Finite Dimensional Realizations of Regime-Switching HJM models

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DRAFT

Abstract

This paper studies Heath-Jarrow-Morton (HJM) type of models with regime-switching stochastic volatility. In this setting the forward rate volatility is allowed to depend on the current forward rate curve as well as on a continuous time Markov chain y with finitely many states. Employing the framework developed in Björk and Svensson (2001) we find necessary and sufficient conditions on the volatility guaranteeing the representation of the forward rate process by a finite dimensional Markovian state space model. These conditions allow us to investigate regime-switching generalizations of some well-known models such as those by Ho-Lee, Hull-White, and Cox-Ingersoll-Ross.

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1 Introduction

The modelling of the term structure of interest rates in the Heath-Jarrow-Morton (HJM) framework (see Heath et al., 1992), usually begins with the specification of the volatility term in the stochastic differential equation (SDE) for the forward rate.[1] By assuming absence of arbitrage it can be shown that the drift term is completely determined by the volatility under the risk neutral measure (the HJM drift condition). Thus, to get a specific model of the term structure in this framework one only has to specify the volatility.

The object that is modelled, the forward rate curve, is an infinite dimensional object, so the SDE that we are specifying is in fact infinite dimensional. From this perspective there is no reason why the solution to this SDE could be realized by a finite dimensional state space model. It can of course happen. To take a concrete example, if we set the volatility to be a constant, then one can show that this gives rise to a one dimensional model equivalent to the Ho and Lee (1986) model for the short rate fitted to the initial term structure. For more complicated volatility functions however we are not à priori certain to get a model that has a finite dimensional representation. In fact, considering the inherent infinite dimensionality of the problem, this should be the exception rather than the rule.

In the end of the 1990's numerous papers address the question of how one should to specify the volatility in order to guaranty that the model they produced have a finite dimensional realization (henceforth FDR): Ritchken and Sankarasubramanian (1995), Bhar and Chiarella (1997), Inui and Kijima (1998), and Chiarella and Kwon (2001).[2] These paper provide examples of volatilities known to yield a FDR. Björk and Svensson (2001), building on earlier results by Björk and Christensen (1999), where the first to provide both necessary and sufficient conditions for the existence of a FDR. To complement this pure existence result, Björk and Landén (2002) address the question of the actual construction of the realization.

The present paper uses the geometric framework developed by Björk and his coauthors to investigate HJM models with regime shifting stochastic volatility. Here the volatility is a function of a continuous time Markov chain with a finite number of states and of the current forward rate curve. The main purpose is to provide necessary and sufficient conditions on the volatility to guaranty a FDR.

From a modeling perspective the use of a Markov chain can be motivated by the need to make the model more flexible without making it intractable. There could also be economic reasons why a interest rate model driven by a Markov chain is attractive. Business cycle expansions and contraction can potentially have a first order impact on inflationary expectations, monetary policy, and nominal interest rates. Such regime shifts could have a major impact on the whole term structure of interest rates.

There is empirical evidence by Brown and Dybvig (1986), to take one example, showing considerable variation of the parameters when a version of the CIR model is estimated on US data. Gibbons and Ramaswany (1993) also have evidence pointing in the same direction. This evidence of poor empirical performance is consistent with regime shifting interest rates.

[1] For an overview of the HJM framework see e.g. chapter 23 in Björk (2004).

[2] See Björk (2003) for an overview.

Previous attempts to introduce regime shifts in the parameters of interest models are quite recent and have been mainly concerned with short rate models (e.g. Landén (2000) and Hansen and Poulsen (2000)).

Valchev (2004) seems to be the first and, to our knowledge, only author to introduce regime shifting coefficients in a HJM type of models. He first provides a semi-martingale representation of the Markov chain modulated volatility. Then he considers some explicit representations of the volatility, and derives the corresponding short rate dynamics. The paper is restricted to Gaussian models, i.e. models where the volatility does not depend on the current forward rate curve. The question of the existence of FDRs is not discussed and no realizations are constructed.

The paper by Björk, Landén, and Svensson (2004) is related to the present one in the sense that it treats FDRs of stochastic volatility models within the same geometric framework. The difference is that the authors consider a volatility driven by a diffusion, and that this is essential for the methods employed. In the present case with Markov chain modulated volatility the geometric picture is different and therefore needs to be tackled by other means.

The rest of the paper is organized as follows. Section 2 provides the theoretical foundations needed in the rest of the paper. In particular the FDR problem is formalized and a very general theorem on the FDR of regime switching volatility models is proved. After these quite abstract results, more concrete matters are discussed. Section 3 treats the case of deterministic volatility in detail. This allow us to obtain concrete results concerning the possible choices of volatilities and on their FDRs. The regime switching generalizations of Ho and Lee (1986) model and of the Hull and White (1990) model are then presented and their FDR derived. Section 4 treats the special case of separable volatility. Possible regime switching models of this class are discussed and finally the example of the generalized Cox, Ingersoll, and Ross (1985) model is analyzed.

