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# Risk Sensitive Benchmarked Asset Management

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**Agenda:**

- Introductory remarks on mathematics of asset management (AM).
- Risk-sensitive control in AM, with benchmark
- Mutual-fund theorems
- Lévy-driven models
- Conclusions and extensions

*Note:* The mathematical development in the first part of this paper follows very closely Kuroda and Nagai (Stochastics and Stochastics Reports 2002).

## Mathematics of asset management

See J. Campbell and L. Viceira *Long-term asset allocation* Oxford UP 2003

## Markowitz

- 1-period model
- Quantifies basic risk-return trade-off
- Huge impact on practical AM

**Stochastic control** (Merton)

- Dynamic theory
- Maximizes expected utility (wealth and/or consumption)
- *No* impact on practical AM, because
  - ** Questionable utility specification
  - ** Too dependent on stylized math model

**Basic problem with both approaches** (or indeed any approach):

Estimation of growth rates is impossible. Example

$$dZ_t = \mu\, dt + \sigma\, dW_t.$$

Assume $\sigma$ is known. Maximum likelihood estimate of $\mu$ is

$$\hat{\mu}_T = \frac{Z_T}{T} \quad \text{with error} \quad \hat{\mu}_T - \mu = \frac{\sigma W_T}{T}.$$

Time for reasonably accurate estimate of $\mu \;\approx$ 1500 years.

**"Solution":** must use some sort of factor approach or quantify investor 'views' (Black-Litterman)

In this paper we suppose $\mu = \mu(X_t)$ where $X_t$ is an observed vector process of factors. (If not directly observed then Kalman filtering ...)

## Risk-Sensitive Control

Control theory: Jacobson, Whittle, Bensoussan, Fleming,..
Asset Management: Bielecki-Pliska, Kuroda-Nagai, Peng-Nagai

Conventional control: $\max \mathbb{E}[F]$ for some performance function $F$.

Risk-sensitive control: maximize

$$-\frac{2}{\theta} \log \mathbb{E}\left[e^{-\frac{\theta}{2}F}\right] = \mathbb{E}[F] - \frac{\theta}{2}\mathrm{var}[F] + o(\theta).$$

Conventional control recovered as $\theta \to 0$.

In risk-sensitive asset management, $F$ is the log-return, i.e. $F = \log V$ where $V$ is portfolio value. Objective is then to maximize

$$-\frac{2}{\theta} \log \mathbb{E}\left[e^{-\frac{\theta}{2}\log V}\right] = -\frac{2}{\theta} \log \mathbb{E}\left[V^{-\theta/2}\right].$$

The optimization problem is then equivalent to maximizing power utility, but has an aspect of ‘risk-return trade-off’ à la Markowitz. As $\theta \to 0$ we revert to the growth-optimal portfolio.

## Analytical Setting

Let $(\Omega, \{\mathcal{F}_t\}, \mathcal{F}, \mathbb{P})$ be the underlying probability space. On this space is defined an $\mathbb{R}^N$-valued $(\mathcal{F}_t)$-Brownian motion $W(t)$ with components $W_k(t)$, $k = 1, \ldots, N$. All processes are defined on $(\Omega, \mathcal{F}, \mathbb{P})$ and are $\mathcal{F}_t$-adapted. Here $N = m + n$.

Growth rates of both assets and benchmark depend on an $n$-vector factor process $X(t)$. We assume that the factors are observable.

Money market account process pays continuously compounding interest

$$r(t) = \eta + \zeta' X(t).$$

Assets market comprises $m$ risky securities $S_i$, $i = 1, \ldots m$.

**Asset dynamics**

The dynamics of the money market account is given by:

$$\frac{dS_0(t)}{S_0(t)} = (\eta + \zeta' X(t))\, dt, \qquad S_0(0) = s_0$$

Dynamics of the $m$ risky securities and $n$ factors can be expressed as:

$$\frac{dS_i(t)}{S_i(t)} = (a + AX(t))_i dt + \sum_{k=1}^{N} \sigma_{ik} dW_k(t), \qquad S_i(0) = s_i, \quad i = 1, \dots, m$$

$$dX(t) = (b + BX(t))dt + \Lambda dW(t), \qquad X(0) = x$$

where $X(t)$ is the $\mathbb{R}^n$-valued factor process with components $X_j(t)$ and the market parameters $a$, $A$, $b$, $B$, $\Sigma := [\sigma_{ij}]$, $i = 1, \dots, m$, $j = 1, \dots, N$, $\Lambda := [\Lambda_{ij}]$, $i = 1, \dots, n$, $j = 1, \dots, N$ are matrices of appropriate dimensions.

**Assumption 1.** *The matrix* $\Sigma\Sigma'$ *is positive definite.*

Let $\mathcal{G}_t := \sigma((S(s), L(s), X(s)), 0 \le s \le t)$ be the sigma-field generated by the security, liability and factor processes up to time $t$.

