A Singular Control Model with Application to the Goodwill Problem*

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March 28, 2007

Abstract

We consider a stochastic system whose uncontrolled state dynamics are modelled by a general one-dimensional Itô diffusion. The control effort that can be applied to this system takes the form that is associated with the so-called monotone follower problem of singular stochastic control. The control problem that we address aims at maximising a performance criterion that rewards high values of the utility derived from the system's controlled state but penalises any expenditure of control effort. This problem has been motivated by applications such as the so-called goodwill problem in which the system's state is used to represent the image that a product has in a market, while control expenditure is associated with raising the product's image, e.g., through advertising. We obtain the solution to the optimisation problem that we consider in a closed analytic form under rather general assumptions. Also, our analysis establishes a number of results that are concerned with analytic as well as probabilistic expressions for the first derivative of the solution to a second order linear non-homogeneous ordinary differential equation. These results have independent interest and can potentially be of use to the solution of other one-dimensional singular control problems.

1 Introduction

We consider a stochastic system whose state is modelled by the controlled, one-dimensional, positive Itô diffusion

$$dX_t = b(X_t)\,dt + dZ_t + \sigma(X_t)\,dW_t, \quad X_0 = x > 0,$$

*Research supported by EPSRC grant no. GR/S22998/01

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where W is a standard one-dimensional Brownian motion and the controlled process Z is a càglàd increasing process. The objective of the optimisation problem is to maximise the performance criterion

$$J_x(Z) = \limsup_{T\to\infty} \mathbb{E}\left[\int_0^T e^{-\Lambda_t^{r(X)}} h(X_t)\,dt - \int_{[0,T]} e^{-\Lambda_t^{r(X)}} k(X_t) \circ dZ_t\right], \tag{1}$$

over all admissible choices of Z, where

$$\Lambda_t^{r(X)} = \int_0^t r(X_v)\,dv,$$

and

$$\int_{[0,T]} e^{-\Lambda_t^{r(X)}} k(X_t) \circ dZ_t = \int_0^T e^{-\Lambda_t^{r(X)}} k(X_t)\,dZ_t^c + \sum_{0\le t\le T} \int_0^{\Delta Z_t} e^{-\Lambda_t^{r(X)}} k(X_t + s)\,ds. \tag{2}$$

This stochastic control problem has been motivated by the following application. Consider a company marketing a given product. This product has an image in a given market where it is being sold that evolves randomly over time. We use the random variable X_t to model the product's image at time t, for $t \ge 0$. The company marketing the product can raise its image by means of costly interventions such as advertising. We model the effect of these actions by means of the controlled process Z. The company's objective is to maximise the expected discounted utility derived from the product's image minus the expected discounted "dis-utility" resulting from intervention costs, which is reflected in the structure of the performance criterion given by (1).

Optimal control problems motivated by applications such as the one discussed briefly above have a long history and can be traced back to Nerlove and Arrow [NA62], who use deterministic dynamics to model the evolution of the product's image and consider a multi-objective performance criterion. Since then, deterministic optimal control models addressing this type of applications have attracted significant interest (see Buratto and Viscolani [BV02] and references therein). Several models in which the product's image evolves randomly over time, which are more realistic, and result in stochastic optimisation problems have also been studied in the literature (see Marinelli [Mar06] and references therein).

Singular stochastic control was introduced by Bather and Chernoff [BC67] who considered a simplified model of spaceship control. In their seminal paper, Beneš, Shepp and Witsenhausen [BSW80] were the first to solve rigorously an example of a finite-fuel singular control problem. Since then, the area has attracted considerable interest in the literature. Alvarez [A99, A01], Chow, Menaldi and Robin [CMR85], Davis and Zervos [DZ98], Fleming and Soner [FS93, Chapter VIII], Harrison and Taksar [HT83], Jacka [J83, J02], Karatzas [Ka83], Købila [Ko93], Ma [M92], Øksendal [Ø01], Shreve, Lehoczky and Gavers [SLG84], Soner and Shreve [SS89], Sun [Su87], Zhu [Z92], provide an incomplete list, in alphabetical order, of further important contributions. Other related contributions include Menaldi and Robin [MR84], Weerasinghe [W02], and Jack and Zervos [JZ06] who solve singular control problems with long-term average rather than discounted criteria. With regard to the structure of the performance criterion that we consider, penalising the expenditure of control effort by means of integrals as in (2) was introduced by Zhu [Z92] and was later adopted by Davis and Zervos [DZ98] and Jack and Zervos [JZ06].

We solve the problem that we consider by constructing a solution to the associated Hamilton-Jacobi-Bellman (HJB) equation in closed analytic form, under general assumptions. This is possible in the generality that we consider because the control problem's state space is one-dimensional. Explicitly solvable control problems have attracted significant interest in the literature for several reasons. First, some of them, such as the one that we solve here, are motivated by real-life applications. Second, they reveal the qualitative nature of the associated optimal control tactics, and they provide special cases that can be used to assess the efficiency of numerical techniques devised to address more complex problems, which is a major issue. The majority of such control problems assume that the system's uncontrolled dynamics are modelled by a Brownian motion with drift or a geometric Brownian motion. To the best of our knowledge, Alvarez [A01] and Jack and Zervos [JZ06] are the only references in the singular stochastic control literature in which closed-form solutions are derived when the system's uncontrolled dynamics are modelled by a general one-dimensional Itô diffusion. The latter reference considers a long-term average rather than an expected discounted performance criterion such as the one given by (1). On the other hand, Alvarez [A01] assumes that $h \equiv 0$ and that $k < 0$ is constant. The introduction of a non-trivial running payoff function h and a non-trivial running control expenditure function k of sign opposite to the one of h gives rise to a genuinely new problem that involves new analysis. At this point, we should mention that we are not aware of any reference addressing the problem that we solve, even for simple stochastic dynamics.

The paper is organised as follows. Section 2 is concerned with the formulation of the singular stochastic control problem that we solve. In this section, we also develop all of the assumptions on the problem data that we make in the paper, and we prove that these are sufficient for our optimisation problem to be well-posed. Section 3 is concerned with properties of the solution to a non-homogeneous second-order linear ordinary differential equation that plays an important role in our analysis. All of the claims that we make there without proof are standard and can be found in several references, including Feller [F54], Breiman [B68], Mandl [Man68], Itô and McKean [IM96], Karlin and Taylor [KT81], Rogers and Williams [RW00], and Borodin and Salminen [BS02]. The results that we prove are new and have independent interest because they can be of use in the solution of other stochastic control problems. Some of the assumptions made here can be relaxed. However, we decided against such relaxations because they would complicate significantly the exposition. In Section 4, we solve the stochastic control problem that we consider. Finally, Section 5 is concerned with special cases that arise when h is a power utility function, k and r are constants, and the uncontrolled state space dynamics are modelled by a geometric Brownian motion (Section 5.1) or a mean-reverting square-root process such as the one in the Cox-Ingersoll-Ross interest rate model (Section 5.2).

2 The singular stochastic control problem

We fix a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ satisfying the usual conditions and carrying a standard one-dimensional $(\mathcal{F}_t)$-Brownian motion W. We consider a stochastic system whose uncontrolled dynamics are modelled by the Itô diffusion associated with the stochastic differential equation (SDE)

$$dX_t^0 = b(X_t^0)\, dt + \sigma(X_t^0)\, dW_t, \quad X_0^0 = x > 0, \tag{3}$$

and we impose the following assumption.

Assumption 1 The functions $b, \sigma :]0, \infty[\to \mathbb{R}$ are C^1, σ' is locally Lipschitz, and $\sigma^2(x) > 0$, for all $x > 0$. □

This assumption implies that (3) has a unique strong solution. It also implies that the scale function p_{X^0} and the speed measure m_{X^0} given by

$$p_{X^0}(c) = 0, \quad p'_{X^0}(x) = \exp\left(-2\int_c^x \frac{b(s)}{\sigma^2(s)}\, ds\right), \tag{4}$$

and

$$m_{X^0}(dx) = \frac{2}{\sigma^2(x) p'_{X^0}(x)}\, dx, \tag{5}$$

respectively, for some $c > 0$ fixed, are well-defined. Additionally, we assume that the solution to (3) is non-explosive, so that, given any initial condition x, $X_t^0 \in]0, \infty[$, for all $t \geq 0$, with probability 1.

Assumption 2 The Itô diffusion X^0 defined by (3) is non-explosive. □

Feller's test for explosions (see Theorem 5.5.29 in Karatzas and Shreve [KS88]) provides a necessary and sufficient condition for this assumption to hold true. Indeed, if we define

$$l_{X^0}(x) = \int_c^x \left[p_{X^0}(x) - p_{X^0}(s)\right] m_{X^0}(ds),$$

then Assumption 2 is satisfied if and only if $\lim_{x\downarrow 0} l_{X^0}(x) = \lim_{x\to\infty} l_{X^0}(x) = \infty$.

Now, we model the system's controlled dynamics by the SDE

$$dX_t = b(X_t)\, dt + dZ_t + \sigma(X_t)\, dW_t, \quad X_0 = x > 0, \tag{6}$$

where the controlled process Z is an increasing process. With each admissible intervention strategy Z (see Definition 1 below), we associate the performance criterion

$$J_x(Z) = \limsup_{T\to\infty} \mathbb{E}\left[\int_0^T e^{-\Lambda_t^{r(X)}} h(X_t)\, dt - \int_{[0,T]} e^{-\Lambda_t^{r(X)}} k(X_t) \circ dZ_t\right], \tag{7}$$

where

$$\Lambda_t^{r(X)} = \int_0^t r(X_v)\, dv, \tag{8}$$

for some functions $h :]0,\infty[\to \mathbb{R}$ and $k, r :]0,\infty[\to \mathbb{R}_+$. The integral with respect to Z is defined by

$$\int_{[0,T]} e^{-\Lambda_t^{r(X)}} k(X_t) \circ dZ_t = \int_0^T e^{-\Lambda_t^{r(X)}} k(X_t)\, dZ_t^c + \sum_{0\le t\le T} \int_0^{\Delta Z_t} e^{-\Lambda_t^{r(X)}} k(X_t + s)\, ds, \tag{9}$$

where Z^c is the continuous part of the increasing process Z.