2 Model

As in Heath, Jarrow, and Morton (1992) we consider a default and arbitrage free bond market modelled on a filtered probability space $(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathcal{F}, Q)$ where Q is the risk neutral martingale measure. The space carries a one dimensional Wiener process W, and a m-dimensional continuous-time Markov chain $\{y_t\}$, $t \geq 0$, whose finite state space is the set $S = \{e_1, \ldots, e_m\}$ of unit vectors on $\mathbb{R}^m$. The restriction to a one dimensional Wiener process is for notational convenience only and has no bearing on the results.

The price at time t of a zero coupon bond maturing at $t + x$ is denoted $p(t, x)$. Notice the Musiela parametrization[3] which uses the time *to* maturity x instead of the usual time *of* maturity T. The instantaneous forward rate is defined as

$$r_t(x) := -\frac{\partial}{\partial x} \log p(t, x),$$

i.e. the price at time t of a zero-coupon bond maturing at $t + x > t$ is given by $p(t, x) = \exp\{-\int_0^x r_t(s)\, ds\}$.

[3]See Musiela (1993). The use of the Musiela parametrization is motivated by the emphasis on the forward rate curve $x \mapsto r_t(x)$ as an infinite dimensional object in the framework of Björk and Svensson (2001).

The modelling will focus on the forward rate curve, $x \mapsto r_t(x)$, rather than on the individual forward rate $r_t(x)$. To stress this fact the forward rate curve is denoted r_t. The forward rate process $\{r_t\}$ is thus viewed a stochastic process taking values in the Hilbert space $\mathcal{H}$ of forward rate curves.[4]

The present paper studies the following HJM model of the forward rate where the volatility is modulated by the continuous time Markov chain y.

Definition 2.1. *The forward rate is given by the following stochastic differential equation*

$$\begin{cases} dr_t(x) & = \mu_0(r_t, y_t, x)\, dt + \sigma(r_t, y_t, x) dW(t), \\ r_0(x) & = r_0^o(x). \end{cases} \tag{1}$$

As we will be using results from differential geometry, for instance the Frobenius theorem, it is convenient to formulate the model of the forward rate in terms of Stratonovich integrals instead of the usual Itō integrals. This allows us to use the theorems directly without having to translate them beforehand into the Itō setting.

Definition 2.2. *The Stratanovich formulation of the model is given by the following stochastic differential equation*

$$\begin{cases} dr_t(x) & = \mu(r_t, y_t, x)\, dt + \sigma(r_t, y_t, x) \circ dW(t), \\ r_0(x) & = r_0^o(x), \end{cases} \tag{2}$$

where $\circ$ indicates a Stratonovich differential. Henceforth we will use this Stratonovich formulation of the model.

We restrict ourselves to a scalar driving Wiener process mainly to simplify notation by avoiding to differentiate between indices related to Wiener processes and states of the Markov chain. The well known HJM drift condition, emanating from the assumption of absence of arbitrage takes the form

$$\mu(r_t, y_t, x) = \frac{\partial}{\partial x} r_t(x) + \sigma(r_t, y_t, x) \int_0^x \sigma(r_t, y_t, s)\, ds - \frac{1}{2} \sigma'_r(r_t, y_t, x)[\sigma(r_t, y_t, x)]. \tag{3}$$

Here $\sigma'_r(r_t, y_t)[\sigma(r_t, y_t)]$ denotes the Fréchet derivative $\sigma'_r(r_t, y_t)$ operating on $\sigma(r_t, y_t)$. Setting

$$\mathbb{F} := \frac{\partial}{\partial x},$$
$$\Phi(r_t, y_t, x) := \sigma(r_t, y_t, x) \int_0^x \sigma(r_t, y_t, s)\, ds,$$
$$\Delta(r_t, y_t, x) = \sigma'_r(r_t, y_t, x)[\sigma(r_t, y_t, x)],$$

we can write the drift more compactly as

$$\mu(r_t, y_t, x) = \mathbb{F} r_t(x) + \Phi(r_t, y_t, x) - \frac{1}{2} \Delta(r_t, y_t, x). \tag{4}$$

[4]For details concerning the choice of space see Björk (2003) and Filipović and Teichmann (2003).

