Arbitrage-free market models for option prices

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based on joint work with Johannes Wissel (ETH Zürich)

The problem

The problem

Basic problem: Construct a model for

- a stock S
- a bond ($\equiv 1$ in discounted units)
- **several call options** $C(K, T)$ on S

in such a way that the model is

- **arbitrage-free,**
- **practically usable.**

Where is the problem ?

Martingale models ?

- Write down model for S, directly under a pricing measure Q.
- Define $C_t(K,T) := E_Q[(S_T - K)^+ | \mathcal{F}_t]$.
- This **martingale model** is obviously arbitrage-free.
- But . . .
- . . . **calibration is practically impossible !**
- This is **no solution !**
- So what now ???

Market models

- **Basic idea:** specify dynamics (SDEs) for **all tradable assets**.
- Joint model for S and all $C(K,T)$ with $K \in \mathcal{K}$ and $T \in \mathcal{T}$.
- **Advantage:** **calibration is automatic** since market prices of options at time 0 are input as initial conditions.
- But: **arbitrage restrictions:**
  - No **static** arbitrage $\longrightarrow$ restrictions on **state space of processes**.
  - No **dynamic** arbitrage $\longrightarrow$ restrictions on **SDE coefficients** (**drift restrictions** à la HJM).
- How to handle these constraints ?

Ideas and questions

The simplest example

- Consider one stock S and **one** call $C(K,T)$. **Restrictions** are
  - **static:** $(S_t - K)^+ \le C_t \le S_t$ and $C_T = (S_T - K)^+$.
  - **dynamic:** S and C both martingales under some $Q \approx P$.
- How to write down **SDE for** (S,C) to ensure this **???**
- Way out ($\longrightarrow$ Lyons 1997, Babbar 2001):
  - **reparametrize:** Instead of C_t, use **implied volatility** $\hat\sigma_t$ via $C_t = c_{BS}\big(S_t, K, (T-t)\hat\sigma_t^2\big)$.
  - more precisely: work with $V_t := (T-t)\hat\sigma_t^2$.
  - **static** arbitrage constraint is equivalent to $0 \le V_t < \infty$ and $V_T = 0$; so **state space** is nice.
  - **dynamic** arbitrage constraint reduces to **drift restriction** for V in SDE model for (S,V).
  - SDE is still tricky (nonlinear), but feasible; explicit examples.

Multiple maturities

- Consider one stock S and **many** calls $C(K,T)$ with one fixed strike K and **maturities** $T \in \mathcal{T}$.
- Use **new parametrization:**
  - **forward implied volatilities** defined by

$$X_t(T) := \frac{\partial}{\partial T}\big((T-t)\hat{\sigma}_t^2(K,T)\big) = \frac{\partial}{\partial T} V_t(T).$$

  - **static** arbitrage constraints are equivalent to
    (i) $V_t(T) \geq 0$ as before.
    (ii) $T \mapsto V_t(T)$ is increasing, i.e., $0 \leq X_t(T) < \infty$.
    So: **state space** is nice.
  - **dynamic** arbitrage constraints reduce to **drift restrictions** for all $X(T)$ in SDE model for $\big(S, X(T)\big)_{T\in\mathcal{T}}$.
- $\longrightarrow$ Schönbucher 1999, Brace et al. 2001, Ledoit et al. 2002

Problems with multiple maturities I

- Structure of model:

$$\begin{aligned} dS_t &= S_t \mu_t \, dt + S_t \sigma_t \, dW_t, \\ dX_t(T) &= \alpha_t(T) \, dt + v_t(T) \, dW_t. \end{aligned}$$

- **Dynamic arbitrage constraints:** (S_t) and all

$$C_t(T) = c_{BS}\left(S_t, K, \int_t^T X_t(s)\, ds\right)$$

must be (local) martingales under some $Q \approx P$.