The allocation of wealth among the assets is defined by an $\mathbb{R}^m$-valued stochastic process $h$, where the $i$th component $h_i(t)$ denotes the proportion of current wealth invested in the $i$th risky security at time $t$, $i = 1, \ldots, m$. The proportion invested in the money market account is $h_0(t) = 1 - \sum_{i=1}^{m} h_i(t)$.

**Definition 1.** *An investment process $h(t)$ is in class $\mathcal{H}$ if the following conditions are satisfied:*

1. *$h(t)$ is progressively measurable with respect to $\{\mathcal{B}([0,t]) \otimes \mathcal{G}_t\}_{t \geq 0}$;*
2. *$P\left(\int_0^T |h(s)|^2 \, ds < +\infty\right) = 1, \quad \forall T > 0.$*

The wealth, $V(t)$, of the asset only portfolio, in response to an investment strategy $h \in \mathcal{H}$, follows the dynamics

$$\frac{dV_t}{V_t} = (\eta + \zeta' X(t))\, dt + h'(t)\left(\hat{a} + \hat{A}X(t)\right) dt + h'(t)\Sigma dW_t$$

where $\hat{a} := a - \eta\mathbf{1}$ and $\hat{A} := A - \mathbf{1}\zeta'$.

## Benchmark Modelling

We assume that the benchmark evolves according to a similar SDE as the asset prices. Specifically,

$$\frac{dL(t)}{L(t)} = (\alpha + \beta' X(t))dt + \gamma' dW(t), \qquad L(0) = l$$

where $\alpha$ is a scalar constant, $\beta$ is a $n$-element column vector, and $\gamma$ is a $N$-element column vector.

Exposure to the short-term interest rate is included: this can easily be seen by expressing this equation as

$$\frac{dL(t)}{L(t)} = \left(\hat{\alpha} + \hat{\beta}' X(t) + \hat{\kappa}\left(\eta + \zeta' X(t)\right)\right) dt + \gamma' dW_k(t), \qquad L(0) = l$$

where $\hat{\alpha} := \alpha - \hat{\kappa}\eta$ and $\hat{\beta} := \beta - \hat{\kappa}\zeta$ for some scalar $\hat{\kappa}$ reflecting the exposure of the benchmark to the short-term interest rate, $r(t) = \eta + \zeta' X(t)$.

### Optimization Criterion

The objective of benchmarked investors is to maximize the risk adjusted growth of their assets relative to the benchmark. We will express this objective through an optimization criterion, representing the log excess return of the asset portfolio over its benchmark, $F(t)$, defined as:

$$F(t) = \ln \frac{V(t)}{L(t)} = \ln V(t) - \ln L(t)$$

By Itô, the log of excess return in response to a strategy $h$ is

$$\begin{aligned}
F(t) &= \ln \frac{v}{l} + \int_0^t d\ln V(s) - \int_0^t d\ln L(s) \\
&= \ln \frac{v}{l} + \int_0^t (\eta + \zeta' X(s)) + h(s)' \left( \hat{a} + \hat{A} X(s) \right) ds - \frac{1}{2} \int_0^t h(s)' \Sigma \Sigma' h(s) ds \\
&\quad + \int_0^t h(s)' \Sigma dW(s) - \int_0^t (\alpha + \beta' X(s)) ds + \frac{1}{2} \int_0^t \gamma' \gamma ds - \int_0^t \gamma' dW(s)
\end{aligned}$$

## Optimization criterion

The optimization criterion is risk-sensitive control on a finite horizon:

$\mathcal{P}_{\theta,T}$: for $\theta \in ]0,+\infty[$, maximize the risk sensitive expected log excess return over a time horizon $T$

$$J_{\theta,T}(v,x;h) := \left(\frac{-2}{\theta}\right) \ln \mathbf{E} e^{-\frac{\theta}{2}F(T;h)}$$

The class of admissible strategies for problem $\mathcal{P}_{\theta,T}$ is $\mathcal{A}(T) \subset \mathcal{H}$ defined below. A strategy $h \in \mathcal{H}$ is in $\mathcal{A}(T)$ if a technical condition related to the Girsanov theorem is satisfied.