Remark 2.1. *When the HJM drift condition, as expressed in* (4)*, is formulated using the usual parametrization and Itō integrals it only consists of the term* Φ*. The present setting yields two more terms. The term* $\mathbb{F}r_t$ *comes from the use of the Musiela parametrization. The term,* $-1/2\,\Delta$*, comes from the use of Stratonovich integrals.*

Remark 2.2. *To simplify notation we define*

$$\mu_i := \mu(r_t, e_i, x),$$
$$\sigma_i := \sigma(r_t, e_i, x),$$
$$\Phi_i := \Phi(r_t, e_i, x).$$

where $i \in \{1, 2, \ldots, n\}$*. Observe that* μ_i *and* σ_i *are smooth vector fields on* $\mathcal{H}$*. We will also use the following vectors*

$$\boldsymbol{\sigma} := (\sigma_1, \ldots, \sigma_m)$$
$$\boldsymbol{\Phi} := (\Phi_1, \ldots, \Phi_m).$$

Remark 2.3. *We will use the following notation when treating linear vector spaces. If* v_1 *and* v_2 *are vectors we will denote the space spanned by* v_1 *and* v_2 *by*

$$\langle v_1, v_2 \rangle.$$

The Lie algebra generated by two vector fields v_1 *and* v_2 *will be denoted*

$$\langle v_1, v_2 \rangle_{LA}.$$

The short hand notation $\langle v_i \rangle$ *will be used to denote* $\langle v_i; i \in I \rangle$ *when the index set* I *is obvious from the context.*

2.1 Problem Formulation

The main question to be answered in the present paper is the when the regime switching volatility model in (2) admits a finite dimensional realization (FDR). To be more specific we make the following definition.

Definition 2.3 (FDR)**.** *The SDE in* (2) *is said to have a* finite dimensional realization *if there exist smooth d-dimensional vector fields a and b, initial points* $z_0 \in R^d$ *and* $y_0 \in I$*, and a (smooth) mapping* $G : R^d \to \mathcal{H}$ *such that* r_t *has a representation of the following form*

$$r_t(x) = G(z_t, x), \tag{5}$$
$$dz_t = a(z_t, y_t)\,dt + b(z_t, y_t) \circ dW_t, \tag{6}$$
$$z_0 = z^0. \tag{7}$$

The forward curve manifold $\mathcal{G}$ *is defined by*

$$\mathcal{G} = \operatorname{Im} G = \{G(z, \cdot) \in \mathcal{H}; z \in \mathbb{R}^d\}. \tag{8}$$

The tangent space *of* $\mathcal{G}$ *at the point* z *is denoted* $T_{\mathcal{G}}(z)$*.*

Remark 2.4. *Notice that, in the definition above, the Markov chain y only appears in the dynamics of the state variable z_t and not in the mapping G. Setting $r_t = G(z_t, y_t)$ would mean that r_t is allowed to jump with the Markov chain but this would be in conflict with the representation of dynamics of r_t in* (2)*, where y_t only affect the coefficients but no jump term is present.*

Remark 2.5. *The model is formulated under the risk neutral probability measure Q in* (2)*. But we could as well formulated the model under the objective probability measure P. If the equality in* (5) *holds Q almost surely it also holds P almost surely as P and Q are equivalent martingale measures. Formulating the model directly under Q has the advantage of leaving the concept of market price of risk, which has no impact on our results, outside of the modelling.*

The framework developed in Björk and Christensen (1999) and Björk and Svensson (2001) allows us to formulate the following theorem which answer the question of the existence of a finite dimensional realizations.

Theorem 2.1. *The forward rate model in* (2) *admits a finite dimensional realization if and only if the Lie algebra $\mathscr{L}$ generated by $\langle \mu_i, \sigma_i : i \in I \rangle$ denoted $\langle \mu_i, \sigma_i : i \in I \rangle_{LA}$ is finite dimensional at r_0. Here μ_i and σ_i are viewed as vector fields on $\mathcal{H}$.*

Before we proove the theorem we have to make some regularity assumptions.

Proof. This proof follows the proofs of theorem 4.1 in Björk and Christensen (1999) and theorem 3.2 in Björk and Svensson (2001) very closely. It turns out that the introduction of the Markov chain y does not alter the original theorem. Assume that the model admits a FDR. Differentiation of (5) with the help of the Itō formula and use of (6) gives

$$dr_t = G'_z(z_t)a(z_t, y_t)\, dt + G'_z(z_t)b(z_t, y_t) \circ dW_t.$$

Comparing this with (2) produces

$$\begin{aligned} \mu(r_t, y_t) &= G'_z(z_t)a(z_t, y_t) \\ \sigma(r_t, y_t) &= G'_z(z_t)b(z_t, y_t). \end{aligned}$$

We observe that $\operatorname{Im}\left(G'_z(z)\right) = T_{\mathcal{G}}(z)$, so for every $z \in \mathbb{R}^d$ and every $y \in S = \{e_1, e_2, \ldots e_m\}$ we have

$$\begin{aligned} \mu(r, y) &\in T_{\mathcal{G}}(z), \\ \sigma(r, y) &\in T_{\mathcal{G}}(z), \end{aligned}$$

where $r = G(z)$. As the Lie bracket of two vector fields tangential to a manifold is also tangential to the manifold, we also have $[\mu_i(r), \sigma_j(r)] \in T_{\mathcal{G}}(z)$ for all $i, j \in I$. We can conclude that the Lie algebra is finite dimensional.

