. Simple calculus gives for $t \leq T$:

$$D_t S_T^0 = \sigma S_T^0, \gamma(S_T^0) = \sigma^2 (S_T^0)^2 T,$$

$$dZ_t = \sigma Z_t dW_t + V_t^0 S_t^0 dW_t, Z_0 = 0 \Rightarrow$$

$$Z_T = S_T^0(-\sigma v T + v W_T).$$

Thus, π is given by:

$$\pi = \delta\Big(\frac{1}{\sigma T}(-\sigma v T + c W_T)\Big) = -v W_T + \frac{v W_T^2}{\sigma T} - \frac{v}{\sigma},$$

since $\delta(W_T) = W_T^2 - T$. Therefore, the approximative price of a European call with strike K is given by:

$$C_0^a \simeq \underbrace{s\mathrm{N}(d_1) - K\mathrm{N}(d_2)} + a\,\underbrace{\mathbb{E}\Big((S_T^0 - K)_+\pi\Big)}.$$

- More generally, an option price can be expressed as:

$$P^a \simeq P^0 + a \times \text{ volga}(0).$$

⇒ approximate option prices by analytical formulas when:
  * distribution of underlying unknown;
  * there is no analytical expression for the option price for $a \neq 0$.

Strong and Weak Taylor Approximations

Let $a \mapsto F_a$ be a curve, where $a \in \mathbb{R}$ and $F_a \in L^2(\Omega)$ is a smooth curve of random variables.

Definition 1. *A strong Taylor approximation of order $n \geq 0$ is a power series defined by:*

$$\mathbf{T}_a^n(F_a) := \sum_{i=0}^{n} \frac{a^i}{i!} \frac{\partial^i}{\partial a^i}\Big|_{a=0} F_a,$$

such that:

$$\mathbb{E}\big(|F_a - \mathbf{T}_a^n(F_a)|\big) = o(a^n),$$

holds true for $a \to 0$.

Definition 2. *A weak Taylor approximation of order* $n \geq 0$ *is a power series, for each bounded measurable* $f : \mathbb{R}^N \to \mathbb{R}$ *defined by:*

$$\mathbf{W}_a^n(f, F_a) := \sum_{i=0}^{n} \frac{a^i}{i!} \mathbb{E}(f(F_0)\pi_i), \tag{3}$$

where $\pi_i \in L^1(\Omega)$ *denote real valued, integrable random variables, such that:*

$$|\mathbb{E}\big(f(F_a)\big) - \mathbf{W}_a^n(f, F_a)| = o(a^n).$$

The random variable π_i*, for* $i \geq 1$*, is called the* i^{th}*-order Malliavin weight.*

Existence of Weights Theorem...

Theorem 3. *Let F_a be smooth and assume that the Malliavin covariance matrix $\gamma(F_a)$ is invertible in an open interval around $a = 0$. Then there is a weak Taylor approximation of any order $n \geq 0$.*

Idea of the proof. We can prove the formula:

$$\frac{d}{da}\mathbb{E}(f(F_a)) = \mathbb{E}\Big(f(F_a)\delta\Big(\big(\frac{dF_a}{da}\big)^{\mathrm{T}}(\gamma^{-1}(F_a))^{\mathrm{T}}D_tF_a\Big)\Big). \tag{4}$$

Make the following notation:

$$\pi_1 := \delta\Big(\big(\frac{dF_a}{da}\big)^{\mathrm{T}}(\gamma^{-1}(F_a))^{\mathrm{T}}D_tF_a\Big). \tag{5}$$

The general result for the n^{th}-order weight is obtained by recursion of (4) and differentiation of the Skorohod integral:

$$v_t := (\frac{dF_a}{da})^{\mathrm{T}}(\gamma^{-1}(F_a))^{\mathrm{T}} D_t F_a, \text{ for } 0 \le t \le T,$$
$$\pi_n := \delta(v_t \pi_{n-1}) + \frac{d}{da}\pi_{n-1}, \; \pi_0 := 1.$$

□

Remark 4. If $\gamma(F_a)$ invertible only for $a = 0$, the Malliavin weights are still calculable, since they depend on $\gamma(F_0)$; thus assumption of invertibility only necessary at $a = 0$.

