Minimizing the risk of a financial product using a put option

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Advances in Mathematics of Finance
2nd general AMaMeF conference
Bedlewo, Poland
30 April - 5 May, 2007

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Definition

A risk measure ρ is a functional

$$\rho : \Gamma \mapsto \mathbb{R}.$$

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cfr. Artzner, Delbaen, Eber, Heath, Mathematical Finance (1999)

coherent risk measure: monotonic, positive homogeneous, translation invariant and subadditive

Some well-known risk measures

- Value-at-Risk at level p: p-quantile risk measure

$$\mathrm{VaR}_p[Y] = F_Y^{-1}(p) = \inf \{x \in \mathbb{R} \mid F_Y(x) \geq p\}$$

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- Conditional Tail Expectation at level p:

$$\text{CTE}_p[Y] = \text{E}[Y \mid Y > F_Y^{-1}(p)]$$

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$$\rho[L] = X(0) + C - hK + (1-h)\rho[-X(T)]$$

Risk minimization

- constrained optimization problem:

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  $\Rightarrow$ linear trade-off between hedging expenditure and risk measure level

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$$P(0, T, K) - (K + \rho[-X(T)]) \frac{\partial P}{\partial K}(0, T, K) = 0$$

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- distinguish two cases: comonotonic and non-comonotonic sum

Convex order and general inverse

Definition

A random variable X is said to precede a random variable Y in the convex order sense, notation $X \leq_{cx} Y$, if and only if

$$\mathrm{E}[X] = \mathrm{E}[Y] \quad \text{and} \quad \mathrm{E}[(X-d)_+] \leq \mathrm{E}[(Y-d)_+], \quad \text{for all } d.$$

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Definition (Kaas, Dhaene & Goovaerts (2000))

The α-inverse of the cdf F_X of a random variable X is defined as a convex combination of the inverses F_X^{-1} and F_X^{-1+} of F_X:

$$F_X^{-1(\alpha)}(p) = \alpha F_X^{-1}(p) + (1-\alpha) F_X^{-1+}(p), \quad p \in (0,1), \ \alpha \in [0,1],$$

$$\text{with} \quad F_X^{-1}(p) = \inf\{x \in \mathbb{R} \mid F_X(x) \geq p\}, \quad p \in [0,1]$$

$$F_X^{-1+}(p) = \sup\{x \in \mathbb{R} \mid F_X(x) \leq p\}, \quad p \in [0,1].$$

Comonotonicity

Definition (Fréchet (1951), Hoeffding (1940))

A random vector $(Y_1, \ldots, Y_n)$ with marginal cdf's $F_{Y_i}(x) = \Pr[Y_i \leq x]$ is said to be comonotonic if for $U \sim Uniform(0,1)$

$$(Y_1, \ldots, Y_n) \stackrel{d}{=} \left(F_{Y_1}^{-1}(U), F_{Y_2}^{-1}(U), \ldots, F_{Y_n}^{-1}(U)\right).$$

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Property

A random vector $(Y_1, \ldots, Y_n)$ is said to be comonotonic, if there exist a random variable Z and non-decreasing (or either non-increasing) functions $g_1, g_2, \ldots, g_n \colon \mathbb{R} \to \mathbb{R}$ such that

$$(Y_1, Y_2, \ldots, Y_n) \stackrel{d}{=} (g_1(Z), g_2(Z), \ldots, g_n(Z)).$$

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with cdf: $F_{S^c}(x) = \sup\left\{p \in [0,1] \mid \sum_{i=1}^n F_{Y_i}^{-1}(p) \leq x\right\}$

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vector $(\mathrm{E}[Y_1|\Lambda], \ldots, \mathrm{E}[Y_n|\Lambda])$ not necessarily comonotonic

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- comonotonic sum

$$S^c = Y_1^c + \cdots + Y_n^c$$

with cdf: $F_{S^c}(x) = \sup \left\{ p \in [0,1] \mid \sum_{i=1}^n F_{Y_i}^{-1}(p) \leq x \right\}$