Assume that that the Lie algebra is finite dimensional. Frobenius' theorem supplies a finite dimensional integral manifold $\mathcal{G}$ (and a mapping G) such that $\mu_i(r) \in T_{\mathcal{G}}(z)$ and $\sigma_i(r) \in T_{\mathcal{G}}(z)$ for all $i \in I$. We first want to find a factor process z_t with dynamics as in (6) and second show that the mapping G actually gives us r as $r = G(z)$. Recall that $T_{\mathcal{G}}(z) = \operatorname{Im} G'_z(z)$, so $\mu_i(r), \sigma_i(r) \in \operatorname{Im} G'_z(z)$. We can therefore find, for each $i \in I$, $a_i(z)$ and $b_i(z)$

such that $\mu_i(r) = G'_z(z)a_i(z)$, and $\sigma_i(r) = G'_z(z)b_i(z)$, or using more compact notation

$$\mu(r, y) = G'_z(z)a(z, y) \tag{9}$$

$$\sigma(r, y) = G'_z(z)b(z, y). \tag{10}$$

The assumed injectivity of G' guarantees the uniqueness of $a_i(z)$ and $b_i(z)$. As $G'_z(z)$ has a smooth left inverse $\left(G'_z(z)\right)^{-1}$ we can solve for $a(z, y)$ and $b(z, y)$ and get

$$a(z, y) = \left(G'_z(z)\right)^{-1}\mu(r, y)$$
$$b(z, y) = \left(G'_z(z)\right)^{-1}\sigma(r, y).$$

where $r = G(z)$. The assumed smoothness of G and of the inverse $\left(G'_z(z)\right)^{-1}$ gives smoothness of $a(z, \cdot)$ and $b(z, \cdot)$ and therefore local Lipschitz continuity yielding a unique strong solution to the SDE (6) for arbitrary initial condition $z_0 \in \mathbb{R}^d$ and $y_0 \in S$. This completes the quest for the factor process z_t. Define $\rho_t = G(z_t)$. Differentiation gives

$$d\rho_t = G'_z(z_t)a(z_t, y_t)\,dt + G'_z(z_t)b(z_t, y_t)\,dt.$$

By using (9) and (10) we see that this equation is identical to (2). Given an arbitrary $r_0 \in \mathcal{G}$, since $\mathcal{G} = \operatorname{Im} G(z)$ and $G(z)$ is injective there is a unique z_0 such that $r_0 = G(z_0)$. If we now set $\rho_0 = z_0$, by uniqueness of the strong solution to (2) we get that $r_t = \rho_t = G(z_t)$. □

Remark 2.6. *The finiteness of state space of the Markov chain is not needed to prove the theorem. When proving necessity the finite dimensionality of the Lie algebra follows from the finite dimensionality of the manifold $\mathcal{G}$, not from the number of states in the Markov chain. When proving sufficiency, we assume the Lie algebra to be finite dimensional, so we implicitly assume that the only finite many μ_i and σ_i are linearly independent, regardless of the number of states in the Markov chain.*

2.2 Constructing realizations

Having settled the question of existence of a finite dimensional realization, the next natural step is to construct such a realization. We will need the following definition to describe the parametrization of forward curve manifold

Definition 2.4. *Let f be a smooth vector field on the space $\mathcal{H}$, and let y be an arbitrary point on $\mathcal{H}$. The solution to the following ODE*

$$\frac{dy_t}{dt} = f(y_t) \qquad y_0 = y,$$

will be denoted

$$y_t = e^{tf}y.$$

We can now describe the manifold using Theorem 4.2 of Björk and Svensson (2001) which we reproduce here for convenience

Theorem 2.2. *Assume that the Lie algebra $\langle \mu, \sigma \rangle_{LA}$ is spanned by the smooth vector fields $f_1, \ldots, f_d$. Then, for the initial point r^0, all forward rate curves produced by the model will belong to the induced tangential manifold $\mathcal{G}$, which can be parametrized as $\mathcal{G} = \mathrm{Im}[G]$, where*

$$G(z_1, \ldots, z_d) = e^{z_d f_d} \cdots e^{z_1 f_1} r^0.$$

and where the operator $e^{z_i f_i}$ is defined as above.

Remark 2.7. *Notice that the preceding theorem shows that the dimension of the realization is equal to the size of the Lie algebra.*

To construct the manifold we follow the procedure outlined in Björk and Landén (2002). Given a volatility $\sigma(r, y, t)$ for which the Lie algebra $\mathscr{L}$ is finite dimensional and given an initial forward rate curve r^0 we perform the following steps

(i) Select a finite number of vector fields $f_1, \ldots, f_d$ spanning the Lie algebra $\mathscr{L}$.