... and Option Prices

Theorem 5. *Assume that $d \leq N$ and that the vector fields in* (1) *written in Stratonovich formulation satisfy condition (H) at $a = 0$. Then the price of an option $\mathbb{E}(f(X_T^{x,a}))$ is given by the first-order weak Taylor approximation:*

$$\mathbf{W}_a^1(f, X_T^{x,a}) \simeq \mathbb{E}(f(X_T^{x,0})) + a\mathbb{E}(f(X_T^{x,0})\pi_1).$$

Idea of the proof. From Theorem 3, there exists a first-order weak Taylor approximation, where the first-order weight π_1 is given as in (5).

By (3), the result is obtained directly. □

The LIBOR Market Model

Discrete tenor structure $0 = T_0 < T_1 < \ldots < T_N < T_{N+1} =: T$, $\alpha := T_{i+1} - T_i$.

$P(t, T_i)$: price at t of a zero coupon bond with maturity T_i.

$L_t(T_i, T_{i+1})$: forward LIBOR rate at t for the period $[T_i, T_{i+1}]$:

$$L_t^i := L_t(T_i, T_{i+1}) = \frac{1}{\alpha}\left(\frac{P(t, T_i)}{P(t, T_{i+1})} - 1\right).$$

$\mathbb{P}^{T_i}$: T_i-forward measure for the bond $P(t, T_i)$ as numéraire.

The model under the terminal measure $\mathbb{P}^T$, $i = 1, ..., N$:

$$dL_t^i = \sigma^i(t) L_t^i \left(- \sum_{j=i+1}^{N} \frac{\alpha L_t^j \sigma^j(t)}{1 + \alpha L_t^j} \rho_{ij} \right) dt + \sigma^i(t) L_t^i dW_t^i.$$

Correcting the Frozen Drift - Strong Taylor

Approximate the random or real drift term by its starting value or frozen drift:

$$\frac{\alpha L_t^j}{1+\alpha L_t^j} \approx \frac{\alpha L_0^i}{1+\alpha L_0^i} \Rightarrow \text{difference in option prices.}$$

Idea: replace the real drift by its first-order strong Taylor approximation:

$$d\hat{L}_t^{(i,\epsilon_1)} = \sigma^i(t)\hat{L}_t^{(i,\epsilon_1)}\Big(-\sum_{j=i+1}^{N} \frac{\alpha\Big(\mathbf{T}^1_{\epsilon_1}(X_t^{(j,\epsilon_1)})\Big)_+ \sigma^j(t)}{1+\alpha\Big(\mathbf{T}^1_{\epsilon_1}(X_t^{(j,\epsilon_1)})\Big)_+}\rho_{ij}dt + dW_t^i\Big),$$

where $\epsilon_1 \in \mathbb{R}$, $dX_t^{(i,\epsilon_1)} = \epsilon_1 dL_t^{(i,\epsilon_1)}$ with $L_0^{(i,\epsilon_1)} = X_0^{(i,\epsilon_1)}$ $\forall i$ and $\forall \epsilon_1$,

$$\mathbf{T}^1_{\epsilon_1}(X_t^{(i,\epsilon_1)}) = X_0^{(i,0)} + \epsilon_1 \frac{\partial}{\partial \epsilon_1}\Big|_{\epsilon_1=0} X_t^{(i,\epsilon_1)}.$$

Example 6. *Let $N = 3$, price a caplet on the LIBOR rate L^1.*

payoff $= \alpha\big(\big(L^1_{T_1} - K\big)_+\big)$,

volatility functions $\sigma^i(t) = \big(a(T_i - t) + d\big) \exp\big(- b(T_i - t)\big) + e$,

correlations $\rho_{ij} = 0.49 + (1 - 0.49) \exp(-0.13|i - j|), i, j = 1, 2, 3$.

strikes	*K=3%*	*K=3.5%*	*K=4%*	*K=5.75%*	*K=6.25%*
price	*11.1831*	*8.5897*	*6.5503*	*3.0349*	*2.4423*
strong Taylor	*11.0687*	*8.5691*	*6.5867*	*3.1448*	*2.5513*
frozen drift	*13.9551*	*11.1822*	*8.8803*	*4.6313*	*3.8506*

Table 1: Caplet values in bps for parameters $\epsilon_1 = 1$, $\alpha = 0.50137$, $c_1 = 3.86777\%$, $c_2 = 3.7574\%$, $c_3 = 3.8631\%$ and $T_1 = 1.53151$ and constants $a = -0.113035$, $b = 0.22911$, $d = -a$, $e = 0.684784$.