**Derivation of the Bellman Equation**

*Criterion Under the Expectation*

Multiplying by $-\frac{\theta}{2}$ and taking the exponential in the expression for $F(t)$ we get

$$\begin{aligned} e^{-\frac{\theta}{2}F(t)} &= f_0^{-\frac{\theta}{2}} \exp\left\{ \frac{\theta}{2}\int_0^t g(X_s, h(s);\theta)ds - \frac{\theta}{2}\int_0^t \left(h(s)'\Sigma - \gamma'\right) dW_s \right. \\ &\left. -\frac{1}{2}\left(\frac{\theta}{2}\right)^2 \int_0^t (h(s)'\Sigma - \gamma')(h(s)'\Sigma - \gamma')'ds \right\} \end{aligned}$$

where $f_0 = \frac{v}{l}$ and

$$\begin{aligned} g(x, h;\theta) &= \frac{1}{2}\left(\frac{\theta}{2}+1\right) h'\Sigma\Sigma'h - \eta - \zeta'x - h'(\hat{a} + \hat{A}x) - \frac{1}{2}\frac{\theta}{2}(h'\Sigma\gamma + \gamma'\Sigma'h) \\ &+(\alpha + \beta'x) + \frac{1}{2}\left(\frac{\theta}{2} - 1\right)\gamma'\gamma \end{aligned}$$

*Change of Measure*

Let $\mathbb{P}_h^\theta$ be the measure on $(\Omega, \mathcal{F})$ defined by $d\mathbb{P}_h/d\mathbb{P}|_{\mathcal{F}_t} = \chi_t$ where

$$\chi_t = \exp\left\{-\frac{\theta}{2}\int_0^t (h(s)'\Sigma - \gamma')\,dW_s - \frac{1}{2}\left(\frac{\theta}{2}\right)^2\int_0^t (h(s)'\Sigma - \gamma')(h(s)'\Sigma - \gamma')'ds\right\}.$$

We denote by $\mathcal{A}(T)$ the set of investment strategies $h \in \mathcal{H}$ on $[0, T]$ such that $\mathbb{P}_h^\theta$ is a probability measure. For $h \in \mathcal{A}(T)$,

$$W_t^\theta = W_t + \frac{\theta}{2}\int_0^t (\Sigma' h(s) - \gamma)ds$$

is a standard Brownian motion under $\mathbb{P}_h^\theta$.

Under $\mathbb{P}_h^\theta$, $X_t$ satisfies the SDE:

$$dX_s = \left(b + BX_s - \frac{\theta}{2}\Lambda\left(\Sigma' h(s) - \gamma\right)\right) ds + \Lambda dW_s^\theta \tag{1}$$

and we introduce the auxiliary criterion function under the measure $\mathbb{P}_h^\theta$:

$$I(f_0, x; h; t, T) = \ln f_0 - \frac{2}{\theta} \ln \mathbf{E}^\theta \left[\exp\left\{\frac{\theta}{2}\int_0^{T-t} g(X_s, h(s); \theta) ds\right\}\right]$$

where $\mathbf{E}^\theta[\cdot]$ denotes the expectation taken with respect to measure $\mathbb{P}_h^\theta$.

*Key points*

- We have replaced the original portfolio optimization problem by a stochastic control problem in the factor process $X_t$.
- If $\Lambda\Sigma' = 0$, i.e. the factor and asset price 'noises' are uncorrelated, then (1) is an *uncontrolled* SDE and we can write down the solution of the optimization problem just by Feynman-Kac.

### The HJB Equation

Let $\Phi$ be the value function for the auxiliary criterion function $I(f_0, x; h; t, T)$. Then $\Phi$ is defined as

$$\Phi(t, x) = \sup_{\mathcal{A}(T-t)} I(f_0, x; h; t, T)$$

and it satisfies the HJB PDE

$$\frac{\partial \Phi}{\partial t} + \sup_{h \in \mathbb{R}^m} L_t^h \Phi = 0$$

where

$$L_t^h \Phi = \left( b + Bx - \frac{\theta}{2} \Lambda(\Sigma' h - \gamma) \right)' D\Phi + \frac{1}{2} \mathrm{tr} \left( \Lambda \Lambda' D^2 \Phi \right) - \frac{\theta}{4} (D\Phi)' \Lambda \Lambda' D\Phi - g(x, h; \theta)$$

The HJB equation has a solution in the form

$$\Phi(x, t) = \frac{1}{2} x' Q(t) x + q'(t) x + k(t)$$

with corresponding optimal investment optimal investment strategy

$$h^* = \frac{2}{\theta + 2} (\Sigma \Sigma')^{-1} \left( \hat{a} + \hat{A} x - \frac{\theta}{2} \Sigma \Lambda' D\Phi + \frac{\theta}{2} \Sigma \gamma \right)$$

$Q(t), q(t), k(t)$ are calculated as follows.