(ii) Determine the manifold $\mathcal{G}$ by using Theorem 2.2 above.

(iii) Set $r_t = G(z_t)$. Make the following *Ansatz* for the dynamics of the state variable z

$$dz_t = a(z_t, y_t)\, dt + b(z_t, y_t) \circ dW_t.$$

Using the Stratonovich version of the Itō formula yields

$$dr_t = dG(z_t) = G'(z_t)\, a(z_t, y_t)\, dt + G'(z_t)\, b(z_t, y_t) \circ dW_t. \tag{11}$$

Identifying the coefficients of the drift and diffusion vector fields of the r process we get

$$G'(z_t) a(z_t, y_t) = \mu(G(z_t), y_t) \qquad G'(z_t) b(z_t, y_t) = \sigma(G(z_t), y_t). \tag{12}$$

These equations can now be used to solve for the vector fields a and b.

Remark 2.8. *It can sometimes be very burdensome to find the* minimal *set of vector fields spanning the Lie algebra. In those cases it is more practical to work with a slightly larger set of vector fields. Occasionally this can cause the equations in* (12) *to have multiple solutions. This is of no consequence since it's enough to find one solution, and any solution will be adequate for our purpose.*

3 Deterministic volatility

We now go on to use the abstract results above to derive concrete results for particular model choices. In this section we consider the simple case where the volatility does not depend on r. We first derive the Lie algebra, then we determine the invariant manifold and find the state variable dynamics. Finally we study regime-switching generalizations of the Ho-Lee and Hull-White models.

3.1 The Lie algebra

We assume that the volatility is of the following form

$$\sigma(r, x, y) = \sigma(x, y), \tag{13}$$

i.e. the volatility does not depend on r. When the Markov chain is in state e_i the volatility $\sigma_i(x)$ is a constant vector field on $\mathcal{H}$.

Using the notation in (4) for the drift μ, the generators of the Lie algebra will be the following vector fields (one for each state of the Markov chain y)

$$\begin{aligned} \sigma(r, x, y) &= \sigma_i \qquad\qquad i \in I \\ \mu(r, x, y) &= \mathbb{F}r + \Phi_i, \end{aligned}$$

Because the σ_i:s are constant as functions of r the Fréchet derivatives $\sigma_i' = 0$ and therefore $\Delta = \sigma_r'[\sigma] = 0$ in (4).

To compute Lie brackets we need the Fréchet derivative of μ_i which is $\mu_i' = \mathbb{F}$ as $\mathbb{F}$ is a linear operator (and in the space $\mathcal{H}$ also a bounded[5] operator).

Taking Lie brackets we get

$$\begin{aligned} [\mu_i, \mu_j] &= \mathbb{F}\big(\mathbb{F}r + \Phi_j\big) - \mathbb{F}\big(\mathbb{F}r + \Phi_i\big) = \mathbb{F}\big(\Phi_j - \Phi_i\big), \\ [\mu_i, \sigma_j] &= \mathbb{F}\sigma_j, \\ [\sigma_i, \sigma_j] &= 0, \end{aligned}$$

where $i, j \in I$. Now the only remaining non-constant vector field is μ_i, so computing the remaining Lie brackets is easy and yields the following system of generators for the Lie algebra

$$\mathscr{L} = \langle \mathbb{F}r + \Phi_i, \mathbb{F}^k\sigma_i, \mathbb{F}^\ell(\Phi_i - \Phi_j); i, j \in I; k, \ell = 0, 1, \ldots\rangle_{LA}. \tag{14}$$

As we are interested in a FDR of the model, we want to know when the Lie algebra $\mathscr{L}$ is finite dimensional (remember Remark 2.7). Now for $\mathscr{L}$ to be finite dimensional, first of all the space $\langle F^k\sigma_i; k = 0, 1, \ldots\rangle$ must be finite dimensional, for every fixed $i \in I$. But for this to occurs, the vector fields $\mathbb{F}^k\sigma_i$ must satisfy a linear relation of the form

$$\mathbb{F}^{n_i}\sigma_i = \sum_{k=0}^{n_i-1} c_k^i \mathbb{F}^k\sigma_i, \tag{15}$$

where $c_k^i \in \mathbb{R}$. We recall that $\mathbb{F} := \partial/\partial x$, so that (15) is in fact an ODE with constant coefficients in the variable σ_i. By general ODE theory σ_i is a solution to (15) if and only if σ_i is a quasi-exponential function.[6]

Definition 3.1 (QE function). *A quasi-exponential function is a function of the following form*

$$f(x) = \sum_i p_i(x)e^{\lambda_i x} + \sum_j e^{\alpha_j x}\big[p_j(x)\cos(\omega_j x) + q_j(x)\sin(\omega_j x)\big],$$

where $\lambda_i, \alpha_j, \omega_j$ are real numbers and p_i, p_j, q_j are polynomials with real coefficients.