$\Rightarrow$ strong Taylor method:

- pathwise approximation;
- performs very well; computationally simpler and faster.

Correcting Frozen Drift Prices - Weak Taylor

Idea: approximate LIBOR option prices by first-order weak Taylor approximation.

Example 7. *Let* $N = 3$, *price a payers swaption,* $\mathbf{L}_{T_1} := (L^1_{T_1}, L^2_{T_1})$.

- *Assume existence of* $\gamma^{-1}(\mathbf{L}^0_{T_1}) \iff \rho_{12} \neq 1$ *if* $\rho_{23} = 1$*;*

$\Rightarrow$ *invertibility of* $\gamma(\mathbf{L}^0_{T_1})$ *natural assumption.*

payoff $g(\mathbf{L}_{T_1}) = \big(- \sum_{k=1}^{2} \alpha_k \prod_{j=k}^{2} (1 + \alpha L^j_{T_1}) - (1 + K\alpha)\big)_+$,

volatility functions $\sigma^i(t) := \sigma_i, i = 1, 2, 3$,

weak Taylor price $= \mathbf{W}^1_a(g, \mathbf{L}^{\epsilon_1}_{T_1}) = \mathbb{E}\big(g(\mathbf{L}^0_{T_1})\big) + \epsilon_1 \mathbb{E}\big(g(\mathbf{L}^0_{T_1})\zeta_{T_1}\big)$,

Malliavin weight $\zeta_{T_1} = \delta\Big((D_t \mathbf{L}^0_{T_1})^{\mathrm{T}} \gamma^{-1}(\mathbf{L}^0_{T_1}) \frac{\partial}{\partial \epsilon_1}\big|_{\epsilon_1 = 0} \mathbf{L}^{\epsilon_1}_{T_1}\Big) = \zeta^1_{T_1} + \zeta^2_{T_2}$.

$$\zeta^1_{T_1} = \rho_{12}\Big(W^1_{T_1}\Big(\frac{\sigma_3\alpha c_3\beta_2 T_1}{2(1+\alpha c_3)} - \frac{(\beta_2+\beta_3)}{T_1}\int_0^{T_1} W^2_t dt\Big) + \frac{\rho_{12}(\beta_2+\beta_3)T_1}{2}\Big) -$$

$$- \rho_{12}\Big(\frac{\rho_{12}\beta_3 T_1}{2} - \frac{\beta_3 W^1_{T_1}}{T_1}\int_0^{T_1} W^2_t dt\Big), \beta_i := \frac{\alpha c_i^2\sigma_i^2}{(1+\alpha c_i)^2}, i = 2, 3.$$

$\zeta^2_{T_1}$ *has similar form, weight* ζ_{T_1} *function of normal variables;*

$\Rightarrow$ no Monte-Carlo required!

strikes	*K=4%*	*K=4.5%*	*K=4.75%*	*K=5%*	*K=5.15%*
price	*10.2240*	*6.5386*	*4.7454*	*3.1060*	*2.2599*
strong Taylor	*10.2240*	*6.5386*	*4.7454*	*3.1060*	*2.2599*
frozen drift	*10.2132*	*6.5326*	*4.7419*	*3.1028*	*2.2582*
weak Taylor	*10.2266*	*6.5407*	*4.7485*	*3.1064*	*2.2593*

Table 2: Swaption values in bps for $\epsilon_1 = 1$, $\alpha = 0.25$, $\sigma_1 = 18\%$, $\sigma_2 = 15\%$, $\sigma_3 = 12\%$, $c_0 = 5.28875\%$, $c_1 = 5.37375\%$, $c_2 = 5.40\%$, $c_3 = 5.40125\%$ and $\rho_{12} = 0.75$.

$\Rightarrow$ achieve pricing for multi-LIBOR options under terminal measure; approximate option price by a deterministic formula.

The Stochastic Volatility LIBOR Market Model

$$dL_t^{(i,\epsilon_1,\epsilon_2)} = \sigma^i(t) L_t^{(i,\epsilon_1,\epsilon_2)} \sqrt{v_t^{\epsilon_2}} \Big(- \sum_{j=i+1}^{N} \frac{\alpha X_t^{(j,\epsilon_1,\epsilon_2)} \sigma^j(t)}{1 + \alpha X_t^{(j,\epsilon_1,\epsilon_2)}} \rho_{ij} \sqrt{v_t^{\epsilon_2}} dt + dW_t^i \Big),$$

$$dv_t^{\epsilon_2} = \kappa(\theta - v_t^{\epsilon_2}) dt + \epsilon_2 \sqrt{v_t^{\epsilon_2}} \big(\rho_i dW_t^i + \sqrt{1 - \rho_i^2} dZ_t^i \big).$$

$\Rightarrow$ no analytic tractability, use the same idea as previously.

