On the problem of optimal bond portfolio choice.

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The problem:

Find $sup_{\phi}\mathbb{E}\Big[U(V_T(x))\Big]$, where $V_T(x)$ is the value at time T of a bond portfolio, with initial endowment x, U is an utility function who satifies usual *Inada's type* conditions, and ϕ_s is an *admissible, self-financing* portfolio strategy.

If the term structure model is a *finite factor* model, the problem is reduced to the usual **Utility Maximization Problem** in a stock market:

Let $\mathrm{d}f(t,T) = \alpha(t,T)\,\mathrm{d}t + \sigma(t,T).\mathrm{d}W_t\,,$ (with W_t n–dimensional) , suppose $T \leq T_1 < \ldots < T_n$ be such that the matrix

$$\left[P(t,T_i)\,.\,\int_t^{T_i} \sigma_j(t,s)\,\mathrm{d}s\right]_{i,j=1,\ldots,n}$$

is invertible (for $t \leq T$) : *then the whole Bond Market until time T is equivalent to the market given by* $P(t,T_1),\ldots,P(t,T_n)$.

The problem of optimal Bond Portfolio choice is interesting (from a mathematical point of view) *only in true infinite–dimensional models*.

Three papers:

- **Ekeland–Taflin, (A.A.P. 2005)**: H.J.B. equation in infinite dimension

- **Carmona–Tehranchi (A.A.P. 2004)** and **Ringer–Tehranchi (F.S. 2006)** : Malliavin calculus infinite–dimensional.

- **De Donno–Pratelli (A.A.P. 2005)** and **Pratelli (Ascona Conference 2005)** : extension of results by Kramkov–Schachermayer.

Main difference among these papers:

- in **E.T.** and **C.T.** the Bond Market model $\mathbf{P}_t$ is taken as a stoch. process *with values in a suitable space of functions*

- in **DD.P** a *stochastic integration theory* specialized for an infinite family of semimartingales is built.

Problems :

- existence results for the optimal strategy
- a *Mutual Fund Theorem* (generalization of Merton's M.F.T.)

Merton's Mutual Fund Theorem: the optimal investor's allocation decision can be separated in two steps:

- an *efficient portfolio* of risky assets is determined (the **M.F.**)
- the investor decides the allocation between this efficient portfolio and the riskless asset.

An outline of our approach:

Elementary strategy:

$$\mathbf{\Phi}_t = \sum_{i=1}^{n} \Phi_t^i \, \delta_{T_i}$$

and

$$\int_0^T \mathbf{\Phi}_s \, \mathrm{d}\mathbf{P}_s = \int_0^T \sum_{i=i}^{n} \Phi_s^i \, \mathrm{d}P(s, T_i)$$

Generalized strategies: processes with values unbounded linear operators.

Definition: $\mathbf{\Phi}_t$ is integrable w.r.t. $\mathbf{P}_t$ if there exists a sequence of *elementary strategies* $(\mathbf{\Phi}_t^n)$ such that

- $\mathbf{\Phi}^n \longrightarrow \mathbf{\Phi}$ a.s.

- $\int \mathbf{\Phi}^n \, \mathrm{d}\mathbf{P} \longrightarrow Y$ in the *semimartingale topology*

By definition, $Y = \int \mathbf{\Phi} \, \mathrm{d}\mathbf{P}$.

Good results with this approach:

- the **Memin's** theorem is satisfied (limit of stochastic integrals is still a stochastic integral)

- the **Ansel–Stricker's** lemma is satisfied (with a suitable definition of admissible strategies)

As a consequence, the following **Delbaen-Schachermayer's** result is satisfied:

$0 \leq f \leq x + \int \mathbf{\Phi}_s \mathrm{d}\widetilde{\mathbf{P}}_s$ for some suitable $\mathbf{\Phi}$

$$\Updownarrow$$

$\sup_{\mathbf{Q}} \mathbb{E}^{\mathbf{Q}}[f] \leq x$, $\mathbf{Q}$ equiv. martingale measure.