[5]The boundedness of $\mathbb{F}$ is the reason underlying the choice of space $\mathcal{H}$. For details consult Björk (2003) and Filipović and Teichmann (2003).

[6]See e.g. Schwartz (1992), chapter IV, §4.

The preceding argument, shows that a necessary condition for the model to posses a FDR is that σ_i is QE. It turns out this condition also is sufficient. To show this we will need the following Lemma which lists some well known the properties of QE functions.

Lemma 3.1. *Quasi exponential functions have the following properties*

(i) A function is QE if and only if it is a component of the solution of a vector-valued linear ODE with constant coefficients.

(ii) A function is QE if and only if it can be written as $f(x) = ce^{Ax}b$.

(iii) If f *is QE, then* f' *is QE.*

(iv) If f *is QE, then its primitive function is QE.*

(v) If f *and* g *are QE, then* fg *is QE.*

We now have the following result.

Proposition 3.1. *Assume that the volatility is deterministic i.e. of the following form*

$$\sigma(r, y, x) = \sigma(y, x).$$

Then the model in (2) *posseses a FDR if and only if* $\sigma(x, y)$ *is a quasi-exponential function in the* x *variable.*

Proof. Having proved necessity, we assume that σ_i is QE for every $i \in I$. Then by (i) we get that $\langle \mathbb{F}^k \sigma_i \rangle$ is finite dimensional. The function $\Phi_i = \sigma \int_0^x \sigma^\top \, ds$ is QE by (v) and (iv) since it is composed of a QE multiplied by the integral of a QE. In addition $\Phi_i - \Phi_j$ is QE as linear combinations of QE are QE. We can therefore also conclude that $\langle \mathbb{F}^{\ell+1}(\Phi_i - \Phi_j) \rangle$ is finite dimensional. □

Knowing that the volatility σ is quasi-exponential in the variable x allows us to give a simpler system of generators for the Lie algebra $\mathscr{L}$ than the one provided in (14). This will facilitate the calculation of the dimension of the Lie algebra and the construction of realization in the next section. We have the following result.

Proposition 3.2. *Assume that the volatility* $\sigma_i(x)$ *has the following form*

$$\sigma_i(x) = \sum_{j=1}^{N_i} p_{i,j}(x)\, e^{-\alpha_{i,j} x},$$

where $p_{i,j}(x)$ *is a polynomial of degree* $n_{i,j}$, $\alpha_{i,j} \in \mathbb{R}$, *and* $N_i \in \mathbb{N}$. *Then*

$$\langle \mu_i, \sigma_i \rangle_{LA} = \langle \mathbb{F}r, \mathbb{F}^k \sigma_i, \mathbb{F}^\ell \Phi_i; i \in I; k = 0, \ldots, \mathfrak{n}_1^i; \ell = 0, \ldots, \mathfrak{n}_2^i \rangle_{LA}, \tag{16}$$

where

$$\mathfrak{n}_1^i = \sum_{j=1}^{N_j} n_j^i + N_i - 1$$
$$\mathfrak{n}_2^i = \sum_{(j,k)} (n_j^i + n_k^i) + N_i^2 - 1$$

We will prove this proposition in four steps, each step giving rise to a lemma. The first lemma shows that the linear space $\langle\ \mathbb{F}^k\sigma_i\rangle$ is spanned by monomials $x^j e^{\alpha_k x}$. Although this lemma is not used explicitly in the proof of the proposition above, it is used to prove the next three lemmas. Before stating the second lemma we need to introduce the vector field $\tilde{\Phi}_i$. The second lemma then shows that the linear space $\langle\mathbb{F}^\ell(\tilde{\Phi}_i - \tilde{\Phi}_j)\rangle$ is spanned by monomials $x^k e^{-(\alpha_{i,j}+\alpha_{i,k})x}$ thereby allowing us to disentangle the vector fields $\mathbb{F}(\tilde{\Phi}_i - \tilde{\Phi}_j)$ into $\mathbb{F}\tilde{\Phi}_i$ and $\mathbb{F}\tilde{\Phi}_j$. The third lemma shows that we can also disentangle the vector field $\mathbb{F}r + \tilde{\Phi}_i$ into $\mathbb{F}r$ and $\tilde{\Phi}_i$. Finally, in the fourth lemma we calculate the size of the Lie algebra $\mathscr{L}$ which enable us to specify $\mathfrak{n}_1^i$ and $\mathfrak{n}_2^i$ mentioned in the proposition.

To avoid cluttering the notation with (unnecessary) double indices we will temporarily remove the dependence on y from the expressions. The first lemma used in the proof of the proposition is as follows.