- $Q(t)$ satisfies the **Riccati equation**

$$\dot{Q}(t) - Q(t)K_0Q(t) + K_1'Q(t) + Q(t)K_1 + \frac{2}{\theta+2}\hat{A}'(\Sigma\Sigma')^{-1}\hat{A} = 0$$

  for $t \in [0, T]$, with *terminal* condition $Q(T) = 0$ and with

$$K_0 = \frac{\theta}{2}\left[\Lambda\left(I - \frac{\theta}{\theta+2}\Sigma'(\Sigma\Sigma')^{-1}\Sigma\right)\Lambda'\right]$$

$$K_1 = B - \frac{\theta}{\theta+2}\Lambda\Sigma'(\Sigma\Sigma')^{-1}\hat{A}$$

- $q(t)$ satisfies a **linear ordinary differential equation**

$$\dot{q}(t) + (K_1' - Q(t)K_0)\,q(t) + Q(t)b + \frac{\theta}{2}Q'(t)\Lambda\gamma + \zeta - \beta$$
$$+\frac{1}{\theta+2}\left(2\hat{A}' - \theta Q'(t)\Lambda\Sigma'\right)(\Sigma\Sigma')^{-1}\left(\hat{a} + \frac{\theta}{2}\Sigma\gamma\right) = 0$$

  with *terminal* condition $q(T) = 0$.

- $k(t) =$ explicit expression involving $Q(\cdot)$, $q(\cdot)$.

*Remarks:*

1. The Riccati equation used to determine the coefficient matrix of the quadratic term in the RSBAM and the asset only case are identical. (The equations for $q(t), k(t)$ contain terms involving the benchmark.)

2. The matrix $I - \frac{\theta}{\theta+2}\Sigma'(\Sigma\Sigma')^{-1}\Sigma$ appearing in the definition of $K_0$ (the quadratic coefficient in the Riccati equation) can be rewritten as:

$$\left(I - \Sigma'(\Sigma\Sigma')^{-1}\Sigma\right) + \frac{2}{\theta+2}\Sigma'(\Sigma\Sigma')^{-1}\Sigma$$

   Both these matrices are projection operators and hence non-negative definite. This is enough to guarantee the Riccati equation has a unique solution for all $t < T$.

**Theorem 1.** *The investment strategy* $h^*(t)$ *defined by*

$$h^*(t) = \frac{2}{\theta+2}(\Sigma\Sigma')^{-1}\left(\hat{a} + \frac{\theta}{2}\Sigma\gamma - \frac{\theta}{2}\Sigma\Lambda' q(t) + \left(\hat{A} - \frac{\theta}{2}\Sigma\Lambda' Q(t)\right) X_t\right) \tag{2}$$

*where* $Q$ *is the solution of the Riccati equation and* $q$ *is a solution of the linear ODE, belongs to* $\mathcal{A}(T)$ *and is optimal in* $\mathcal{A}(T)$ *for the finite time horizon problem*

$$J_{\theta,T}(v,x;h) := \left(\frac{-2}{\theta}\right) \ln \boldsymbol{E} e^{-\frac{\theta}{2}F(t;h)} = \frac{1}{2}x'Q(0)x + q'(0)x + k(0)$$

*where* $k$ *is given as above.*

*Proof.* The proof is articulated around two main ideas. First, we need to verify that indeed $h^*(t) \in \mathcal{A}(T)$. This follows from an argument proposed by Bensoussan. Then, we must prove the optimality of $h^*$. The argument needed here can be found in Kuroda and Nagai. □

## Mutual Fund Theorem

For a given time $t$ and state vector $X(t)$, the efficient frontier can be entirely parameterized using the risk-sensitivity, $\theta$.

**Theorem 2.** *(Benchmark and Assets Mutual Fund Theorem). Given a time $t$ and a state vector $X(t)$, any portfolio can be expressed as a linear combination of investments into two "mutual funds" with respective risky asset allocations:*

$$h^K(t) = (\Sigma\Sigma')^{-1}\left(\hat{a} + \hat{A}X(t)\right) \tag{3}$$

$$h^C(t) = (\Sigma\Sigma')^{-1}\left[\Sigma\gamma - \Sigma\Lambda'\left(q(t) + Q(t)X(t)\right)\right] \tag{4}$$

*and respective allocation to the money market account given by:*

$$h_0^K(t) = 1 - \mathbf{1}'(\Sigma\Sigma')^{-1}\left(\hat{a} + \hat{A}X(t)\right) \tag{5}$$

$$h_0^C(t) = 1 - \mathbf{1}'(\Sigma\Sigma')^{-1}\left[\Sigma\gamma - \Sigma\Lambda'\left(q(t) + Q(t)X(t)\right)\right] \tag{6}$$

*Moreover, if an investor has a risk sensitivity $\theta$, then the respective weights of each mutual fund in the investor's portfolio are equal to $\frac{2}{\theta+2}$ and $\frac{\theta}{\theta+2}$.*

*Proof.* In the asset only case, the optimal risk-sensitive asset allocation is given by:

$$h^*(t) = \frac{2}{\theta+2}(\Sigma\Sigma')^{-1}\left(\hat{a} + \frac{\theta}{2}\Sigma\gamma - \frac{\theta}{2}\Sigma\Lambda' q(t) + \left(\hat{A} - \frac{\theta}{2}\Sigma\Lambda' Q(t)\right)X(t)\right)$$