Definition 1 The family $\mathcal{A}$ of all *admissible intervention strategies* is the set of all $(\mathcal{F}_t)$-adapted càglàd processes Z with increasing sample paths such that $Z_0 = 0$, (6) has a unique non-explosive strong solution, and

$$\mathbb{E}\left[\int_{[0,T]} e^{-\Lambda_t^{r(X)}} k(X_t) \circ dZ_t\right] < \infty, \quad \text{for all } T > 0.$$

The objective of our control problem is to maximise J_x over all admissible strategies. Accordingly, we define the problem's value function v by

$$v(x) = \sup_{Z\in\mathcal{A}} J_x(Z), \quad \text{for } x > 0.$$

For our optimisation problem to be well-posed and to admit a solution that conforms with economic intuition, we need to make additional assumptions.

Assumption 3 The discounting factor r is C^1, r' is locally Lipschitz, and there exists a constant $r_0 > 0$ such that $r(x) \ge r_0$, for all $x > 0$. □

Our analysis will also involve the SDE

$$dY_t^0 = \mu(Y_t^0)\, dt + \sigma(Y_t^0)\, dW_t, \quad Y_0^0 = x > 0, \tag{10}$$

where

$$\mu(x) = b(x) + \sigma(x)\sigma'(x) - \tfrac{1}{2}\sigma^2(x)\frac{r'(x)}{r(x)}. \tag{11}$$

In the presence of Assumptions 1 and 3, this SDE has a unique strong solution. Also, a straightforward calculation shows that the scale function p_{Y^0} of this diffusion satisfies

$$p'_{Y^0}(x) := \exp\left(-\int_c^x \frac{2\mu(s)}{\sigma^2(s)}\,ds\right) = \frac{\sigma^2(c)}{r(c)}\frac{r(x)}{\sigma^2(x)}p'_{X^0}(x), \tag{12}$$

where p_{X^0} is the scale function of the diffusion X^0 defined by (4). We make the following additional assumption.

Assumption 4 The Itô diffusion Y^0 defined by (10) is non-explosive. □

Throughout the paper, we denote by K the function defined by

$$K(x) = \int_0^x k(s)\,ds, \tag{13}$$

and, given a C^2 function w, we denote by $\mathcal{L}_X w$ the function given by

$$\mathcal{L}_X w(x) = \tfrac{1}{2}\sigma^2(x)w''(x) + b(x)w'(x) - r(x)w(x). \tag{14}$$

Also, we define

$$Q(x) = h(x) + \mathcal{L}_X K(x). \tag{15}$$

We can now complete the list of assumptions made in the paper.

Assumption 5 The following conditions hold:
(a) The running payoff function h is C^1 and bounded from below.
(b) The running cost function k is C^1. Also, $k(x) \geq 0$, for all $x > 0$, and the function K defined by (13) is real-valued.
(c) The problem data is such that

$$\rho(x) := \frac{r^2(x) - r(x)b'(x) + r'(x)b(x)}{r(x)} \geq r_0, \quad \text{for all } x > 0, \tag{16}$$

where the constant r_0 is the same as in Assumption 3, without loss of generality.
(d) There exists a real $x^* \geq 0$ such that

$$\mathcal{D}_r Q(x) := \frac{r(x)Q'(x) - r'(x)Q(x)}{r(x)} \begin{cases} \geq 0, & \text{for } x \leq x^*, \text{ if } x^* > 0, \\ < 0, & \text{for } x > x^*, \end{cases}$$

where the function Q is defined by (15). Also, if $x^* = 0$, then $\lim_{x\downarrow 0} Q(x)/r(x) < \infty$.
(e) The integrability condition

$$\mathbb{E}\left[\int_0^\infty e^{-\Lambda_t^{r(X^0)}}\left[\left|h(X_t^0)\right| + \left|\mathcal{L}_X K(X_t^0)\right|\right] dt\right] < \infty \tag{17}$$

is satisfied, and

$$K(x) = -\mathbb{E}\left[\int_0^\infty e^{-\Lambda_t^{r(X^0)}} \mathcal{L}_X K(X_t^0)\, dt\right], \tag{18}$$

for every initial condition $x > 0$. □

Remark 1 It is worth noting that (18) is in essence an integrability condition. Indeed, an application of Itô's formula yields

$$e^{-\Lambda_T^{r(X^0)}} K(X_{T+}^0) = K(x) + \int_0^T e^{-\Lambda_t^{r(X^0)}} \mathcal{L}_X K(X_t^0)\, dt + \int_0^T e^{-\Lambda_t^{r(X^0)}} \sigma(X_t^0) k(X_t^0)\, dW_t. \tag{19}$$

If we assume that the stochastic integral appearing in this identity is a martingale and that the so-called transversality condition

$$\liminf_{T\to\infty} \mathbb{E}\left[e^{-\Lambda_T^{r(X^0)}} K(X_{T+}^0)\right] = 0$$

holds, then we can take expectations in (19) and then pass to the limit $T \to \infty$ to obtain (18). □

Remark 2 For future reference, we observe that the integrability condition (17) implies

$$\mathbb{E}\left[\int_0^\infty e^{-\Lambda_t^{r(X^0)}} \left|Q(X_t^0)\right| dt\right] < \infty.$$

Furthermore, if we define

$$R_{X^0,h}(x) = \mathbb{E}\left[\int_0^\infty e^{-\Lambda_t^{r(X^0)}} h(X_t^0)\, dt\right], \tag{20}$$

then

$$R_{X^0,h}(x) - K(x) = \mathbb{E}\left[\int_0^\infty e^{-\Lambda_t^{r(X^0)}} Q(X_t^0)\, dt\right] =: R_{X^0,Q}(x), \tag{21}$$

thanks to (18). □

The following result is concerned with the well-posedness of our optimisation problem.

Lemma 1 *Suppose that Assumptions 1, 2, 3, 5.(a), 5.(b) and 5.(d) hold true, and fix any initial condition $x > 0$. The performance index $J_x(Z)$ is well-defined for every admissible intervention strategy $Z \in \mathcal{A}$, and $v(x) < \infty$.*

Proof. Fix any initial condition $x > 0$ and any admissible intervention process $Z \in \mathcal{A}$. Using Itô's formula, the fact that $\Delta X_t = \Delta Z_t$ and the definition (13) of the function K, we obtain

$$
\begin{aligned}
e^{-\Lambda_T^{r(X)}} K(X_{T+}) &= K(x) + \int_0^T e^{-\Lambda_t^{r(X)}} \mathcal{L}_X K(X_t)\, dt + \int_0^T e^{-\Lambda_t^{r(X)}} k(X_t)\, dZ_t \\
&\quad + \sum_{t\in[0,T]} e^{-\Lambda_t^{r(X)}} \left[K(X_{t+}) - K(X_t) - k(X_t)\Delta X_t\right] + M_T \\
&= K(x) + \int_0^T e^{-\Lambda_t^{r(X)}} \mathcal{L}_X K(X_t)\, dt + \int_0^T e^{-\Lambda_t^{r(X)}} k(X_t)\, dZ_t^c \\
&\quad + \sum_{t\in[0,T]} e^{-\Lambda_t^{r(X)}} \int_{X_t}^{X_{t+}} k(s)\, ds + M_T,
\end{aligned}
$$

where

$$
M_T = \int_0^T e^{-\Lambda_t^{r(X)}} \sigma(X_t) k(X_t)\, dW_t.
$$

In view of (9) and the definition (15) of Q, this calculation implies

$$
\begin{aligned}
\int_0^T e^{-\Lambda_t^{r(X)}} h(X_t)\, dt - \int_{[0,T]} e^{-\Lambda_t^{r(X)}} k(X_t) \circ dZ_t &= K(x) - e^{-\Lambda_T^{r(X)}} K(X_{T+}) \\
&\quad + \int_0^T e^{-\Lambda_t^{r(X)}} Q(X_t)\, dt + M_T. \qquad (22)
\end{aligned}
$$

Now Assumption 5.(d) implies that the function Q/r is bounded from above. It follows that

$$
\int_0^T e^{-\Lambda_t^{r(X)}} Q(X_t)\, dt \le \sup_{s>0} \frac{Q(s)}{r(s)} \int_0^\infty e^{-\Lambda_t^{r(X)}} r(X_t)\, dt = \sup_{s>0} \frac{Q(s)}{r(s)}. \qquad (23)
$$

Also, the assumption that h is bounded from below implies

$$
\int_0^T e^{-\Lambda_t^{r(X)}} h(X_t)\, dt \ge \inf_{s>0} h(s) \int_0^\infty e^{-\Lambda_t^{r(X)}}\, dt \ge \frac{1}{r_0} \inf_{s>0} h(s),
$$

where r_0 is as in Assumption 3. In light of these inequalities, we can see that

$$
\inf_{t\le T} M_t \ge \frac{1}{r_0} \inf_{s>0} h(s) - \sup_{s>0} \frac{Q(s)}{r(s)} - \int_0^x k(s)\, ds - \int_{[0,T]} e^{-\Lambda_t^{r(X)}} k(X_t) \circ dZ_t.
$$

The admissibility of Z implies that the right-hand side of this inequality belongs to $\mathcal{L}^1$, for all $T > 0$. It follows that M is a supermartingale. Combining this observation with (23), we can see that (22) implies

$$
\mathbb{E}\left[\int_0^T e^{-\Lambda_t^{r(X)}} h(X_t)\, dt - \int_{[0,T]} e^{-\Lambda_t^{r(X)}} k(X_t) \circ dZ_t\right] \le \sup_{s>0} \frac{Q(s)}{r(s)} + \int_0^x k(s)\, ds.
$$

However, this inequality establishes the claims made. □

3 The solution to a second order linear ODE

In this section, we review some properties of the solution to the ODE

$$\mathcal{L}_X w(x) + G(x) \equiv \tfrac{1}{2}\sigma^2(x)w''(x) + b(x)w'(x) - r(x)w(x) + G(x) = 0, \tag{24}$$

that is associated with the Itô diffusion (3), and we establish some new results that are concerned with appropriate analytic and probabilistic expressions for the derivative w' of a solution w to (24). Indeed, if G is absolutely continuous, then differentiating the ODE (24) yields

$$\tfrac{1}{2}\sigma^2(x)w'''(x) + [b(x) + \sigma(x)\sigma'(x)]\, w''(x) - [r(x) - b'(x)]\, w'(x) - r'(x)w(x) + G'(x) = 0.$$

Using (24) once again to eliminate $w(x)$ from this equation, we can see that w' solves the ODE

$$\mathcal{L}_Y u(x) + \mathcal{D}_r G(x) := \tfrac{1}{2}\sigma^2(x)u''(x) + \mu(x)u'(x) - \rho(x)u(x) + \mathcal{D}_r G(x) = 0, \tag{25}$$

where μ is defined by (11), ρ is defined by (16), and

$$\mathcal{D}_r G(x) := \frac{r(x)G'(x) - r'(x)G(x)}{r(x)} = r(x)\frac{d}{dx}\left(\frac{G(x)}{r(x)}\right). \tag{26}$$

It follows that w' satisfies a second order linear ODE that is similar to (24) and is associated with the SDE (10).