Lemma 3.2. *Assume that the volatility $\sigma(x)$ has the following form*

$$\sigma(x) = \sum_{j=1}^{N} p_j(x) e^{-\alpha_j x},$$

where $p_j(x)$ is a polynomial of degree n_j, $\alpha_j \in \mathbb{R}$, and $N \in \mathbb{N}$. Then

$$\langle\mathbb{F}^k\sigma(x); k = 0, 1, \ldots\rangle = \langle\mathbb{F}^k\big(x^{n_j} e^{-\alpha_j x}\big); k = 0, 1, \ldots; j = 1, \ldots, N.\rangle. \quad (17)$$

Proof. We start by showing that

$$\langle\mathbb{F}^k\sigma(x); k = 0, 1, \ldots\rangle = \langle x^i\, e^{-\alpha_j x}; j = 1, \ldots, N; i = 0, \ldots, n_j\rangle \quad (18)$$

The equality (17) then follows from the trivial fact that

$$\langle\mathbb{F}^k\big(x^{n_j}\, e^{-\alpha_j x}\big); k = 0, 1, \ldots; j = 1, \ldots N\rangle = \langle x^i\, e^{-\alpha_j x}; j = 1, \ldots, N; i = 0, \ldots, n_j\rangle.$$

To prove (18) we first have to show that every element in the left hand side (LHS) can be written as a linear combination of the elements on the right hand side (RHS). But this is trivial since we know that σ is quasi-exponential. Conversely, we have to show that every monomial $x^i e^{-\alpha_j x}$ on the RHS of (18) can be written as a linear combination of the vector fields $\mathbb{F}^k\sigma$ on the LHS. To this end we will use the operator $(\mathbb{F} + \alpha_j)$ which lowers the degree of the polynomial $p_j(x)$. To make things as transparent as possible, we start by assuming that $N = 1$ i.e. that $\sigma = p(x)e^{-\alpha x}$, where $p(x)$ is a polynomial of degree n. We then have that $(\mathbb{F} + \alpha)\sigma = p'(x)e^{-\alpha x} + p(x)(-\alpha)e^{-\alpha x} + \alpha p(x)e^{-\alpha x} = p'(x)e^{-\alpha x}$ where $p'(x) = dp(x)/dx$.

Applying powers of the operator $\mathbb{F} + \alpha_1$ yields

$$\begin{aligned}
(\mathbb{F} + \alpha_1)^0\sigma(x) &= p_n(x)e^{-\alpha x} \\
(\mathbb{F} + \alpha_1)^1\sigma(x) &= p_{n-1}(x)e^{-\alpha x} \\
&\vdots \\
(\mathbb{F} + \alpha_1)^n\sigma(x) &= p_0(x)e^{-\alpha x},
\end{aligned}$$

where $p_i(x)$ denotes a polynomial of degree i. Having at our disposal one polynomial of each degree allow us to perform Gaussian elimination and thereby express each monomial $x^i e^{-\alpha x}$ as a linear combination of different powers of the operator $\mathbb{F} + \alpha$. It only remains to show that every power of the operator $\mathbb{F} + \alpha$ can itself be expressed as a linear combination of the vector fields $\mathbb{F}^k \sigma$. A simple application of the binomial theorem shows that

$$(\mathbb{F} + \alpha)^k \sigma = \sum_{i=0}^{n} \binom{n}{i} \alpha^{n-i} \mathbb{F}^i \sigma,$$

and we are done with the case $N = 1$.

Next we have to show that we can obtain the same result when $N > 1$. To keep the notation simple we now treat the case $N = 2$. To show that the result also holds for higher values of N is trivial. So now we assume that $\sigma = q_1(x)e^{-\alpha_1 x} + q_2(x)e^{-\alpha_2 x}$, where the $q_i(x)$ denotes a polynomial of degree n_i. Now notice that the operator $\mathbb{F} + \alpha_1$ only lowers the degree in front of $e^{\alpha_1 x}$ leaving the degree of the polynomial in front of $e^{\alpha_2 x}$ unchanged as can be seen in the following calculation

$$(\mathbb{F} + \alpha_1)\sigma = q_1'(x)e^{-\alpha_1 x} + [q_2'(x) + (\alpha_1 - \alpha_2)q_2(x)]e^{-\alpha_2 x}.$$

This means that we can recover the monomials $x^i e^{\alpha_1 x}$ by first annihilating the polynomial $q_2(x)$ from σ as $(\mathbb{F} + \alpha_2)^{n_2+1}\sigma = \pi_1(x)e^{\alpha_1 x}$, where $\pi_1(x)$ is a polynomial of degree n_1. Then we just apply the same procedure as in the $N = 1$ case, by operating with powers of the operator $\mathbb{F} + \alpha_1$ thereby recovering all monomials $x^i e^{-\alpha_1 x}$. We can then apply the same procedure to recover all monomials $x^i e^{-\alpha_2 x}$ by first annihilating $q_1(x)$ with the help of $(\mathbb{F} + \alpha_1)^{n_1}$ □

Before stating the next lemma we will replace the vector fields Φ_i in the generator set of $\mathscr{L}$ in (14) with the vector fields $\tilde{\Phi}_i$. This is because Φ_i is composed of monomials of the form $x^i e^{-(\alpha_j + \alpha_k)x}$ as well as $x^i e^{-\alpha_j x}$ and the latter are redundant in the sense that they already are generated by $\mathbb{F}^k \sigma$. The vector fields $\tilde{\Phi}$ will therefore only involve the first kind of monomials.