Denote

$$\begin{aligned}
h^K(t) &= (\Sigma\Sigma')^{-1}\left(\hat{a} + \hat{A}X(t)\right) \\
h^C(t) &= (\Sigma\Sigma')^{-1}\Sigma\gamma - (\Sigma\Sigma')^{-1}\Sigma\Lambda'\left(q(t) + Q(t)X(t)\right)
\end{aligned}$$

the risky asset allocation of funds $K$ and $C$. Now

$$h(t) = \frac{2}{\theta+2}h^K(t) + \frac{\theta}{\theta+2}h^C(t)$$

so by the budget equation

$$\begin{aligned}
h_0(t) &= 1 - \mathbf{1}'h(t) \\
&= \frac{2}{\theta+2}\left(1 - \mathbf{1}'h^K(t)\right) + \frac{\theta}{\theta+2}\left(1 - \mathbf{1}'h^C(t)\right) \\
&= \frac{2}{\theta+2}h_0^K(t) + \frac{\theta}{\theta+2}h_0^C(t)
\end{aligned}$$

where $h_0^K(t)$ is given by (5) and $h_0^C(t)$ is given by (6). $\square$

**Corollary 1.** *(Geometric Brownian Motion.) When the risky assets follow a Geometric Brownian Motion with drift vector $\mu$ and the money market account is risk-free (i.e. $\eta = r$ and $\zeta = 0$), then any optimal portfolio can be expressed as a linear combination of investments into two "mutual funds" with respective asset allocations*

$$\begin{aligned} h^K(t) &= (\Sigma\Sigma')^{-1}(\mu - r\mathbf{1}) \\ h^C(t) &= (\Sigma\Sigma')^{-1}\Sigma\gamma \end{aligned} \tag{7}$$

*and respective allocation to the money market account given by:*

$$\begin{aligned} h_0^K(t) &= 1 - \mathbf{1}'(\Sigma\Sigma')^{-1}(\mu - r\mathbf{1}) \\ h_0^C(t) &= 1 - \mathbf{1}'(\Sigma\Sigma')^{-1}\Sigma\gamma \end{aligned}$$

*Moreover, if an investor has a risk sensitivity $\theta$, then the respective weights of each mutual fund in the investor's portfolio are equal to $\frac{2}{\theta+2}$ and $\frac{\theta}{\theta+2}$.*

The next result is related to the asset only setting considered by Kuroda and Nagai.

**Corollary 2.** *(Asset Only Mutual Fund Theorem). Given a time $t$ and a state vector $X(t)$, any portfolio can be expressed as a linear combination of investments into two "mutual funds" with respective risky asset allocations:*

$$
\begin{aligned}
h^K(t) &= (\Sigma\Sigma')^{-1}\left(\hat{a} + \hat{A}X(t)\right) \\
h^C(t) &= -(\Sigma\Sigma')^{-1}\Sigma\Lambda'\left(q(t) + Q(t)X(t)\right)
\end{aligned}
\tag{8}
$$

*and respective allocation to the money market account given by:*

$$
\begin{aligned}
h_0^K(t) &= 1 - \mathbf{1}'(\Sigma\Sigma')^{-1}\left(\hat{a} + \hat{A}X(t)\right) \\
h_0^C(t) &= 1 + \mathbf{1}'(\Sigma\Sigma')^{-1}\Sigma\Lambda'\left(q(t) + Q(t)X(t)\right)
\end{aligned}
$$

*Moreover, if an investor has a risk sensitivity $\theta$, then the respective weights of each mutual fund in the investor's portfolio are equal to $\frac{2}{\theta+2}$ and $\frac{\theta}{\theta+2}$.*

**Remark 1:** as was expected, the asset allocation within funds $K$ and $C$ is independent from the investor's risk aversion. As we saw in Subsection 6.4, when $\theta \to 0$ the optimal portfolio becomes fund $K$. Also, as $\theta \to +\infty$ the optimal portfolio becomes fund $C$. Fund $C$ can be interpreted as a trading strategy trading on the comovement of assets and valuation factors.

**Remark 2:** when we assume that there are no underlying valuation factors, the risky securities follow geometric Brownian motions with drift vector $\mu$ and the money market account becomes the risk-free asset (i.e. $\eta = r$ and $\zeta = 0$). In this case $\Sigma\Lambda' = 0$ and we can then easily see that fund $C$ is fully invested in the risk-free asset. As a result, we recover Merton's Mutual Fund Theorem for $m$ risky assets and a risk-free asset.