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put option $P_i(0,T,K_i)$ with X_i as underlying, maturity T, strike K_i

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$$K_i = F_{X_i(T)}^{-1(\alpha)}(F_{X(T)}(K)) \text{ with } \sum_{i=1}^{n} a_i F_{X_i(T)}^{-1(\alpha)}(F_{X(T)}(K)) = K$$

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  from where

$$\alpha = \frac{K - \sum_{i=1}^{n} a_i F^{-1+}_{X_i(T)}(F_{X(T)}(K))}{\sum_{i=1}^{n} a_i (F^{-1}_{X_i(T)}(F_{X(T)}(K)) - F^{-1+}_{X_i(T)}(F_{X(T)}(K))}$$

  when $F^{-1}_{X_i(T)}(F_{X(T)}(K)) \neq F^{-1+}_{X_i(T)}(F_{X(T)}(K))$ and without loss of generality $\alpha = 1$ otherwise

- decomposition of derivative of put option price

$$\frac{\partial P}{\partial K}(0, T, K) = \sum_{i=1}^{n} a_i \frac{\partial P_i(0, T, K_i)}{\partial K_i} \frac{\partial K_i}{\partial K}$$

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$$\frac{\partial P_i(0,T,K_i)}{\partial K_i} = \text{disc} \cdot F_{X_i(T)}(K_i) = disc \cdot F_{X(T)}(K)$$

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Two-additive-factor Gaussian model G2++

- short-rate dynamics, see e.g. Brigo and Mercurio (2001)

$$r(t) = x(t) + y(t) + \varphi(t), \quad r(0) = r_0$$

where processes $\{x(t) : t \geq 0\}$ and $\{y(t) : t \geq 0\}$ satisfy

$$dx(t) = -ax(t)dt + \sigma d\hat{W}_1(t), \quad x(0) = 0$$
$$dy(t) = -by(t)dt + \eta d\hat{W}_2(t), \quad y(0) = 0$$

with $(\hat{W}_1, \hat{W}_2)$ a two-dimensional Brownian motion with instantaneous correlation ρ:

$$d\hat{W}_1(t)d\hat{W}_2(t) = \rho dt$$

- zero-coupon bond has lognormal distribution

$$\mathrm{ZB}(t,S) = A(t,S)\exp[-B(a,t,S)x(t) - B(b,t,S)y(t)]$$
$$\stackrel{d}{=} e^{\Pi^{\mathbb{Q}}(0,t,S)+\Sigma(0,t,S)Z}, \quad Z \sim N(0,1),\ \mathbb{Q} = Q, Q^T$$

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- put option with zero-coupon bond as underlying and $T < S$

$$\mathrm{ZBP}(0,T,S,K) = \mathrm{ZB}(0,T)K\Phi\left(\frac{\ln\frac{K\mathrm{ZB}(0,T)}{\mathrm{ZB}(0,S)}}{\Sigma(0,T,S)} + \frac{1}{2}\Sigma(0,T,S)\right) - \mathrm{ZB}(0,S)\Phi\left(\frac{\ln\frac{K\mathrm{ZB}(0,T)}{\mathrm{ZB}(0,S)}}{\Sigma(0,T,S)} - \frac{1}{2}\Sigma(0,T,S)\right)$$

- price coupon-bearing bond at T with coupons $\mathcal{C} = \{c_1, \ldots, c_n\}$ paid out at times $\mathcal{T} = \{S_1, \ldots, S_n\}$ larger than T

$$\mathrm{CB}(T, \mathcal{T}, \mathcal{C}) = \sum_{i=1}^{n} c_i \mathrm{ZB}(T, S_i) \quad \text{sum of lognormals}$$