In the presence of Assumptions 1, 2 and 3, the general solution to the homogeneous ODE $\mathcal{L}_X w(x) = 0$, which is associated with (24), is given by

$$w(x) = A\phi(x) + B\psi(x),$$

for some constants $A, B \in \mathbb{R}$, where ϕ and ψ are C^2 functions such that

$$0 < \phi(x) \quad \text{and} \quad \phi'(x) < 0, \quad \text{for all } x > 0, \tag{27}$$
$$0 < \psi(x) \quad \text{and} \quad \psi'(x) > 0, \quad \text{for all } x > 0, \tag{28}$$

and

$$\lim_{x \downarrow 0} \phi(x) = \lim_{x \to \infty} \psi(x) = \infty. \tag{29}$$

These functions are unique, modulo multiplicative constants. To simplify the notation we assume, without loss of generality, that

$$\phi(c) = \psi(c) = 1, \tag{30}$$

where $c > 0$ is the same constant as the one that we used in the definition (4) of the scale function p_{X^0}. Also, these functions satisfy

$$\phi(x)\psi'(x) - \phi'(x)\psi(x) = Cp'_{X^0}(x), \tag{31}$$

where

$$C := [\psi'(c) - \phi'(c)] > 0. \tag{32}$$

Furthermore, we can use the fact that ϕ and ψ satisfy the ODE $\mathcal{L}_X w(x) = 0$ to verify that

$$\begin{aligned} \phi''(x)\psi'(x) - \phi'(x)\psi''(x) &= \frac{2r(x)}{\sigma^2(x)} [\phi(x)\psi'(x) - \phi'(x)\psi(x)] \\ &= \frac{2Cr(x)}{\sigma^2(x)} p'_{X^0}(x). \end{aligned} \tag{33}$$

Similarly, Assumptions 1, 3, 4 and 5.(c) guarantee that the general solution to the homogeneous ODE $\mathcal{L}_Y u(x) = 0$, which is associated with (25), is given by

$$u(x) = \tilde{A}\tilde{\phi}(x) + \tilde{B}\tilde{\psi}(x),$$

for some constants $\tilde{A}, \tilde{B} \in \mathbb{R}$, where $\tilde{\phi}$ and $\tilde{\psi}$ are C^2 functions satisfying

$$0 < \tilde{\phi}(x) \quad \text{and} \quad \tilde{\phi}'(x) < 0, \quad \text{for all } x > 0, \tag{34}$$

$$0 < \tilde{\psi}(x) \quad \text{and} \quad \tilde{\psi}'(x) > 0, \quad \text{for all } x > 0, \tag{35}$$

and

$$\lim_{x \downarrow 0} \tilde{\phi}(x) = \lim_{x \to \infty} \tilde{\psi}(x) = \infty, \tag{36}$$

Once again, we assume, without loss of generality, that

$$\tilde{\phi}(c) = \tilde{\psi}(c) = 1, \tag{37}$$

for notational simplicity. Also, we note that

$$\tilde{\phi}(x)\tilde{\psi}'(x) - \tilde{\phi}'(x)\tilde{\psi}(x) = \tilde{C}p'_{Y^0}(x),$$

where

$$\tilde{C} := \tilde{\psi}'(c) - \tilde{\phi}'(c) > 0. \tag{38}$$

The following example shows that Assumption 5.(c) is indispensable if we want the functions $\tilde{\phi}$ and $\tilde{\psi}$ to have the properties (34)–(36), which are essential for constructing the solutions to many one-dimensional stochastic control problems, including the one that we study in this paper.

Example 1 Suppose that $b(x) = 2x$, $\sigma(x) = \sqrt{2}x$ and $r(x) = \frac{3}{4}$. In this case, $\rho(x) = -\frac{5}{4}$ and Assumption 5.(c) is not satisfied. A simple calculation reveals that the functions

$$\phi(x) = x^{-\frac{3}{2}}, \quad \psi(x) = x^{\frac{1}{2}}$$

span the solution space of the ODE $\mathcal{L}_X w(x) = 0$, and that the functions

$$\tilde{\phi}(x) = x^{-\frac{5}{2}}, \quad \tilde{\psi}(x) = x^{-\frac{1}{2}}$$

span the solution space of the ODE $\mathcal{L}_Y u(x) = 0$. Clearly, $\tilde{\psi}$ does not satisfy (35) and (36).

The next result expresses the functions $\tilde{\phi}$ and $\tilde{\psi}$ in terms of the functions ϕ and ψ.

Proposition 2 *Suppose that Assumptions 1, 2, 3, 4 and 5.(c) hold. If ϕ, ψ and $\tilde{\phi}$, $\tilde{\psi}$ are the functions satisfying (27)–(30) and (34)–(37), and spanning the solution space of the homogeneous ODEs $\mathcal{L}_X w(x) = 0$ and $\mathcal{L}_Y u(x) = 0$ associated with (24) and (25), respectively, then*

$$\tilde{\phi}(x) = \frac{1}{\phi'(c)}\phi'(x) \quad \text{and} \quad \tilde{\psi}(x) = \frac{1}{\psi'(c)}\psi'(x). \tag{39}$$

Also, if C and $\tilde{C}$ are the constants defined by (32) and (38), respectively, then

$$\tilde{C} = -\frac{2r(c)}{\sigma^2(c)}\frac{1}{\phi'(c)\psi'(c)}C. \tag{40}$$

Proof. We first show that $\tilde{\psi} = C_\psi \psi'$, for some constant $C_\psi > 0$. To this end, we define the function $\hat{\psi}$ by

$$\hat{\psi}(x) = \int_1^x \tilde{\psi}(s)\, ds. \tag{41}$$

Given an absolutely continuous function $f :]0, \infty[\to \mathbb{R}$ with compact support, we can use the integration by parts formula to calculate

$$\int_0^\infty f'(s)\mathcal{L}_X\hat{\psi}(s)\, ds = -\int_0^\infty f(s)\left[\mathcal{L}_Y\hat{\psi}'(s) + \frac{r'(s)}{r(s)}\mathcal{L}_X\hat{\psi}(s)\right] ds$$

Since $\hat{\psi}' = \tilde{\psi}$ satisfies the ODE $\mathcal{L}_Y u(x) = 0$, it follows that

$$\int_0^\infty \frac{r(s)f'(s) + r'(s)f(s)}{r(s)}\mathcal{L}_X\hat{\psi}(s)\, ds = 0. \tag{42}$$

Now, fix any $a > 0$ and any $z \in]0, a[$, and define

$$q_{a,z}(x) = \begin{cases} 1, & \text{if } x \in [a - z, a], \\ -1, & \text{if } x \in]a, a + z], \\ 0, & \text{otherwise.} \end{cases}$$

If we choose

$$f(x) = \frac{1}{r(x)} \int_0^x q_{a,z}(s)\, ds,$$

then f is absolutely continuous with compact support, and (42) yields

$$\int_{a-z}^{a} \frac{\mathcal{L}_X \hat{\psi}(s)}{r(s)}\, ds = \int_a^{a+z} \frac{\mathcal{L}_X \hat{\psi}(s)}{r(s)}\, ds.$$

Since $z \in]0, a[$ has been arbitrary, we can differentiate this expression with respect to z to obtain

$$\frac{\mathcal{L}_X \hat{\psi}(a - z)}{r(a - z)} = \frac{\mathcal{L}_X \hat{\psi}(a + z)}{r(a + z)}, \quad \text{for all } z \in]0, a[.$$

It follows that the function $\mathcal{L}_X \hat{\psi}/r$ has even symmetry around the point a. However, since $a > 0$ has been arbitrary, this can be true only if $\mathcal{L}_X \hat{\psi}/r$ is constant. Therefore, there exists a constant $C_1 \in \mathbb{R}$ such that $\mathcal{L}_X \left(\hat{\psi}(x) + C_1 \right) = 0$. If we combine this observation with the facts that

$$\hat{\psi} \text{ is strictly increasing}, \quad \lim_{x \downarrow 0} \hat{\psi}(x) \in \mathbb{R} \quad \text{and} \quad \lim_{x \to \infty} \hat{\psi}(x) = \infty,$$

which follow from the definition (41) of $\hat{\psi}$ and the properties (35)–(36) of the function $\tilde{\psi}$, and the properties (27)–(29) of the functions ϕ, ψ that span the solution space of $\mathcal{L}_X w(x)$, we can conclude that $\hat{\psi} + C_1 = C_\psi \psi$, for some constant $C_\psi > 0$. However, this conclusion establishes the identity $\tilde{\psi} = C_\psi \psi'$.