**Special Case: Traded Benchmark**

*Benchmark as a Portfolio of Risky Assets* We will first consider the case when the benchmark is a constant proportion strategy invested in a combination of traded assets. The benchmark dynamics can be expressed as:

$$\frac{dL_t}{L_t} = \nu'(a + AX(t))dt + \nu'\Sigma dW_t$$

where $\nu$ is a $m$-element allocation vector satisfying the budget equation:

$$\mathbf{1}'\nu = 1$$

The corresponding optimal asset allocation for the finite-time horizon problem is

$$\begin{aligned} h^*(t) &= \frac{2}{\theta+2}(\Sigma\Sigma')^{-1}\left(\hat{a} + \frac{\theta}{2}\Sigma\Sigma'\nu - \frac{\theta}{2}\Sigma\Lambda' q(t) + \left(\hat{A} - \frac{\theta}{2}\Sigma\Lambda' Q(t)\right)X_t\right) \\ &= \frac{2}{\theta+2}(\Sigma\Sigma')^{-1}\left(\hat{a} + \hat{A}X_t\right) + \frac{\theta}{\theta+2}\nu \\ &\quad -\frac{\theta}{\theta+2}(\Sigma\Sigma')^{-1}\Sigma\Lambda'\left(q(t) + Q(t)X_t\right) \end{aligned}$$

As $\theta \to 0$,

$$h^*(t) \;\to\; (\Sigma\Sigma')^{-1}\left(\hat{a} + \hat{A}X_t\right)$$

and we recover the log utility optimal portfolio.

As $\theta \to +\infty$,

$$h^*(t) \;\to\; \nu - (\Sigma\Sigma')^{-1}\Sigma\Lambda'\left(q(t) + Q(t)X_t\right)$$

The resulting investment strategy can be decomposed into two elements. The first one, $\nu$, replicates the index. The second element is a risk adjustment trade, which, when combined with the allocation to the money market account can be interpreted as a "long-short macro hedge fund" with zero net weight, so that

$$h_0^*(t) - \mathbf{1}'(\Sigma\Sigma')^{-1}\Sigma\Lambda'\left(q(t) + Q(t)X_t\right) = 0$$

The $n$-element vector $q(t)$ satisfies the linear ordinary differential equation

$$\begin{aligned} &\dot{q}(t) + \left(K_1' - Q(t)K_0\right)q(t) + Q(t)b + \frac{\theta}{\theta+2}Q'(t)\Lambda\Sigma'\nu - \frac{2}{\theta+2}\hat{A}'\nu \\ &+\frac{1}{\theta+2}\left(2\hat{A}' - \theta Q'(t)\Lambda\Sigma'\right)(\Sigma\Sigma')^{-1}\hat{a} = 0 \qquad (9) \end{aligned}$$

with *terminal* condition $q(T) = 0$

In summary:

**Corollary 3.** *(Fund Separation Theorem with a Constant Proportion Benchmark (I)). Any optimal portfolio can be expressed as a linear combination of investments into a "mutual funds", an index fund and a "long-short hedge fund" with respective risky asset allocations:*

$$\begin{aligned} h^K(t) &= (\Sigma\Sigma')^{-1}\left(\hat{a} + \hat{A}X(t)\right) \\ h^I(t) &= \nu \\ h^H(t) &= -(\Sigma\Sigma')^{-1}\Sigma\Lambda'\left(q(t) + Q(t)X(t)\right) \end{aligned} \tag{10}$$

*and respective allocation to the money market account given by:*

$$\begin{aligned} h_0^K(t) &= 1 - \mathbf{1}'(\Sigma\Sigma')^{-1}\left(\hat{a} + \hat{A}X(t)\right) \\ h_0^I(t) &= 0 \\ h_0^H(t) &= \mathbf{1}'(\Sigma\Sigma')^{-1}\Sigma\Lambda'\left(q(t) + Q(t)X(t)\right) \end{aligned} \tag{11}$$

*Moreover, if an investor has a risk sensitivity $\theta$, then the respective weights of each mutual fund in the investor's portfolio are equal to $\frac{2}{\theta+2}$, $\frac{\theta}{\theta+2}$ and $\frac{\theta}{\theta+2}$.*

**‘Fractional Kelly’**

Log-optimal or ‘Kelly’ strategies are known to be ‘risky’. Various authors have proposed ‘fractional Kelly’ strategies of the form

$$h(t) = (1 - \lambda)h^K(t) + \lambda h^{MM}(t)$$

where $h^{MM}$ invests only in the money market account. Our results show that such strategies are sub-optimal because they fail to exploit the opportunities for hedging.

## A Lévy-driven model with diffusion factors

Extend the previous model to discontinuous asset prices, but maintain factor process $X_t$ as a linear diffusion.