- **Drift restrictions:**

$$\begin{aligned} \mu_t &= -\sigma_t b_t, \\ \sigma_t &= f\big(X_t(t), S_t, v_t(\cdot)\big), \\ \alpha_t(T) &= g\big(X_t(T), S_t, v_t(\cdot), X_t(\cdot)\big). \end{aligned}$$

Problems with multiple maturities II

- Structure of model with **drift restrictions:**

$$
\begin{aligned}
dS_t &= S_t f\big(X_t(t), S_t, v_t(\cdot)\big)(dW_t - b_t\,dt), \\
dX_t(T) &= g\big(X_t(T), S_t, v_t(\cdot), X_t(\cdot)\big)\,dt + v_t(T)\,dW_t.
\end{aligned}
$$

- f and g are **nonlinear**; so even if v is Lipschitz and of linear growth, dt-coefficients and σ are not !
- **Existence problem for (infinite, nonlinear) SDE system !**
- No results in the literature (except classical HJM, with severe conditions: bounded and Lipschitz).

Multiple strikes: even more problems

- Consider one stock S and **many** calls $C(K,T)$ with one fixed maturity T and **strikes** $K \in \mathcal{K}$.
- Arbitrage constraints:
  - **dynamic:** as usual some **drift restrictions.**
  - **static:** $K \mapsto C_t(K,T)$ is convex and satisfies

$$-1 \leq \frac{\partial}{\partial K} C_t(T,K) \leq 0.$$

  - **state space** for $C(T,K)$ very complicated.
  - using (classical or forward) implied volatilities does not help either.
- Before even thinking about SDEs: **How to choose parametrization ??**

Results

Infinite SDE systems

- **Key mathematical tool:** consider SDE system

$$dX_t(T) = \alpha_t\big(T, X_\cdot(\cdot)\big)\,dt + \sigma_t\big(T, X_\cdot(\cdot)\big)\,dW_t$$

with $0 \leq T \leq T^*$ and $0 \leq t \leq T_0 \leq T^*$.

- $\longrightarrow$ **J. Wissel 2006:**
  - **Existence and uniqueness** result for **strong solution** under only **local Lipschitz-type conditions** on α, σ.
  - Includes sufficient conditions on growth for non-explosion.
  - Key idea: work on product space $[0, T^*] \times \Omega$.

- **Important:** global Lipschitz condition is too strong for the required applications.

Multiple maturities

- SDE system with **drift restrictions** is

$$\begin{aligned} dS_t &= S_t f\big(X_t(t), S_t, v_t(\cdot)\big)(dW_t - b_t\,dt), \\ dX_t(T) &= g\big(X_t(T), S_t, v_t(\cdot), X_t(\cdot)\big)\,dt + v_t(T)\,dW_t. \end{aligned}$$

- **Theorem:** Sufficient conditions on v for **existence and uniqueness** of solution.
- **Not direct** from general SDE results, because
  - only v can be chosen here.
  - in addition, must have $X \geq 0$.
- Classes of explicit examples for such models, for **first time in literature.**
- $\longrightarrow$ **S/Wissel 2006**

Explicitly:

$$
\begin{aligned}
\alpha_t(T) \quad = \quad & -\frac{1}{2}\left(\big(\mathcal{R}_t(T)\big)^2 - \frac{1}{\mathcal{Z}_t(T)} - \frac{1}{4}\right) v_t(T) \cdot \int_t^T v_t(s)\,ds \\
& + \frac{1}{2}\left(\big(\mathcal{R}_t(T)\big)^2 - \frac{1}{2}\frac{1}{\mathcal{Z}_t(T)}\right) \frac{X_t(T)}{\mathcal{Z}_t(T)} \left|\int_t^T v_t(s)\,ds\right|^2 \\
& + \left(\mathcal{R}_t(T) - \frac{1}{2}\right) \sigma_t v_t^1(T) \\
& - \mathcal{R}_t(T) \frac{X_t(T)}{\mathcal{Z}_t(T)} \sigma_t \int_t^T v_t^1(s)\,ds - b_t \cdot v_t(T)
\end{aligned}
$$

with

$$
Y_t(t) := \log S_t, \quad \mathcal{R}_t(T) := \frac{Y_t(t) - \log K}{\mathcal{Z}_t(T)}, \quad \mathcal{Z}_t(T) := \int_t^T X_t(s)\,ds.
$$