Now, since ϕ' and $\psi' = C_\psi^{-1} \tilde{\psi}$ are independent solutions to the ODE $\mathcal{L}_Y u(x) = 0$ and $\mathcal{L}_Y \tilde{\phi}(x) = 0$, there exist constants C_ϕ and Γ such that $\tilde{\phi} = -C_\phi \phi' + \Gamma \tilde{\psi}$. However, the limits

$$\lim_{x \to \infty} \tilde{\phi}(x) \in [0, \infty[, \quad \lim_{x \to \infty} \phi'(x) = 0 \quad \text{and} \quad \lim_{x \to \infty} \tilde{\psi}(x) = \infty$$

imply that $\Gamma = 0$, which proves that $\tilde{\phi} = -C_\phi \phi'$, for some constant $C_\phi > 0$.

Finally, we note that

$$-C_\phi = \frac{1}{\phi'(c)} \quad \text{and} \quad C_\psi = \frac{1}{\psi'(c)}$$

are the only choices for the constants C_ϕ and C_ψ that are compatible with (37). Also, (40) follows by a simple calculation involving the definitions of the constants C and $\tilde{C}$, (39) and (33). □

To proceed further, consider the solution X^0 to the SDE (3) and we define

$$\Lambda_t^{r(X^0)} = \int_0^t r(X_v^0)\, dv.$$

We recall that, if G is a measurable function, then

$$\mathbb{E}\left[\int_0^\infty e^{-\Lambda_t^{r(X^0)}} \left|G(X_t^0)\right| dt\right] < \infty \tag{43}$$

if and only if

$$\int_0^x \frac{|G(s)|\psi(s)}{\sigma^2(s)p'_{X^0}(s)}\, ds + \int_x^\infty \frac{|G(s)|\phi(s)}{\sigma^2(s)p'_{X^0}(s)}\, ds < \infty. \tag{44}$$

In the presence of these equivalent integrability conditions, the function $R_{X^0,G}$ defined by

$$R_{X^0,G}(x) = \mathbb{E}\left[\int_0^\infty e^{-\Lambda_t^{r(X^0)}} G(X_t^0)\, dt\right], \tag{45}$$

admits the analytic expression

$$R_{X^0,G}(x) = \frac{2}{C}\phi(x)\int_0^x \frac{G(s)\psi(s)}{\sigma^2(s)p'_{X^0}(s)}\, ds + \frac{2}{C}\psi(x)\int_x^\infty \frac{G(s)\phi(s)}{\sigma^2(s)p'_{X^0}(s)}\, ds, \tag{46}$$

where $C > 0$ is the constant defined by (32), and is a special solution to (24).

At this point, we establish the following technical result that we will need.

Lemma 3 *Suppose that Assumptions 1, 2 and 3 hold. If G is a function satisfying (43) and (44), then*

$$\liminf_{x\downarrow 0} \frac{|G(x)|\psi'(x)}{r(x)p'_{X^0}(x)} = \liminf_{x\to\infty} \frac{|G(x)|\phi'(x)}{r(x)p'_{X^0}(x)} = 0. \tag{47}$$

Proof. In view of (31), we can see that

$$0 < \frac{\phi(x)\psi'(x)}{Cp'_{X^0}(x)} < 1 \quad \text{and} \quad 0 < -\frac{\phi'(x)\psi(x)}{Cp'_{X^0}(x)} < 1,$$

which, combined with (29) implies

$$\lim_{x\downarrow 0}\frac{\psi'(x)}{p'_{X^0}(x)} = \lim_{x\to\infty}\frac{\phi'(x)}{p'_{X^0}(x)} = 0. \tag{48}$$

Also, the calculation

$$\frac{d}{dx}\left(\frac{1}{p'_{X^0}(x)}\right) = \frac{2b(x)}{\sigma^2(x)p'_{X^0}(x)}$$

and the fact that ϕ satisfies the ODE $\mathcal{L}_X w(x) = 0$, imply

$$\begin{aligned}\frac{d}{dx}\left(\frac{\phi'(x)}{p'_{X^0}(x)}\right) &= \frac{2}{\sigma^2(x)p'_{X^0}(x)}\left[\tfrac{1}{2}\sigma^2(x)\phi''(x) + b(x)\phi'(x)\right]\\ &= \frac{2r(x)\phi(x)}{\sigma^2(x)p'_{X^0}(x)}.\end{aligned} \tag{49}$$

Similarly, we can show that

$$\frac{d}{dx}\left(\frac{\psi'(x)}{p'_{X^0}(x)}\right) = \frac{2r(x)\psi(x)}{\sigma^2(x)p'_{X^0}(x)}. \tag{50}$$

Now, we consider any sequence (x_n) such that

$$0 < x_n < \frac{1}{n} \quad \text{and} \quad \frac{|G(x_n)|}{r(x_n)} \leq \inf_{x\in\left]0,\frac{1}{n}\right[}\frac{|G(x)|}{r(x)} + 1.$$

Using (48) and (49), we calculate

$$\begin{aligned}\int_0^{x_n}\frac{2|G(s)|\psi(s)}{\sigma^2(s)p'_{X^0}(s)}\,ds &\geq \left(\frac{|G(x_n)|}{r(x_n)} - 1\right)\int_0^{x_n} d\left(\frac{\psi'(s)}{p'_{X^0}(s)}\right)\\ &= \left(\frac{|G(x_n)|}{r(x_n)} - 1\right)\frac{\psi'(x_n)}{p'_{X^0}(x_n)}.\end{aligned}$$

In view of (48) and the fact that $\lim_{n\to\infty}\int_0^{x_n}\frac{2|G(s)|\psi(s)}{\sigma^2(s)p'_{X^0}(s)}\,ds = 0$, we can pass to the limit $n\to\infty$ in this inequality to obtain

$$\lim_{n\to\infty}\frac{|G(x_n)|\psi'(x_n)}{r(x_n)p'_{X^0}(x_n)} = 0,$$

which proves that $\liminf_{x\downarrow 0}\frac{|G(x)|\psi'(x)}{r(x)p'_{X^0}(x)} = 0$. Using similar arguments, we can also show that $\liminf_{x\to\infty}\frac{|G(x)|\phi'(x)}{r(x)p'_{X^0}(x)} = 0$. $\square$

Remark 3 It is worth noting that the conclusions of the preceding result cannot be strengthened. To see this, suppose that $b(x) = -x$, $\sigma(x) = \sqrt{2}x$ and $r(x) = 3$, so that

$$\phi(x) = 1/x, \quad \psi(x) = x^3 \quad \text{and} \quad p'_{X^0}(x) = x.$$

Also, let G be the positive function given by

$$G(x) = \sum_{n=1}^{\infty} G^{(n)}(x),$$

where $G^{(n)}$ is the tent-like function defined, for $n \geq 1$ by

$$G^{(n)}(x) = \begin{cases} n^4 x + n - n^3, & \text{for } x \in \left[\frac{1}{n} - \frac{1}{n^3}, \frac{1}{n}\right[, \\ -n^4 x + n + n^3, & \text{for } x \in \left[\frac{1}{n}, \frac{1}{n} + \frac{1}{n^3}\right[, \\ 0, & \text{otherwise.} \end{cases}$$

Given any $x > 0$, we calculate

$$\begin{aligned} \int_0^x \frac{|G(s)|\psi(s)}{\sigma^2(s)p'_{X^0}(s)} ds &\leq \int_0^2 G(s) ds \\ &= \sum_{n=1}^{\infty} \frac{1}{2} n \left[\frac{1}{n} + \frac{1}{n^3} - \left(\frac{1}{n} - \frac{1}{n^3}\right)\right] \\ &< \infty, \end{aligned}$$

and we can immediately see that

$$\int_x^{\infty} \frac{|G(s)|\phi(s)}{\sigma^2(s)p'_{X^0}(s)} ds < \infty.$$

It follows that the assumptions of Lemma 3 are satisfied. However,

$$\limsup_{x \downarrow 0} \frac{|G(x)|\psi'(x)}{r(x)p'_{X^0}(x)} \geq \lim_{n \to \infty} \frac{1}{n} G^{(n)}\left(\frac{1}{n}\right) > 0.$$

□

In what follows, we assume that G is absolutely continuous. Also, we consider the solution Y^0 to the SDE (10) and we define

$$\Lambda_t^{\rho(Y^0)} = \int_0^t \rho(Y_v^0)\, dv,$$

If $\mathcal{D}_r G$ is defined by (26), then the standard theory discussed above Lemma 3 implies that

$$\mathbb{E}\left[\int_0^\infty e^{-\Lambda_t^{\rho(Y^0)}} \left|\mathcal{D}_r G(Y_t^0)\right| dt\right] < \infty \tag{51}$$

if and only if

$$\int_0^x \frac{|\mathcal{D}_r G(s)|\tilde{\psi}(s)}{\sigma^2(s)p'_{Y^0}(s)}\, ds + \int_x^\infty \frac{|\mathcal{D}_r G(s)|\tilde{\phi}(s)}{\sigma^2(s)p'_{Y^0}(s)}\, ds < \infty. \tag{52}$$

In light of (39) in Proposition 2 and (12), we can see that (52) is true if and only if

$$\int_0^x \frac{|\mathcal{D}_r G(s)|\psi'(s)}{r(s)p'_{X^0}(s)}\, ds + \int_x^\infty \frac{|\mathcal{D}_r G(s)|[-\phi'](s)}{r(s)p'_{X^0}(s)}\, ds < \infty. \tag{53}$$

Furthermore, if these equivalent conditions hold true, then the function $R_{Y^0,\mathcal{D}_r G}$ defined by

$$R_{Y^0,\mathcal{D}_r G}(x) = \mathbb{E}\left[\int_0^\infty e^{-\Lambda_t^{\rho(Y^0)}} \mathcal{D}_r G(Y_t^0)\, dt\right]$$

admits the analytic expressions

$$\begin{aligned} R_{Y^0,\mathcal{D}_r G}(x) &= \frac{2}{\tilde{C}}\tilde{\phi}(x)\int_0^x \frac{(\mathcal{D}_r G)(s)\tilde{\psi}(s)}{\sigma^2(s)p'_{Y^0}(s)}\, ds + \frac{2}{\tilde{C}}\tilde{\psi}(x)\int_x^\infty \frac{(\mathcal{D}_r G)(s)\tilde{\phi}(s)}{\sigma^2(s)p'_{Y^0}(s)}\, ds \\ &= \frac{1}{C}[-\phi'](x)\int_0^x \frac{(\mathcal{D}_r G)(s)\psi'(s)}{r(s)p'_{X^0}(s)}\, ds + \frac{1}{C}\psi'(x)\int_x^\infty \frac{(\mathcal{D}_r G)(s)[-\phi'](s)}{r(s)p'_{X^0}(s)}\, ds, \end{aligned} \tag{54}$$

where C and $\tilde{C}$ are defined by (32) and (38), respectively, and provides a solution to the ODE $\mathcal{L}_Y u(x) + \mathcal{D}_r G(x) = 0$. Once again, the second equality here follows from Proposition 2 and (12).