Let $(\mathbf{Z}, \mathcal{B}_{\mathbf{Z}})$ be a Borel space. Let $p$ be a $(\mathcal{F}_t)$-adapted $\sigma$-finite Poisson point process on $\mathbf{Z}$ whose underlying point functions are map from a countable set $\mathbf{D}_p \subset (0, \infty)$ into $\mathbf{Z}$. Define

$$\mathfrak{Z}_p := \{U \in \mathcal{B}(Z), \mathbb{E}\left[N_p(t, U)\right] < \infty \ \forall t\} \tag{12}$$

Assume $p$ is of class (QL) and consider $N_p(dt, dz)$, the Poisson random measure on $(0, \infty) \times Z$ induced by $p$. Then there exists $\hat{N}_p = \left(\hat{N}_p(t, U)\right)$ such that

(i.) for $U \in \mathfrak{Z}_p$, $t \mapsto \hat{N}_p(t, U)$ is a continuous $(\mathcal{F}_t)$-adapted increasing process;

(ii.) for each $t$ and a.a. $\omega \in \Omega$, $U \mapsto \hat{N}_p(t, U)$ is a $\sigma$-finite measure on $(Z, \mathcal{B}(Z))$;

(iii.) for $U \in \mathfrak{Z}_p$, $t \mapsto \tilde{N}_p(t, U) = N_p(t, U) - \hat{N}_p(t, U)$ is an $(\mathcal{F}_t)$-martingale;

The random measure $\left\{\hat{N}_p(t, U)\right\}$ is called the *compensator* of the point process $p$. Here $\hat{N}_p(t, U) = \nu(U)t$ where $\nu$ is the $\sigma$-finite characteristic measure of the Poisson point process $p$

Finally, fix a set $\mathbf{Z}_0 \subset \mathcal{B}_{\mathbf{Z}}$ such that $\nu(\mathbf{Z}\backslash\mathbf{Z}_0) < \infty$ and define the Poisson random measure $\bar{N}_p(dt, dz)$ as

$$\begin{aligned} &\bar{N}_p(dt, dz) \\ = &\begin{cases} N_p(dt, dz) - \hat{N}_p(dt, dz) = N_p(dt, dz) - \nu(dz)dt =: \tilde{N}_p(dt, dz) & \text{if } z \in \mathbf{Z}_0 \\ N_p(dt, dz) & \text{if } z \in \mathbf{Z}\backslash\mathbf{Z}_0 \end{cases} \end{aligned}$$

**Factor Dynamics**

As before, these are given by

$$dX(t) = (b + BX(t))dt + \Lambda dW(t), \qquad X(0) = x \tag{13}$$

where $X(t)$ is the $\mathbb{R}^n$-valued factor process with components $X_j(t)$ and $b \in \mathbb{R}^n$, $B \in \mathbb{R}^{n \times n}$, $\Lambda := [\Lambda_{ij}]$, $i = 1, \ldots, n$, $j = 1, \ldots, N$.

**Asset Market Dynamics**

*Money market account:*

$$\frac{dS_0(t)}{S_0(t)} = (\eta + \zeta' X(t))\, dt, \qquad S_0(0) = s_0 \tag{14}$$

*Asset prices:* The price $S_i(t)$ of the $i$th security satisfies

$$\begin{aligned} \frac{dS_i(t)}{S_i(t^-)} &= (a + AX(t))_i dt + \sum_{k=1}^{N} \sigma_{ik} dW_k(t) + \int_{\mathbf{Z}} \gamma_i(z) \bar{N}_p(dt, dz), \\ & S_i(0) = s_i, \quad i = 1, \ldots, m \end{aligned} \tag{15}$$

where $a \in \mathbb{R}^m$, $A \in \mathbb{R}^{m \times n}$, $\Sigma := [\sigma_{ij}]$, $i = 1, \ldots, m$, $j = 1, \ldots, M$ and $\gamma(z) \in \mathbb{R}^m$ with $-1 \leq \gamma_i^{min} \leq \gamma_i(z) \leq \gamma_i^{max} < \infty$ for $i = 1, \ldots, m$.

*Portfolio dynamics:*

$$\begin{aligned} \frac{dV(t)}{V(t^-)} &= (\eta + \zeta' X(t))\, dt + h'(t) \left( \hat{a} + \hat{A} X(t) \right) dt + h'(t) \Sigma dW_t \\ &\quad + \int_{\mathbf{Z}} h'(t) \gamma(z) \bar{N}_p(dt, dz) \end{aligned} \tag{16}$$

Note for admissibility we must have $h'(s)\gamma(s) \geq -1$. More precisely, define

$$\mathbf{S} = \mathrm{supp}(\nu) \in \mathcal{B}_{\mathbf{Z}}, \qquad \tilde{\mathbf{S}} = \mathrm{supp}(\nu \circ \gamma^{-1}) \in \mathcal{B}(\mathbb{R}^m).$$

Let

$$J := \left\{ h \in \mathbb{R}^m : -1 - h'\tilde{s} < 0 \quad \forall \tilde{s} \in \tilde{\mathbf{S}} \right\}$$

and

$$\mathcal{K} := \{ h(t) \in \mathcal{H} : h(t) \in J \quad \forall t \text{ a.s.} \}$$

The set $J$ is a convex cone in $\mathbb{R}^m$. Note that the set $\tilde{\mathbf{S}}$ is a key feature of the model and determines admissibility of strategies.