In particular, the **Kramkov–Schachermayer (A.A.P. 2003)** results can be extended in this framework, thus obtaining:

- an *existence result* of the optimal portfolio $\widehat{V}_T(x)$

- if $\mathbf{Q}$ is unique, a characterization

$$\widehat{V}_T(x) = \Big(U'\Big)^{-1}\Big(y\,\frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}\mathbf{P}}\Big)$$

(x and y are linked by the duality relation like in **K.S. 2003**)

Drawbacks with this approach:

- the *generalized strategies* are not easy to characterize in practice

- the **self-financing condition** is not clear with this approach

Self-financing condition:

$$V_t = \Phi_t^0 B_t + \mathbf{\Phi}_t \mathbf{P}_t$$

$$\mathrm{d}V_t = \Phi_t^0 \mathrm{d}B_t + \mathbf{\Phi}_t \mathrm{d}\mathbf{P}_t$$

or, equivalently,

$$\mathrm{d}\widetilde{V}_t = \mathbf{\Phi}_t \mathrm{d}\widetilde{\mathbf{P}}_t$$

Problem (with our approach): find conditions under which the optimal strategy can be characterized explicitly, the self-financing condition is satisfied, and a **Mutual Fund Theorem** can be determined.

Three steps towards a **M.F.T.**(under uniqueness of $\mathbf{Q}$) :

- $\frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}\mathbf{P}}$ is measurable w.r.t to a smaller filtration $\mathcal{G}_t \subseteq \mathcal{F}_t$

- for the filtration $(\mathcal{G}_t)$ there exist a $\mathbf{Q}$–martingale (k–dimensional) (N_t) with the *representation property*

- $N_t = \int_0^t \mathbf{\Phi}_s \, \mathrm{d}\widetilde{\mathbf{P}}_s$ with an *explicit characterization* of $\mathbf{\Phi}_t$ (the **mutual fund**)

In this situation

$$\widehat{V}_T(x) = \left(U'\right)^{-1}\left(y\,\frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}\mathbf{P}}\right) = \int_0^T H_s \mathrm{d}N_s = \int_0^T \left(H_s \mathbf{\Phi}_s\right)\mathrm{d}\widetilde{\mathbf{P}}_s$$

Explicit results can be obtained if **the noise is an infinite–dimensional Wiener process**.

The noise:

Ekeland–Taflin

$$\mathsf{d}\widetilde{\mathbf{P}}_t = \sum_i \Gamma_i\big(t, \widetilde{\mathbf{P}}_t\big)\mathsf{d}W_t^i$$

where (W^i) is a sequence of independent Wiener processes.

Carmona–Tehranchi

$$\mathsf{d}\widetilde{\mathbf{P}}_t = \boldsymbol{\sigma}\big(t, \widetilde{\mathbf{P}}_t\big)\mathsf{d}\mathbf{W}_t$$

where $\mathbf{W}$ is a cylindrical Wiener process on a separable Hilbert space G and $\boldsymbol{\sigma}(.\,,.)$ with values $\mathcal{L}_{HS}\big(G, H_w^2\big)$, H_w^2 is a suitable weighted Sobolev space of functions.

In both papers

$$\frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}\mathbf{P}} = \exp\Big(\int_0^T \boldsymbol{\lambda}_s \mathrm{d}\mathbf{W}_s - \frac{1}{2}\int_0^T \|\boldsymbol{\lambda}_s\|_G^2 \, \mathrm{d}s \Big)$$

Consider the continuous martingale $M_t = \int_0^t \boldsymbol{\lambda}_s \mathrm{d}\mathbf{W}_s$ (a time–changed Wiener process): when (M_t) has the *stochastic integral representation property*? (**Revuz-Yor**).

A **sufficient condition**: $\langle M \rangle_t = \int_0^t \|\boldsymbol{\lambda}_s\|_G^2 \, \mathrm{d}s$ is deterministic.

Ekeland–Taflin obtain an *existence result* and a *Mutual Fund Theorem* under a condition satisfied if $\|\boldsymbol{\lambda}_s\|_G$ is deterministic.

This result, obtained with an *accurate investigation* of an infinite–dimensional H.J.B. equation, can be derived as an exercise in the framework of **DD.P.** results.

Carmona–Tehranchi's point of view.

Let F^i_w the *weighted Sobolev space* of functions f defined on $[0,+\infty[$, i-times derivable (sense of distributions) such that $f^j(+\infty) = 0$ for $j \le i-1$, with norm $\|f\|^2 = \int_0^{+\infty} f^i(s)^2\, w(s)\, \mathrm{d}s$; **C.T.** consider $\mathbf{P}_t$ as a process with values F^2_w.