Proposition 4 *Suppose that Assumptions 1, 2, 3, 4 and 5.(c) hold true, and let G be a function satisfying (43) and (44). Also, suppose that there exists a constant $\varepsilon > 0$ such that*

$$\text{either } \mathcal{D}_r G(s) \le 0, \text{ for all } s \le \varepsilon, \text{ or } \mathcal{D}_r G(s) \ge 0, \text{ for all } s \le \varepsilon$$

and

$$\text{either } \mathcal{D}_r G(s) \le 0, \text{ for all } s \ge 1/\varepsilon, \text{ or } \mathcal{D}_r G(s) \ge 0, \text{ for all } s \ge 1/\varepsilon.$$

Under these conditions, the integrability condition (53) holds true, the function $R_{Y^0,\mathcal{D}_r G}$ is well-defined and real-valued, and

$$R'_{X^0,G}(x) = R_{Y^0,\mathcal{D}_r G}(x), \quad \text{for all } x > 0.$$

Proof. In view of Lemma 3, (49), (50) and the relationship (26) of the functions G and $\mathcal{D}_r G$, we can use the integration by parts formula to obtain

$$\int_0^x \frac{(\mathcal{D}_r G)(s)\psi'(s)}{r(s)p'_{X^0}(s)}\,ds = \frac{G(x)}{r(x)}\frac{\psi'(x)}{p'_{X^0}(x)} - 2\int_0^x \frac{G(s)\psi(s)}{\sigma^2(s)p'_{X^0}(s)}\,ds$$

and

$$\int_x^\infty \frac{(\mathcal{D}_r G)(s)\,[-\phi'](s)}{r(s)p'_{X^0}(s)}\,ds = \frac{G(x)}{r(x)}\frac{\phi'(x)}{p'_{X^0}(x)} + 2\int_x^\infty \frac{G(s)\phi(s)}{\sigma^2(s)p'_{X^0}(s)}\,ds.$$

However, combining these expressions and the expressions (46) and (54) for $R_{X^0,G}$ and $R_{Y^0,\mathcal{D}_r G}$ with the assumptions on $\mathcal{D}_r G$ and its local integrability we can see that all of the statements made hold true. □

Remark 4 If we remove the assumption that $\mathcal{D}_r G(x)$ has constant sign for all x sufficiently small and for all x sufficiently large, then the conclusions of the result above, (52) in particular, do not necessarily hold. To see this, suppose that b, σ and r are as in Remark 3, and let G be the function given by

$$G(x) = \sum_{n=1}^{\infty} G^{(n)}(x),$$

where $G^{(n)}$ is the tent-like function defined by

$$G^{(n)}(x) = \begin{cases} 2n(2n+1)x - 2n, & \text{for } x \in \left[\frac{1}{2n+1}, \frac{1}{2n}\right], \\ -2n(2n-1)x + 2n, & \text{for } x \in \left[\frac{1}{2n}, \frac{1}{2n-1}\right], \\ 0, & \text{otherwise.} \end{cases}$$

Plainly, G satisfies (43) and (44) because it is bounded. However, the calculation

$$\begin{aligned} \int_0^x \frac{|\mathcal{D}_r G(s)|\psi'(s)}{r(s)p'_{X^0}(s)} ds &= \int_0^x s|\mathcal{D}_r G(s)|\,ds \\ &= \sum_{n=1}^{\infty} 2n(2n+1)\int_{\frac{1}{2n+1}}^{\frac{1}{2n}} s\,ds + \sum_{n=1}^{\infty} 2n(2n-1)\int_{\frac{1}{2n}}^{\frac{1}{2n-1}} s\,ds \\ &= \sum_{n=1}^{\infty} \frac{4n+1}{4n(2n+1)} + \sum_{n=1}^{\infty} \frac{4n-1}{4n(2n-1)} \\ &= \infty \end{aligned}$$

shows that (53) is not satisfied. □

The optimal strategy of the control problem that we solve in the next section reflects the state process in a given point $a > 0$ in the positive direction. Such a strategy involves the construction of a continuous increasing process Z^a such that, if X^a is the associated solution to the SDE (6) with $x \geq a$, then

$$X_t^a \geq a \quad \text{and} \quad Z_t^a = \int_0^t \mathbf{1}_{\{X_s^a = a\}} \, dZ_s^a, \quad \text{for all } t \geq 0. \tag{55}$$

Such a construction is standard and can be found in El Karoui and Chaleyat-Maurel [EC78] (see also Schmidt [Sc89]).

Lemma 5 *Suppose that Assumptions 1, 2, 3, 4 and 5.(c) hold. Given a real number $a > 0$, consider the continuous increasing process Z^a and the solution X^a to the SDE (6) that satisfy (55). If $G :]0, \infty[\to \mathbb{R}$ is a measurable function satisfying (43) and (44), then*

$$\mathbb{E}\left[\int_0^\infty e^{-\Lambda_t^{r(X^a)}} |G(X_t^a)| \, dt\right] < \infty, \tag{56}$$

and

$$U_{X^a,G}(x) := -\frac{R'_{X^0,G}(a)}{\phi'(a)}\phi(x) + R_{X^0,G}(x) = \mathbb{E}\left[\int_0^\infty e^{-\Lambda_t^{r(X^a)}} G(X_t^a) \, dt\right], \tag{57}$$

where $R_{X^0,G}$ is defined by (46). Furthermore,

$$\lim_{n\to\infty} \mathbb{E}\left[e^{-\Lambda_{T_n^a}^{r(X^a)}}\right] R_{X^0,G}(n) = 0, \tag{58}$$

where T_n^a is the first hitting time of $\{n\}$ defined by

$$T_n^a = \inf\{t \geq 0 \mid \ X_t^a = n\}. \tag{59}$$

Proof. In view of the continuity of Z^a and the fact that it increases on the set $\{X_t^a = a\}$ (see (55)), we can see that Itô's formula and the identities $\mathcal{L}_X U_{X^a,G}(x) + G(x) = 0$ and $U'_{X^a,G}(a) = 0$ imply

$$\begin{aligned} e^{-\Lambda_T^{r(X^a)}} U_{X^a,G}(X_T^a) &= U_{X^a,G}(x) + \int_0^T e^{-\Lambda_t^{r(X^a)}} \mathcal{L}_X U_{X^a,G}(X_t^a) \, dt \\ &\quad + \int_0^T e^{-\Lambda_t^{r(X^a)}} U'_{X^a,G}(X_t^a) \, dZ_t^a + M_T^a \\ &= U_{X^a,G}(x) - \int_0^T e^{-\Lambda_t^{r(X^a)}} G(X_t^a) \, dt + M_T^a, \end{aligned} \tag{60}$$

where

$$M_T^a = \int_0^T e^{-\Lambda_t^{r(X^a)}} \sigma(X_t^a) U'_{X^a,G}(X_t^a)\, dW_t.$$

Also, Itô's isometry and the continuity of σ and $U'_{X^a,G}$ imply

$$\begin{aligned}\mathbb{E}\left[\left(M_{T_n^a}^a\right)^2\right] &= \mathbb{E}\left[\int_0^\infty \mathbf{1}_{\{t \le T_n^a\}} \left[e^{-\Lambda_t^{r(X^a)}} \sigma(X_t^a) U'_{X^a,G}(X_t^a)\right]^2 dt\right] \\ &\le \frac{1}{2r_0} \sup_{s\in[a,n]} \left[\sigma(s) U'_{X^a,G}(s)\right]^2 \\ &< \infty,\end{aligned}$$

where r_0 is as in Assumption 3. It follows that the stopped process $(M^a)^{T_n^a}$ is a uniformly square integrable martingale, and therefore, $\mathbb{E}\left[(M^a)_\infty^{T_n^a}\right] \equiv \mathbb{E}\left[M_{T_n^a}^a\right] = 0$. In view of this observation and (60), we obtain

$$\mathbb{E}\left[e^{-\Lambda_{T_n^a}^{r(X^a)}}\right] U_{X^a,G}(n) = U_{X^a,G}(x) - \mathbb{E}\left[\int_0^{T_n^a} e^{-\Lambda_t^{r(X^a)}} G(X_t^a)\, dt\right]. \tag{61}$$

Now, if G is bounded, then (57) follows immediately by passing to the limit $n \to \infty$ in (61) using the dominated convergence theorem, the fact that the restriction of $U_{X^a,G}$ in $[a,\infty[$ is bounded and Assumption 5.(c). If G is positive, we define $G^{(m)} = G \wedge m$, for $m \ge 1$, and we note that

$$U_{X^a,G^{(m)}}(x) \equiv -\frac{R'_{X^0,G^{(m)}}(a)}{\phi'(a)}\phi(x) + R_{X^0,G^{(m)}}(x) = \mathbb{E}\left[\int_0^\infty e^{-\Lambda_t^{r(X^a)}} G^{(m)}(X_t^a)\, dt\right] \tag{62}$$

because $G^{(m)}$ is bounded. In view of (46), the calculation

$$R'_{X^0,G}(x) = \frac{2}{C}\phi'(x) \int_0^x \frac{G(s)\psi(s)}{\sigma^2(s) p'_{X^0}(s)}\, ds + \frac{2}{C}\psi'(x) \int_x^\infty \frac{G(s)\phi(s)}{\sigma^2(s) p'_{X^0}(s)}\, ds,$$

and the monotone convergence theorem, we can see that

$$\lim_{m\to\infty} R_{X^0,G^{(m)}}(x) = R_{X^0,G}(x), \quad \lim_{m\to\infty} R'_{X^0,G^{(m)}}(a) = R'_{X^0,G}(a)$$

and

$$\lim_{m\to\infty} \mathbb{E}\left[\int_0^\infty e^{-\Lambda_t^{r(X^a)}} G^{(m)}(X_t^a)\, dt\right] = \mathbb{E}\left[\int_0^\infty e^{-\Lambda_t^{r(X^a)}} G(X_t^a)\, dt\right].$$

However, these limits and (62) imply (57). The general case now follows by considering the minimal decomposition $G = G^+ - G^-$ of the function G to the difference of two positive functions and by linearity.