Multiple strikes: new parametrization

- Recall key difficulty: **how to parametrize ?**
- Call option prices **admissible** if for each t, $K \mapsto C_t(K)$
  - is C^2,
  - is strictly convex,
  - satisfies $-1 < C_t'(K) < 0$ for all K,
  - satisfies $\lim_{K \to \infty} C_t(K) = 0$.
- (This is slight strengthening of **static arbitrage constraints.**)
- **New concept: local implied volatilities**

$$X_t(K) := \frac{1}{\sqrt{T-t}\, K C_t''(K)} \varphi\Big(\Phi^{-1}\big(-C_t'(K)\big)\Big)$$

and, for fixed K_0, **price level**

$$Y_t := \sqrt{T-t}\, \Phi^{-1}\big(-C_t'(K_0)\big).$$

- **Theorem:** There is a bijection between **admissible option price models** and all pairs (X, Y) of **positive local implied volatility curves** X and **real-valued price levels** Y.
- In other words:
  - State space of (X, Y) is nice ...
  - ...and yet captures exactly the static arbitrage constraints !
- Also
  - interpretation for the $X_t(T)$ as "local implied volatilities".
  - explicit formulae relating the classical and the above new local implied volatilities.
  - recovers standard volatility in Black-Scholes setting.
- So: **good solution to parametrization problem with multiple strikes !**

Multiple strikes: structure of models

- **Dynamic arbitrage constraints:**

$$S_t = C_t(0) = \int_0^\infty \Phi\left(\frac{Y_t - \int_{K_0}^k \frac{1}{hX_t(h)}\,dh}{\sqrt{T-t}}\right) dk$$

and all call prices

$$C_t(K) = \int_K^\infty \Phi\left(\frac{Y_t - \int_{K_0}^k \frac{1}{hX_t(h)}\,dh}{\sqrt{T-t}}\right) dk$$

must be (local) martingales under some $Q \approx P$.

- **Drift restrictions** on SDEs for X and Y **?**

- Model for local implied volatilities X and price level Y:

$$
\begin{aligned}
dX_t(K) &= X_t(K)u_t(K)\,dt + X_t(K)v_t(K)\,dW_t,\\
dY_t &= \beta_t\,dt + \gamma_t\,dW_t.
\end{aligned}
$$

- **Drift restrictions** from **dynamic arbitrage constraints:**

$$
\begin{aligned}
\beta_t &= -\gamma_t \cdot b_t + \frac{1}{2}\frac{Y_t}{T-t}\left(|\gamma_t|^2 - 1\right),\\
u_t(K) &= -v_t(K)\cdot b_t + \frac{1}{T-t}\Bigg(\frac{1}{2}\left(1 - |\gamma_t + \mathcal{I}_t^v(K)|^2\right)\\
&\quad + \left(Y_t + \mathcal{I}_t^1(K)\right)\left(\gamma_t + \mathcal{I}_t^v(K)\right)\cdot v_t(K)\Bigg) + |v_t(K)|^2
\end{aligned}
$$

with

$$
\mathcal{I}_t^1(K) := \int_{K_0}^{K} \frac{1}{hX_t(h)}\,dh, \qquad \mathcal{I}_t^v(K) := \int_{K_0}^{K} \frac{v_t(h)}{hX_t(h)}\,dh.
$$

Multiple strikes: existence of models

- **Theorem:** Sufficient conditions on v for **existence and uniqueness** of solution.
- Again, **not direct** from general SDE results, because
  - only v can be chosen here.
  - in addition, must have $X > 0$.
- Up to now, **no result on existence** of such models in the literature.
- First **tractable parametrization** to tackle this problem at all !
- In addition, explicit class of **examples** for models.
- $\longrightarrow$ **S/Wissel 2007**

Towards the end

Open problems

- Model and parametrization for **full option surface** (all maturities T and all strikes K): **??**
- Practical implementation ?
- Analogous results for finite family of options ?
- Recalibration ?
- ...

Some related work

- **Dupire:** **local volatility** model:
  - can also fit any initial term structure of option prices ...
  - ...but not rich enough for recalibration over time.
  - No explicit formulas, only PDEs for $C_t(K, T)$ with $t > 0$.
  - extensions to more stochastic factors ...
  - ...but there no existence results for models.
- **Bühler:** market models for **variance swaps:**
  - only maturity parameter T; no strike structure.
  - special payoff function yields easy infinite SDE system.
  - some more explicit results.
- **Durrleman:** links between **spot and implied volatilities:**
  - classical martingale modelling, no market models.
  - results for at-the-money options and shortly before maturity.
  - practical usability less clear.

References

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The end (for the time being . . .)

Thank you for your attention !