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- put option with maturity T and strike K (Brigo and Mercurio, 2001)

$$\mathrm{CBP}\left(0, T, \mathcal{T}, \mathcal{C}, K\right) = \mathrm{ZB}(0, T) \int_{-\infty}^{+\infty} \frac{e^{-\frac{1}{2}\left(\frac{x-\mu_x}{\sigma_x}\right)^2}}{\sigma_x\sqrt{2\pi}} \left[K\Phi(-h_1(x, K)) - \sum_{i=1}^{n} \lambda_i(x) e^{\kappa_i(x)} \Phi(-h_{2i}(x, K))\right] dx$$

where $h_1(x, K) := \left(\frac{\hat{y}(x,K)-\mu_y}{\sigma_y} - \frac{\rho_{xy}(x-\mu_x)}{\sigma_x}\right) / \sqrt{1 - \rho_{xy}^2}$ with $\hat{y} = \hat{y}(x, K)$ solution of following equation for each fixed x

$$\sum_{i=1}^{n} c_i A(T, S_i) e^{-B(a,T,S_i)x - B(b,T,S_i)\hat{y}} = K$$

- optimal strike K^* from equation

$$\mathrm{CBP}(0,T,\mathcal{T},\mathcal{C},K)-(K+\rho[-\mathrm{CB}(T,\mathcal{T},\mathcal{C})])\frac{\partial \mathrm{CBP}}{\partial K}(0,T,\mathcal{T},\mathcal{C},K)=0$$

or equivalently

$$\mathrm{ZB}(0,T)\int_{-\infty}^{+\infty}\frac{e^{-\frac{1}{2}\left(\frac{x-\mu_x}{\sigma_x}\right)^2}}{\sigma_x\sqrt{2\pi}}\left[\rho[-\mathrm{CB}(T,\mathcal{T},\mathcal{C})]\Phi(-h_1(x,K))\right.$$
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- solve numerically for K^* and use as benchmark

- approximate problem for $\nu = \ell, c$ with $\alpha = 1$

$$\sum_{i=1}^{n} a_i \mathrm{ZBP}^{\nu}(0, T, S_i, F^{-1}_{\mathrm{ZB}^{\nu}(T,S_i)}(A^{\nu}_K))$$

$$= \mathrm{disc} \cdot A^{\nu}_K \sum_{i=1}^{n} a_i (F^{-1}_{\mathrm{ZB}^{\nu}(T,S_i)}(A^{\nu}_K) + \rho[-\mathrm{ZB}^{\nu}(T, S_i)])$$

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- explicit expressions for inverse cdfs $F^{-1}_{\mathrm{ZB}^{\nu}(T,S_i)}(p)$, $p \in [0, 1]$

$$\nu = \ell \qquad F^{-1}_{\mathrm{E}^{\mathbb{Q}}[\mathrm{ZB}(T,S_i)|\Lambda^{\mathbb{Q}}]}(p)$$
$$= e^{\Pi^{\mathbb{Q}}(0,T,S_i) + r_i^{\mathbb{Q}} \Sigma(0,T,S_i)\Phi^{-1}(p) + \frac{1}{2}(1-(r_i^{\mathbb{Q}})^2)\Sigma(0,T,S_i)^2}$$
$$\text{with} \qquad r_i^{\mathbb{Q}} = \mathrm{corr}(\Lambda^{\mathbb{Q}}, \ln \mathrm{ZB}(T, S_i)) > 0 \text{ for all } i$$

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- choices of conditioning r.v. $\Lambda^{\mathbb{Q}}$:

$$\Lambda^{\mathbb{Q}} = \sum_{i=1}^{n} \gamma_i^{\mathbb{Q}} \Sigma(0, T, S_i) Z_i$$