To complete the proof, we note that, if G satisfies (44), then $U_{X^a,|G|}(x) < \infty$, which, combined with (57), implies (56). Also, (61), (56) and the dominated convergence theorem imply $\lim_{n\to\infty} \mathbb{E}\left[e^{-\Lambda^{r(X^a)}_{T^a_n}}\right] U_{X^a,G}(n) = 0$. However, this limit, the fact that the restriction of ϕ in $[a,\infty[$ is bounded and Assumption 5.(c) imply (58). □

4 The solution to the control problem

With regard to standard theory of stochastic control, we expect that the value function v identifies with an appropriate solution w to the Hamilton-Jacobi-Bellman (HJB) equation

$$\max\left\{\mathcal{L}_X w(x) + h(x),\ w'(x) - k(x)\right\} = 0, \tag{63}$$

where $\mathcal{L}_X$ is the operator defined by (14). We conjecture that the optimal strategy of our problem can take one of two qualitatively different forms, depending on the problem data. The first of these arises when it is never optimal to exert any control effort. In this case, we expect that the function $R_{X^0,h}$ that is defined by (20) in Remark 2 should satisfy (63).

The second one is characterised by a boundary point $a > 0$ and can be described as follows. If the system's initial condition x is less than a, then maximal control should be exercised to immediately reposition the system's state at level a (i.e., cause a "jump" of size $a - x$ in the positive direction at time 0). After this possible initial jump, minimal control should be exercised so that the system's state process is reflected at the boundary point a in the positive direction. In view of the heuristic arguments that explain the structure of the HJB equation (63), if this strategy is indeed optimal, then we should look for a function w and a point $a > 0$ such that

$$\mathcal{L}_X w(x) + h(x) = 0, \quad \text{for } x \geq a, \tag{64}$$

and

$$w(x) = w(a) - \int_x^a k(s)\, ds, \quad \text{for } x < a. \tag{65}$$

Now, every solution to the ODE in (64) is given by

$$w(x) = A\phi(x) + B\psi(x) + R_{X^0,h}(x), \tag{66}$$

for some constants $A, B \in \mathbb{R}$, where the functions ϕ and ψ are defined as in Section 3. It turns out that the arguments that we use to establish Theorem 7 below, which is our main result, remain valid only for the choice $B = 0$. For this reason, we look for a solution to the HJB equation (63) of the form

$$w(x) = \begin{cases} A\phi(x) + R_{X^0,h}(x), & \text{for } x \geq a, \\ w(a) - \int_x^a k(s)\, ds, & \text{for } x < a. \end{cases} \tag{67}$$

To specify the parameter A and the free-boundary point a, we appeal to the so-called "smooth pasting" condition of singular stochastic control that requires that the value function should be C^2, in particular, at the free-boundary point a. This requirement gives rise to the system of equations

$$R'_{X^0,h}(a) + A\phi'(a) = k(a),$$
$$R''_{X^0,h}(a) + A\phi''(a) = k'(a),$$

which is equivalent to

$$A = \frac{k(a) - R'_{X^0,h}(a)}{\phi'(a)} = \frac{k'(a) - R''_{X^0,h}(a)}{\phi''(a)}, \tag{68}$$

$$\phi'(a)\left[R''_{X^0,h}(a) - k'(a)\right] - \phi''(a)\left[R'_{X^0,h}(a) - k(a)\right] = 0. \tag{69}$$

Taking note of (21) in Remark 2, we can see that

$$R'_{X^0,h}(x) - k(x) = R'_{X^0,Q}(x), \tag{70}$$

so (69) is equivalent to

$$\phi'(a)R''_{X^0,Q}(a) - \phi''(a)R'_{X^0,Q}(a) = 0. \tag{71}$$

Now, Proposition 4 and (54) imply

$$\begin{aligned} R'_{X^0,Q}(x) &= R_{Y^0,\mathcal{D}_rQ}(x) \\ &= -\frac{1}{C}\phi'(x)\int_0^x \frac{(\mathcal{D}_rQ)(s)\psi'(s)}{p'_{X^0}(s)}\,ds - \frac{1}{C}\psi'(x)\int_x^\infty \frac{(\mathcal{D}_rQ)(s)\phi'(s)}{p'_{X^0}(s)}\,ds, \end{aligned} \tag{72}$$

where the function $\mathcal{D}_rQ$ is as in Assumption 5.(d). Combining this observation with the calculation

$$R''_{X^0,Q}(x) = -\frac{1}{C}\phi''(x)\int_0^x \frac{(\mathcal{D}_rQ)(s)\psi'(s)}{p'_{X^0}(s)}\,ds - \frac{1}{C}\psi''(x)\int_x^\infty \frac{(\mathcal{D}_rQ)(s)\phi'(s)}{p'_{X^0}(s)}\,ds, \tag{73}$$

we can see that (71) is equivalent to

$$[\phi''(a)\psi'(a) - \phi'(a)\psi''(a)]\int_a^\infty \frac{(\mathcal{D}_rQ)(s)\phi'(s)}{p'_{X^0}(s)}\,ds = 0. \tag{74}$$

In view of the fact that $\phi''\psi' - \phi'\psi''$ is strictly positive (see also (33)), we conclude that (69) is equivalent to

$$g(a) := \int_a^\infty \frac{(\mathcal{D}_rQ)(s)\phi'(s)}{p'_{X^0}(s)}\,ds = 0. \tag{75}$$

The following result is concerned with the considerations above regarding the solvability of the HJB equation (63).

Lemma 6 *Suppose that Assumptions 1–5 are satisfied. The equation $g(a) = 0$ has a unique solution $a > 0$ if and only if*

$$x^* > 0 \quad \textit{and} \quad \lim_{x \downarrow 0} g(x) < 0, \tag{76}$$

where $x^* \geq 0$ is as in Assumption 5.(d). Furthermore, the following cases hold true:

(I) If (76) is not true, then the function $R_{X^0,h}$ satisfies the HJB equation (63).

(II) If (76) is true, $a > 0$ is the unique solution to the equation $g(a) = 0$ and $A \geq 0$ is the constant given by (68), then the function w defined by (67) is C^2 and solves the HJB equation (63).

In either case, the associated solution w to the HJB equation (63) is bounded from below.

Proof. In view of the definition (75) of g and Assumption 5.(d), we calculate

$$g'(x) = -\frac{(\mathcal{D}_r Q)(x)\phi'(x)}{p'_{X^0}(x)} \begin{cases} \geq 0, & \text{for } x \leq x^*, \text{ if } x^* > 0, \\ < 0 & \text{for } x > x^*. \end{cases} \tag{77}$$

Combining this calculation with the fact that $\lim_{x\to\infty} g(x) = 0$, we can see that the equation $g(a) = 0$ has a unique solution $a > 0$ if and only if (76) is true. For future reference, we also note that

$$\text{if there exists } a > 0 \text{ such that } g(a) = 0, \text{ then } a < x^* \text{ and } g(x) > 0, \text{ for all } x > a. \tag{78}$$

In this case, (77) and the fact that $\phi' < 0 < \psi'$ imply

$$\frac{(\mathcal{D}_r Q)(x)\psi'(x)}{p'_{X^0}(x)} \geq 0, \quad \text{for all } x \leq a,$$

which, combined with (70), (72) and (75), implies

$$A = -\frac{R'_{X^0,Q}(a)}{\phi'(a)} = \frac{1}{C}\int_0^a \frac{(\mathcal{D}_r Q)(s)\psi'(s)}{p'_{X^0}(s)}\, ds \geq 0.$$

On the other hand,

$$\text{if the equation } g(a) = 0 \text{ has no solution } a > 0, \text{ then } g(x) > 0, \text{ for all } x > 0. \tag{79}$$

With regard to Case I we will prove that $R_{X^0,h}$ satisfied the HJB equation (63) if we show that $R'_{X^0,h}(x) - k(x) \leq 0$, for all $x > 0$, which is equivalent to showing that

$$R_{Y^0,\mathcal{D}_r Q}(x) \leq 0, \quad \text{for all } x > 0, \tag{80}$$

thanks to (70) and (72). To this end, we use (72) and (73) to calculate

$$\frac{d}{dx}\left(\frac{R_{Y^0,\mathcal{D}_rQ}(x)}{-\phi'(x)}\right) = -\frac{\phi''(x)\psi'(x) - \phi'(x)\psi''(x)}{C[\phi'(x)]^2}g(x). \tag{81}$$

The right-hand side of this identity is strictly negative for all $x > 0$, thanks to the strict positivity of the function $\phi''\psi' - \phi'\psi''$ (see (33)) and (79). Also, in view of (79) and the fact that $\phi' < 0 < \psi'$, we can see that

$$\begin{aligned}\lim_{x\downarrow 0}\frac{R_{Y^0,\mathcal{D}_rQ}(x)}{-\phi'(x)} &= \frac{1}{C}\lim_{x\downarrow 0}\left[\int_0^x \frac{(\mathcal{D}_rQ)(s)\psi'(s)}{p'_{X^0}(s)}ds + \frac{\psi'(x)}{\phi'(x)}g(x)\right] \\ &\le 0.\end{aligned}$$

However, these observations imply (80).