*Wealth equation:*

$$
\begin{aligned}
e^{-\theta \ln V(t)} &= v^{-\theta} \exp\left\{\theta \int_0^t g(X_s, h(s);\theta)ds - \theta \int_0^t h(s)'\Sigma dW_s \right.\\
&\quad -\frac{1}{2}\theta^2 \int_0^t h(s)'\Sigma\Sigma' h(s)ds \\
&\quad + \int_0^t \int_{\mathbf{Z}} \ln\left(1 - H(h(s);\theta)\right) \tilde{N}_p(ds, dz) \\
&\quad \left. + \int_0^t \int_{\mathbf{Z}} \left\{\ln\left(1 - H(h(s);\theta)\right) + H(h(s);\theta)\right\} \nu(dz)ds \right\}
\end{aligned}
$$

where

$$
\begin{aligned}
g(x, h;\theta) &= \frac{1}{2}(\theta+1)h'\Sigma\Sigma'h - \eta - \zeta'x - h'(\hat{a} + \hat{A}x) \\
&\quad + \int_{\mathbf{Z}} \left\{\frac{1}{\theta}\left[(1 + h'\gamma(z))^{-\theta} - 1\right] + h'\gamma(z)\mathbf{1}_{\mathbf{Z}_0}(z)\right\} \nu(dz) \qquad (17)
\end{aligned}
$$

$$
H(h;\theta) = 1 - (1 + h'\gamma(z))^{-\theta}
$$

*Change of Measure:* Let $\mathbb{P}_h^\theta$ be the measure on $(\Omega, \mathcal{F})$ defined as

$$\begin{aligned}
\chi_t &:= \left.\frac{d\mathbb{P}_h^\theta}{d\mathbb{P}}\right|_{\mathcal{F}_t} \\
&= \exp\left\{-\theta\int_0^t h(s)'\Sigma dW_s - \frac{1}{2}\theta^2\int_0^t h(s)'\Sigma\Sigma' h(s)ds\right. \\
&\quad + \int_0^t\int_{\mathbf{Z}} \ln\left(1 - H(h(s);\theta)\right)\tilde{N}_p(ds,dz) \\
&\quad \left. + \int_0^t\int_{\mathbf{Z}} \left\{\ln\left(1 - H(h(s);\theta)\right) + H(h(s);\theta)\right\}\nu(dz)ds\right\}, \quad \forall t \geq 0
\end{aligned}$$

We denote by $\mathcal{A}$ the set of investment strategies $h \in \mathcal{K}$ on $[0,T]$ such that $\mathbb{P}_h^\theta$ is a probability measure. For $h(t) \in \mathcal{A}$, $X_t$ satisfies the SDE:

$$dX_s = \left(b + BX_s - \theta\Lambda\Sigma' h(s)\right)ds + \Lambda dW_s^\theta$$

We can now introduce the auxiliary criterion function under the measure $\mathbb{P}_h^\theta$:

$$I(v,x;h;t,T) = \ln v - \frac{1}{\theta}\ln \mathbf{E}^\theta\left[\exp\left\{\theta\int_0^{T-t} g(X_s,h(s);\theta)ds\right\}\right] \tag{18}$$

where $\mathbf{E}^\theta\left[\cdot\right]$ denotes the expectation taken with respect to measure $\mathbb{P}_h^\theta$.

**The HJB Equation**

Let $\Phi$ be the value function for the auxiliary criterion function $I(v, x; h; t, T)$. Then $\Phi$ is defined as

$$\Phi(t, x) = \sup_{\mathcal{A}} I(v, x; h; t, T)$$

and it satisfies the HJB PDE

$$\frac{\partial \Phi}{\partial t} + \sup_{h \in \bar{J}} L_t^h \Phi = 0 \tag{19}$$

where

$$L_t^h \Phi = \left(b + Bx - \theta \Lambda \Sigma' h\right)' D\Phi + \frac{1}{2} \mathrm{tr}\left(\Lambda \Lambda' D^2 \Phi\right) - \frac{\theta}{2} (D\Phi)' \Lambda \Lambda' D\Phi - g(x, h; \theta)$$

and subject to terminal condition

$$\Phi(T, x) = \ln v \tag{20}$$

.

*Next step:* show that the value function is the unique viscosity solution of (19),(20).

## Concluding remarks

- Risk-sensitive control seems to combine the virtues of Markowitz and Merton by providing a fully dynamic theory with a clear risk-return trade-off interpretation.
- Conventional performance objectives of outperforming a benchmark are readily handled within this framework.
- Current work is aimed at completing the Lévy assets/diffusion factor theory and extending to Lévy factor.
- Main application: expand asset universe to include credit risk