Example 8. *Let $N = 2$, price a payers swaption, payoff and volatility functions as before.*

- *Assume invertibility of $\gamma(\mathbf{L}_{T_1}^{0,0}) \iff \rho_{12} \neq 1 \Rightarrow$ natural condition.*

weak Taylor price $=$

$$\mathbf{W}_a^1(g, \mathbf{L}_{T_1}^{\epsilon_1,\epsilon_2}) = \mathbb{E}\big(g(\mathbf{L}_{T_1}^{0,0})\big) + \epsilon_1 \mathbb{E}\big(g(\mathbf{L}_{T_1}^{0,0}) \zeta_{T_1}\big) + \epsilon_2 \mathbb{E}\big(g(\mathbf{L}_{T_1}^{0,0}) \pi_{T_1}\big),$$

Malliavin weights

$$\zeta_{T_1} = \zeta_{T_1}^1 + \zeta_{T_1}^2, \;\; \pi_{T_1} = \pi_{T_1}^1 + \pi_{T_1}^2$$

$\Rightarrow \zeta_{T_1}^1 = -\frac{\beta_2 \rho_{12}}{c}\Big(X_1 Y - \mathbb{C}ov(X_1, Y)\Big), \zeta_{T_1}^2$ *similar*, $\pi_{T_1}^1 = \pi_{T_1}^{1,1} - \pi_{T_1}^{1,2}$,
$\pi_{T_1}^{1,2} = \frac{\rho_{12}}{2c}\Big(X_1(\rho_2 D_2 + \sqrt{1-\rho_2^2} Z_2) + \frac{\sigma_1 B X_1}{\kappa} + \frac{\rho_{12} B}{\kappa} - \frac{\sigma_1 \rho_1 e}{\kappa}\Big)$
with X_i, Y, B *normal variables,* D_i, Z_i *double stochastic integrals.*

strikes	*K=3.5%*	*K=4%*	*K=5%*	*K=6%*	*K=7%*
price	*3.8984*	*2.9221*	*1.2588*	*0.3858*	*0.1019*
frozen drift and vol	*3.8951*	*2.9053*	*1.2705*	*0.3966*	*0.0942*
weak Taylor	*3.8990*	*2.9159*	*1.2694*	*0.3791*	*0.1042*

Table 3: SV-swaption values in bps for $\epsilon_1 = 1$, $\alpha = 1.5$, $\sigma_1 = 25\%$, $\sigma_2 = 15\%$, $c_0 = 5.28875\%$, $c_1 = 5.4\%$, $c_2 = 5.39\%$, $v_0 = 1$, $\rho_1 = -0.75$, $\rho_2 = -0.6$, $\kappa = 2.3767$, $\theta = 0.2143$, $\epsilon_2 = 25\%$, $\rho_{12} = 0.63$.

$\Rightarrow$ end up with approximative semi-deterministic pricing formula; weights given by simple Itô integrals!

Concluding remarks

- Strong Taylor method:
  - ∗ very good approximation of the price;
  - ∗ the drift part only depends on the Brownian path ⇒ complexity of real drift reduced ⇒ easier and faster to simulate.
  - ∗ could be used with *any extension* of the log normal LMM, e.g. Lévy LIBOR.
- Weak Taylor method:
  - ∗ nice results close to the price;
  - ∗ weights quickly evaluated;
    - ◇ universal, i.e. same for all payoff functions;
    - ◇ tractable, i.e. either deterministic integrals or when not, need no extra computation than that of the underlying.

Future research

- Higher-order strong and weak Taylor approximations;
- Greeks in the LMM and SVLMM;
- Models with jumps - Lévy LIBOR;
- Other stochastic volatility models or general multi-factor models; look for approximative closed-form formulas.

Talk Contents

- M. Siopacha, J. Teichmann (2007). Weak and strong Taylor methods for numerical solutions of stochastic differential equations, preprint, available at arXiv.org.

- M. Siopacha (2006). Taylor expansions of option prices by means of Malliavin calculus, Doctoral Thesis, Vienna University of Technology, Austria.

Thank you for your attention!