By construction, Case II will follow if we show that

$$\tfrac{1}{2}\sigma^2(x)k'(x) + b(x)k(x) - r(x)\left[w(a) - \int_x^a k(s)\,ds\right] + h(x) \le 0, \quad \text{for all } x \le a, \tag{82}$$

$$w'(x) - k(x) \le 0, \quad \text{for all } x > a. \tag{83}$$

In view of the definitions (13) and (15) of the function K and Q, respectively, we can see that (82) is equivalent to

$$Q(x) + r(x)K(a) - r(x)w(a) \le 0, \quad \text{for all } x \le a.$$

Also, the facts that w is C^2 and satisfies (64) imply

$$\begin{aligned}r(a)w(a) &= \tfrac{1}{2}\sigma^2(a)k'(a) + b(a)k(a) + h(a) \\ &= Q(a) + r(a)K(a).\end{aligned}$$

These observations and the strict positivity of r imply that (82) is equivalent to

$$\frac{Q(x)}{r(x)} \le \frac{Q(a)}{r(a)}, \quad \text{for all } x \le a.$$

However, this inequality follows immediately from the fact that (78) and Assumption 5.(d) imply that $(Q/r)'(x) \ge 0$, for all $x \le a$.

Substituting the first expression in (68) for A in (67), and using (70) and (72), we can see that (83) is equivalent to

$$\frac{R_{Y^0,\mathcal{D}_rQ}(a)}{-\phi'(a)} \ge \frac{R_{Y^0,\mathcal{D}_rQ}(x)}{-\phi'(x)}, \quad \text{for all } x > a.$$

However, this inequality follows immediately once we combine (81) with the strict positivity of $\phi''\psi' - \phi'\psi''$ and (78) to obtain

$$\frac{d}{dx}\left(\frac{R_{Y^0,\mathcal{D}_r Q}(x)}{-\phi'(x)}\right) < 0, \quad \text{for all } x > a.$$

Finally, we note that the assumption that h is bounded from below (see Assumption 5.(a)) and (20) in Remark 2 imply trivially that $R_{X^0,h}$ is bounded from below. However, this observation and the structure of the solution w to the HJB equation (63) associated with either of the two cases considered imply that w is bounded from below. □

We can now prove the main result of the paper.

Theorem 7 *Suppose that Assumptions 1–5 hold. The value function v of our control problem identifies with the solution w to the HJB equation (63) derived in Lemma 6. In particular, the following two cases hold true:*

(I) If the problem data are such that (76) is false, then it is optimal not to exert any control effort at all times.

(II) If the problem data are such that (76) is true, then the optimal intervention strategy involves a jump of size $(a-x)^+$ at time 0 and then reflects the state process X in the boundary point $a > 0$ in the positive direction.

Proof. Fix any initial condition $x > 0$ and any admissible intervention strategy $Z \in \mathcal{A}$. Using Itô's formula and the fact that $\Delta X_t = \Delta Z_t$, we calculate

$$\begin{aligned} e^{-\Lambda_T^{r(X)}} w(X_{T+}) &= w(x) + \int_0^T e^{-\Lambda_t^{r(X)}} \mathcal{L}_X w(X_t)\, dt + \int_0^T e^{-\Lambda_t^{r(X)}} w'(X_t)\, dZ_t \\ &\quad + \sum_{t\in[0,T]} e^{-\Lambda_t^{r(X)}} \left[w(X_{t+}) - w(X_t) - w'(X_t)\Delta X_t\right] + M_T \\ &= w(x) + \int_0^T e^{-\Lambda_t^{r(X)}} \mathcal{L}_X w(X_t)\, dt + \int_0^T e^{-\Lambda_t^{r(X)}} w'(X_t)\, dZ_t^c \\ &\quad + \sum_{t\in[0,T]} e^{-\Lambda_t^{r(X)}} \left[w(X_t + \Delta Z_t) - w(X_t)\right] + M_T, \end{aligned}$$

where the operator $\mathcal{L}_X$ is defined by (14), and

$$M_T = \int_0^T e^{-\Lambda_t^{r(X)}} \sigma(X_t) w'(X_t)\, dW_t.$$

In view of this calculation, (9) and the fact that w satisfies the HJB equation (63), we obtain

$$\begin{aligned}
&\int_0^T e^{-\Lambda_t^{r(X)}} h(X_t)\,dt - \int_{[0,T]} e^{-\Lambda_t^{r(X)}} k(X_t) \circ dZ_t \\
&\qquad = w(x) - e^{-\Lambda_T^{r(X)}} w(X_{T+}) + \int_0^T e^{-\Lambda_t^{r(X)}} \left[\mathcal{L}_X w(X_t) + h(X_t)\right] dt \\
&\qquad\quad + \int_0^T e^{-\Lambda_t^{r(X)}} \left[w'(X_t) - k(X_t)\right] dZ_t^c \\
&\qquad\quad + \sum_{t\in[0,T]} e^{-\Lambda_t^{r(X)}} \int_0^{\Delta Z_t} \left[w'(X_t+s) - k(X_t+s)\right] ds + M_T \\
&\qquad \le w(x) - e^{-\Lambda_T^{r(X)}} w(X_{T+}) + M_T.
\end{aligned} \tag{84}$$

To proceed further, let (τ_n) be a localising sequence for the stochastic integral M. Taking expectations in (84), we obtain

$$\mathbb{E}\left[\int_0^{\tau_n\wedge T} e^{-\Lambda_t^{r(X)}} h(X_t)\,dt - \int_{[0,\tau_n\wedge T]} e^{-\Lambda_t^{r(X)}} k(X_t) \circ dZ_t\right] \le w(x) + \mathbb{E}\left[e^{-\Lambda_{\tau_n\wedge T}^{r(X)}} w^-(X_{(\tau_n\wedge T)+})\right], \tag{85}$$

where $w^- = -(w \wedge 0)$. Now, the assumption that h is bounded from below and Fatou's lemma imply

$$\mathbb{E}\left[\int_0^T e^{-\Lambda_t^{r(X)}} h(X_t)\,dt\right] \le \liminf_{n\to\infty} \mathbb{E}\left[\int_0^{\tau_n\wedge T} e^{-\Lambda_t^{r(X)}} h(X_t)\,dt\right],$$

while the monotone convergence theorem implies

$$\mathbb{E}\left[\int_{[0,T]} e^{-\Lambda_t^{r(X)}} k(X_t) \circ dZ_t\right] = \lim_{n\to\infty} \mathbb{E}\left[\int_{[0,\tau_n\wedge T]} e^{-\Lambda_t^{r(X)}} k(X_t) \circ dZ_t\right],$$

Also, the fact that w^- is bounded (see Lemma 6) and the dominated convergence theorem imply

$$\lim_{n\to\infty} \mathbb{E}\left[e^{-\Lambda_{\tau_n\wedge T}^{r(X)}} w^-(X_{(\tau_n\wedge T)+})\right] = \mathbb{E}\left[e^{-\Lambda_T^{r(X)}} w^-(X_{T+})\right].$$

In view of these observations, we can pass to the limit $n \to \infty$ in (85) to obtain

$$\mathbb{E}\left[\int_0^T e^{-\Lambda_t^{r(X)}} h(X_t)\,dt - \int_{[0,T]} e^{-\Lambda_t^{r(X)}} k(X_t) \circ dZ_t\right] \le w(x) + \mathbb{E}\left[e^{-\Lambda_T^{r(X)}} w^-(X_{T+})\right].$$

Combining this inequality with the limit

$$\lim_{T\to\infty} \mathbb{E}\left[e^{-\Lambda_T^{r(X)}} w^-(X_{T+})\right] = 0,$$

which follows from Assumption 5.(c) and the fact that w^- is bounded, we can see that

$$J_x(Z) \equiv \limsup_{T\to\infty} \mathbb{E}\left[\int_0^T e^{-\Lambda_t^{r(X)}} h(X_t)\, dt - \int_{[0,T]} e^{-\Lambda_t^{r(X)}} k(X_t) \circ dZ_t\right] \le w(x). \tag{86}$$

If (76) is false, then the control strategy $Z^0 \equiv 0$ has payoff

$$J_x(Z^0) = R_{X^0,h}(x) = w(x).$$

The first equality here follows from the definition (20) of $R_{X^0,h}$ and the dominated convergence theorem (see (17) in Assumption 5.(e)), while the second one is just Case I of Lemma 6. However, these identities and (86) establish Case I of the theorem.

Now, let us assume that (76) is true. Let Z^a be the càglàd process that has a jump of size $(a-x)^+$ at time 0 and then reflects the state process in the boundary point $a > 0$ in the positive direction (see the discussion preceding Lemma 5). Also, let X^a be the associated solution to the SDE (3) and let T_n^a be the first hitting time of $\{n\}$, which is given by (59) in Lemma 5). In this case, we can check that (84) holds with equality, so

$$\int_0^T e^{-\Lambda_t^{r(X^a)}} h(X_t^a)\, dt - \int_{[0,T]} e^{-\Lambda_t^{r(X^a)}} k(X_t^a) \circ dZ_t^a = w(x) - e^{-\Lambda_T^{r(X^a)}} w(X_{T+}^a) + M_T^a, \tag{87}$$

where

$$M_T^a = \int_0^T e^{-\Lambda_t^{r(X^a)}} \sigma(X_t^a) w'(X_t^a)\, dW_t.$$

Using Itô's isometry and the fact that σ and w' are continuous, we can see that, for $n \ge a \vee x$,

$$\begin{aligned}\mathbb{E}\left[\left(M_{T_n^a}^a\right)^2\right] &= \mathbb{E}\left[\int_0^\infty \mathbf{1}_{\{t\le T_n^a\}} \left[e^{-\Lambda_t^{r(X^a)}} \sigma(X_t^a) w'(X_t^a)\right]^2 dt\right] \\ &\le \frac{1}{2r_0} \sup_{s\in[a,n]} \left[\sigma(s) w'(s)\right]^2 \\ &< \infty,\end{aligned}$$

where r_0 is as in Assumption 3. It follows that the stopped process $(M^a)^{T_n^a}$ is a uniformly square integrable martingale, so $\mathbb{E}\left[(M^a)_\infty^{T_n^a}\right] \equiv \mathbb{E}\left[M_{T_n^a}^a\right] = 0$. This observation, (87) and the fact that, apart from a possible jump at time 0, Z^a is continuous, imply

$$\mathbb{E}\left[\int_0^{T_n^a} e^{-\Lambda_t^{r(X^a)}} h(X_t^a)\, dt - \int_{[0,T_n^a[} e^{-\Lambda_t^{r(X^a)}} k(X_t^a) \circ dZ_t^a\right] = w(x) - \mathbb{E}\left[e^{-\Lambda_{T_n^a}^{r(X^a)}}\right] w(n), \tag{88}$$

for all $n > a \vee x$.