On this space δ_T and δ'_T are defined (as elements of $(F^2_W)'$) and hence:

$$P(t,T) = \delta_T \mathbf{P}_t$$

$$f(t,T) = -\frac{\partial}{\partial T} \log P(t,T) = -\frac{\delta'_T \mathbf{P}_t}{\delta_T \mathbf{P}_t}$$

Strategies are processes Φ_t with values in $(F_u^1)'$ where u and w are chosen in such a way that $F_w^2 \hookrightarrow F_u^1$ and therefore $(F_u^1)' \hookrightarrow (F_w^2)'$.

Why these conditions? A *parsimonious* choice of allowed strategies gives *uniqueness* of strategies under suitable assumptions (δ_t' in not allowed, observe that $\delta_t' = r_t \delta_t$).

Why Malliavin Calculus? This family of allowed strategies is evidently restricted (only *approximate completeness*) but **every contingent claim** a time T of the form

$g\big(P(t,T_1),\ldots,P(t,T_n)\big)$, with $T \leq T_1 < \ldots < T_n$ and g **lipschitz can be hedged with these strategies**.

Idea of the proof:

- in the **C.T.** model every $P(t,T)$ has a Malliavin derivative

- therefore $H = g\big(P(t,T_1),\dots,P(t,T_n)\big)$ has a derivative

- **Clark–Ocone–Karatzas** formula

$$H = \mathbb{E}^{\mathbf{Q}}[H] + \int_0^T \mathbb{E}^{\mathbf{Q}}\big[D_t H \big| \mathcal{F}_t\big]\, \mathrm{d}\mathbf{W}_t$$

and

$$\int_0^T \mathbb{E}^{\mathbf{Q}}\big[D_t H \big| \mathcal{F}_t\big]\,\mathrm{d}\mathbf{W}_t = \int_0^T \big(\boldsymbol{\sigma}^*(t,\widetilde{\mathbf{P}}_t)\mathbb{E}^{\mathbf{Q}}\big[D_t H \big| \mathcal{F}_t\big]\big)\,\mathrm{d}\widetilde{\mathbf{P}}_t$$

Ideas of the paper **Ringer-Tehranchi**:

- the **model** is (more or less) the **C.T.** model with an extra (ad hoc) hypothesis:

$$\frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}\mathbf{P}} = \exp\Big(\int_0^T \boldsymbol{\lambda}_s \mathrm{d}\mathbf{W}_s - \frac{1}{2}\int_0^T \|\boldsymbol{\lambda}_s\|_G^2\,\mathrm{d}s\Big)$$

where $\boldsymbol{\lambda}_s = \boldsymbol{\lambda}(s,\widetilde{\mathbf{P}}_s) = \boldsymbol{\sigma}^*(s,\widetilde{\mathbf{P}}_s)\boldsymbol{\Gamma}(s,\widetilde{\mathbf{P}}_s)$ with a suitable $\boldsymbol{\Gamma}(.\,,.)$.

- the **existence** of optimal solution is obtained with a method similar to the paper **Karatzas–Ocone 1991**

- the **optimal strategy** is of the form $\Phi_t^1 + \Phi_t^2 + \Phi_t^3$

where Φ_t^3 is essentially the component with respect to the *riskless asset* (money–market account), $\Phi_t^2 = 0$ if $\boldsymbol{\lambda}_t$ is deterministic and Φ_t^1 is proportional to a fixed *universal* strategy (the **M.F.**).

Where the terms $\mathbf{\Phi}_t^1$ and $\mathbf{\Phi}_t^2$ come from?

- the value of the optimal portfolio is

$$\Big(U'\Big)^{-1}\Big(y \exp\big(\int_0^T \boldsymbol{\lambda}_s \mathrm{d}\mathbf{W}_s - \frac{1}{2}\int_0^T \|\boldsymbol{\lambda}_s\|^2 \,\mathrm{d}s\big)\Big)$$

- chain rule for Malliavin derivative

- Clark–Ocone–Karatzas formula

Moreover, if $\boldsymbol{\lambda}_t$ is deterministic:

$$D_t\big(\int_0^T \boldsymbol{\lambda}_s \mathrm{d}\mathbf{W}_s\big) = \boldsymbol{\lambda}_t$$

If $\boldsymbol{\lambda}_t$ is stochastic (adapted):

$$D_t\big(\int_0^T \boldsymbol{\lambda}_s \mathrm{d}\mathbf{W}_s\big) = \boldsymbol{\lambda}_t + \int_t^T D_t\boldsymbol{\lambda}_s \ \mathrm{d}\mathbf{W}_s$$