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- corresponding correlations:

$$r_i^{\mathbb{Q}} = \frac{\sum_{j=1}^{n} \gamma_j^{\mathbb{Q}} \Sigma(0, T, S_j) \mathrm{cov}(Z_i, Z_j)}{\sqrt{\sum_{i=1}^{n} \sum_{j=1}^{n} \gamma_i^{\mathbb{Q}} \gamma_j^{\mathbb{Q}} \Sigma(0, T, S_i) \Sigma(0, T, S_j) \rho_{ij}}}$$

with

$$\rho_{ij} = \frac{\mathrm{cov}(\ln \mathrm{ZB}(T, S_i), \ln \mathrm{ZB}(T, S_j))}{\Sigma(0, T, S_i) \Sigma(0, T, S_j)}$$

explicit coefficients $\gamma_i^{\mathbb{Q}}$

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$$\gamma_i^{\mathbb{Q}} = \frac{c_i}{\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} c_i c_j \Sigma(0,T,S_i)\Sigma(0,T,S_j)\rho_{ij}}}$$

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$$\gamma_i^{\mathbb{Q}} = \frac{c_i}{\sqrt{\sum_{i=1}^{n} \sum_{j=1}^{n} c_i c_j \Sigma(0, T, S_i)\Sigma(0, T, S_j)\rho_{ij}}}$$

3. Maximal variance (MV): $\gamma_i^{\mathbb{Q}} = c_i e^{\Pi^{\mathbb{Q}}(0,T,S_i)+\frac{1}{2}\Sigma(0,T,S_i)^2}$

explicit coefficients $\gamma_i^{\mathbb{Q}}$

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4. Maximal CTE (MCTE):

$$\gamma_i^{\mathbb{Q}} = c_i e^{\Pi^{\mathbb{Q}}(0,T,S_i)+\frac{1}{2}\Sigma(0,T,S_i)^2} \cdot e^{\frac{1}{2}(\Phi^{-1}(1-p)-r_i^{\mathbb{Q},\mathrm{MV}}\Sigma(0,T,S_i))^2}$$

where $r_i^{\mathbb{Q},\mathrm{MV}} = \mathrm{corr}(\ln(\mathrm{ZB}(T,S_i)),\mathrm{MV})$ and with p level of the risk measure $\mathrm{CTE}_p[-X(T)] = \mathrm{TVaR}_p[-X(T)]$

- consider VaR and TVaR (=CTE)

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$$\text{TVaR}_p[-\text{ZB}^c(T,S_i)] = -\frac{e^{\Pi^Q(0,T,S_i)+\frac{1}{2}\Sigma(0,T,S_i)^2}}{1-p}\Phi(\Phi^{-1}(1-p)-\Sigma(0,T,S_i))$$

$$\text{TVaR}_p[-\text{ZB}^\ell(T,S_i)] = -\frac{e^{\Pi^Q(0,T,S_i)+\frac{1}{2}\Sigma(0,T,S_i)^2}}{1-p}\Phi(\Phi^{-1}(1-p)-r_i^Q\Sigma(0,T,S_i))$$

conditioning r.v. Λ^Q via r_i^Q

Numerical results

- parameters in G2++ model taken from Brigo and Mercurio (2001)

$$a = 0.773511777, \ b = 0.082013014,$$
$$\sigma = 0.022284644, \ \eta = 0.010382461, \ \rho = -0.701985206$$

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- yearly coupon $= 5.75\%$, maturity $T = 0.5$ year, market price of risk $= 0$
- VaR and TVaR for $p = 0.95$ and $p = 0.99$

	K_ℓ^* (TB)	K^* (S.E.)	K_c^*
$\text{VaR}_{0.95}$	1.007884	1.007883	1.007798
		(0.00001035)	
$\text{VaR}_{0.99}$	0.9969816	0.9969794	0.9968373
		(0.00001806)	
$\text{TVaR}_{0.95}$	1.001106	1.001111	1.000984
		(0.00001249743)	
$\text{TVaR}_{0.99}$	0.9918555	0.9918334	0.9916838
		(0.00002382441)	

$$n = 3$$

	K^*_ℓ (TB)	K^*_c
$\text{VaR}_{0.95}$	1.00677	1.006098
$\text{VaR}_{0.99}$	0.9754842	0.9743659
$\text{TVaR}_{0.95}$	0.987328	0.9863788
$\text{TVaR}_{0.99}$	0.961104	0.9597774

$$n = 10$$

Thank you for your attention!