Combining the fact that w is the sum of $R_{X^0,h}$ and a bounded function with Assumption 5.(c) and (58) in Lemma 5, we can see that

$$\lim_{n\to\infty} \mathbb{E}\left[e^{-\Lambda^{r(X^a)}_{T^a_n}}\right] w(n) = 0.$$

Also, Lemma 5 and the dominated convergence theorem imply

$$\lim_{n\to\infty} \mathbb{E}\left[\int_0^{T^a_n} e^{-\Lambda^{r(X^a)}_t} h(X^a_t)\, dt\right] = \mathbb{E}\left[\int_0^{\infty} e^{-\Lambda^{r(X^a)}_t} h(X^a_t)\, dt\right] \in \mathbb{R}, \tag{89}$$

while the monotone convergence theorem implies

$$\lim_{n\to\infty} \mathbb{E}\left[\int_{[0,T^a_n[} e^{-\Lambda^{r(X^a)}_t} k(X^a_t) \circ dZ^a_t\right] = \mathbb{E}\left[\int_{[0,\infty[} e^{-\Lambda^{r(X^a)}_t} k(X^a_t) \circ dZ^a_t\right].$$

These limits and (88) imply

$$\mathbb{E}\left[\int_0^{\infty} e^{-\Lambda^{r(X^a)}_t} h(X^a_t)\, dt\right] - \mathbb{E}\left[\int_{[0,\infty[} e^{-\Lambda^{r(X^a)}_t} k(X^a_t) \circ dZ^a_t\right] = w(x).$$

If we combine this conclusion with (89), then we can see that

$$\mathbb{E}\left[\int_{[0,\infty[} e^{-\Lambda^{r(X^a)}_t} k(X^a_t) \circ dZ^a_t\right] < \infty,$$

so Z^a is admissible. Furthermore, if we combine it with (86), then we can see that $v(x) = w(x)$ and that Z^a is optimal. $\square$

5 Special cases

We now consider special cases that arise when the running payoff function h is a power utility function and the running cost function k as well as the discounting factor r are constant. In particular, we assume that

$$h(x) = \lambda x^{\nu}, \quad k(x) = \kappa \quad \text{and} \quad r(x) = r_1,$$

for some constants $\kappa, \lambda, r_1 > 0$ and $\nu \in \,]0, 1[$. Also, we assume that the uncontrolled system's state dynamics are modelled by a geometric Brownian motion (Section 5.1) or by a mean-reverting square-root process such as the one in the Cox-Ingersoll-Ross interest rate model (Section 5.2).

5.1 Geometric Brownian motion

Suppose that X^0 is a geometric Brownian motion, so that

$$dX_t^0 = bX_t^0\,dt + \sigma X_t^0\,dW_t, \quad X_0^0 = x > 0,$$

for some constants b and $\sigma \neq 0$, and assume that $r_1 > b$. In this case, it is a standard exercise to verify that, if we choose $c = 1$, then

$$\phi(x) = x^m, \quad \psi(x) = x^n \quad \text{and} \quad p'_{X^0}(x) = x^{n+m-1},$$

where the constants $m < 0 < n$ are the solution to the quadratic equation

$$\tfrac{1}{2}\sigma^2 l^2 + \left(b - \tfrac{1}{2}\sigma^2\right) l - r_1 = 0.$$

Also, it is well-known that

$$r_1 > b \quad \Leftrightarrow \quad n > 1. \tag{90}$$

In this context, Assumptions 1, 2, 3, 4 and 5.(a)–5.(c) are plainly satisfied. Also, in the presence of (90), we can calculate

$$\mathbb{E}\left[\int_0^\infty e^{-r_1 t} X_t\,dt\right] = \frac{x}{r_1 - b}.$$

However, using this observation and the facts that

$$0 \leq h(x) \leq \lambda(1 + x) \quad \text{and} \quad \mathcal{L}_X K(x) = -(r_1 - b)\kappa x,$$

we can verify that Assumption 5.(e) is satisfied. Furthermore, we can use the calculation

$$\mathcal{D}_r Q(x) = Q'(x) = \lambda\nu x^{-(1-\nu)} - \kappa(r_1 - b)$$

to verify that $x^* > 0$ and that Assumption 5.(d) is satisfied as well.

In view of the fact that $n > 1$, we can check that the function g defined by (75) admits the expression

$$g(x) = \frac{m\lambda\nu}{n - \nu} x^{\nu - n} - \frac{m\kappa(r_1 - b)}{n - 1} x^{1-n}, \quad \text{for } x > 0.$$

It follows that Case II of Theorem 7 is always true and that the unique solution $a > 0$ to the equation $g(a) = 0$ that characterises the optimal strategy is given by

$$a = \left[\frac{\lambda\nu(n-1)}{\kappa(r_1 - b)(n - \nu)}\right]^{1/(1-\nu)}.$$

5.2 Mean-reverting square-root process

Suppose that X^0 is a mean-reverting square-root process, so that

$$dX_t^0 = \alpha(\theta - X_t^0)\, dt + \sigma\sqrt{X_t^0}\, dW_t, \quad X_0^0 = x > 0,$$

for some constants $\alpha, \theta, \sigma > 0$, and assume that

$$\alpha\theta - \tfrac{1}{2}\sigma^2 > 0, \tag{91}$$

which is a necessary and sufficient condition for X^0 to be non-explosive. In this case, we can check that the associated diffusion Y^0 satisfies the SDE

$$dY_t^0 = \alpha\left(\left[\theta + \frac{\sigma^2}{2\alpha}\right] - Y_t^0\right) dt + \sigma\sqrt{Y_t^0}\, dW_t, \quad Y_0^0 = x > 0,$$

and that Y^0 is non-explosive because $\alpha\theta > 0$, which is plainly true, is the condition corresponding to (91). We can therefore see that Assumptions 1, 2, 3, 4 and 5.(a)–5.(c) are all satisfied.

Now, we calculate

$$\begin{aligned}\mathbb{E}\left[\int_0^\infty e^{-r_1 t} X_t\, dt\right] &= \int_0^\infty e^{-r_1 t}\left[\theta + (x-\theta)e^{-\alpha t}\right] dt \\ &= \frac{\alpha\theta + r_1 x}{r_1(\alpha + r_1)}.\end{aligned}$$

Combining these identities with the inequalities

$$0 \le h(x) \le \lambda(1+x) \quad \text{and} \quad |\mathcal{L}_X K(x)| = |\alpha\theta\kappa - \kappa(\alpha + r_1)x| \le \alpha\theta\kappa + \kappa(\alpha + r_1)x,$$

we can verify that Assumption 5.(e) is satisfied. Also, in view of the calculation

$$\mathcal{D}_r Q(x) = Q'(x) = \lambda\nu x^{-(1-\nu)} - (\alpha + \kappa r_1),$$

we can see that $x^* > 0$ and that Assumption 5.(d) holds.

Making the transformations $y = 2\alpha x/\sigma^2$ and $\hat{w}(y) = w(x)$ in the ODE $\mathcal{L}_X w(x) = 0$, we obtain

$$\hat{w}''(y) + \left(\frac{2\alpha\theta}{\sigma^2} - y\right)\hat{w}'(y) - \frac{r_1}{\alpha}\hat{w}(y) = 0,$$

which is Kummer's equation. With reference to Abramowitz and Stegun [AS72, Chapter 13] or Magnus, Oberhettinger and Soni [MOS66, Chapter VI], it follows that, if we choose $c = 1$, then

$$\phi(x) = \frac{U\left(\frac{r_1}{\alpha}, \frac{2\alpha\theta}{\sigma^2}; \frac{2\alpha}{\sigma^2}x\right)}{U\left(\frac{r_1}{\alpha}, \frac{2\alpha\theta}{\sigma^2}; \frac{2\alpha}{\sigma^2}\right)}, \quad \psi(x) = \frac{{}_1F_1\left(\frac{r_1}{\alpha}, \frac{2\alpha\theta}{\sigma^2}; \frac{2\alpha}{\sigma^2}x\right)}{{}_1F_1\left(\frac{r_1}{\alpha}, \frac{2\alpha\theta}{\sigma^2}; \frac{2\alpha}{\sigma^2}\right)}$$

and

$$p'_{X^0}(x) = x^{-2\alpha\theta/\sigma^2} e^{2\alpha(x-1)/\sigma^2}.$$

Note that ${}_1F_1$ is the well-known confluent hypergeometric function.

Now, in view of the differentiation formula

$$U'(a,b;x) = -aU(a+1,b+1;x)$$

(see Abramowitz and Stegun [AS72, 13.4.21]), we can see that the function g defined by (75) is given by

$$g(x) = -\frac{2r_1 e^{2\alpha/\sigma^2}}{U\left(\frac{r_1}{\alpha}, \frac{2\alpha\theta}{\sigma^2}; \frac{2\alpha}{\sigma^2}\right)} \left[\lambda\nu \int_x^\infty s^{\frac{2\alpha\theta}{\sigma^2}+\nu-1} e^{-\frac{2\alpha}{\sigma^2}s} U\left(\frac{r_1}{\alpha}+1, \frac{2\alpha\theta}{\sigma^2}+1; \frac{2\alpha}{\sigma^2}s\right) ds \right.$$
$$\left. - (\alpha + \kappa r_1) \int_x^\infty s^{\frac{2\alpha\theta}{\sigma^2}} e^{-\frac{2\alpha}{\sigma^2}s} U\left(\frac{r_1}{\alpha}+1, \frac{2\alpha\theta}{\sigma^2}+1; \frac{2\alpha}{\sigma^2}s\right) ds \right].$$

This expression is complex, so we have to resort to numerical techniques to determine the solution of the equation $g(a) = 0$.

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