Let us for a moment put aside the dependence on y and study $\Phi(x)$. To this end we partially integrate the integral of $\sigma(x)$ in the expression for $\Phi(x)$ as follows

$$\begin{aligned}\Phi(x) &= \Big(\sum_{i=1}^{N} p_i(x)e^{-\alpha_i s}\Big)\Big(\int_0^x \sum_{j=1}^{N} p_j(s)e^{-\alpha_j s} ds\Big) \\ &= \Big(\sum_{i=1}^{N} p_i(x)e^{-\alpha_i s}\Big)\Big(\sum_{j=1}^{N} q_j(x)e^{-\alpha_j x} + K\Big),\end{aligned}$$

where

$$\begin{aligned}q_j(x) &= -\Big(\frac{p_j(x)}{\alpha_j} + \frac{\mathbb{F}p_j(x)}{\alpha_j^2} + \dots \frac{\mathbb{F}^n p_j(x)}{\alpha_j^{n+1}}\Big) \\ K &= \sum_{j=1}^{N}\Big(\sum_{i=0}^{N} \frac{1}{\alpha_j^{i+1}} \mathbb{F}^i p_j^i(0)\Big).\end{aligned}$$

By expanding the product we get

$$\Phi(x) = \sum_{(i,j)} p_i(x) q_j(x) e^{-(\alpha_i+\alpha_j)x} + K\,\sigma(x). \tag{19}$$

We notice that the last term of this expression for Φ involves σ and therefore monomials of the form $x^i e^{\alpha_j x}$. We now define

$$\tilde{\Phi}(x) = \Phi(x) - K\sigma(x) \tag{20}$$
$$= \sum_{(i,j)} p_i(x) q_j(x) e^{-(\alpha_i+\alpha_j)x}. \tag{21}$$

We can now replace every occurrence of Φ_i in (14) with $\tilde{\Phi}_i$. This can be shown by simple use of the relation $\tilde{\Phi}_i = \Phi_i + K_i\sigma_i$. The vector fields $\mathbb{F}r + \tilde{\Phi}_i = \mathbb{F}r + \Phi_i + K_i\sigma_i \in \mathscr{L}$. We can therefore add them to the set of generators of $\mathscr{L}$. Next as $\tilde{\Phi}_i - \tilde{\Phi}_j = \Phi_i - \Phi_j + K_i\sigma_i - K_j\sigma_j$ we see that $\mathbb{F}^k(\tilde{\Phi}_i - \tilde{\Phi}_j) = \mathbb{F}^k(\Phi_i - \Phi_j) - K_i\mathbb{F}^k\sigma_i - K_j\mathbb{F}^k\sigma_j \in \mathscr{L}$. We can therefore add $\mathbb{F}^k(\tilde{\Phi}_i - \tilde{\Phi}_j)$ to the set of generators of $\mathscr{L}$.

In the next lemma we treat the generators $\mathbb{F}^\ell(\tilde{\Phi}_i - \tilde{\Phi}_j)$. We are therefore forced to reintroduce the index for the Markov chain y in the notation.

Lemma 3.3. *Assume that the volatility σ_i has the following form*

$$\sigma_i(x) = \sum_{j=1}^{N_i} p_{i,j}(x) e^{-\alpha_{i,j}x},$$

where $i \in I$, $p_{i,j}$ is a polynomial of degree $n_{i,j}$, $\alpha_{i,j} \in \mathbb{R}$, and $N \in \mathbb{N}$. Then

$$\begin{aligned}\langle \mathbb{F}^\ell(\tilde{\Phi}_i - \tilde{\Phi}_j); i,j \in I; \ell = 0,1,\ldots\rangle \\ = \langle \mathbb{F}^\ell \tilde{\Phi}_i; i \in I; \ell = 0,1,\ldots, n_{i,j} + n_{i,k}\rangle \\ = \langle x^\ell e^{-(\alpha_i^j + \alpha_i^k)x}; i \in I; j,k = 0,1,\ldots; \ell = 0,\ldots, n_{i,j} + n_{i,k}\rangle.\end{aligned}$$

Proof. As $\tilde{\Phi}_i(x)$ has exactly the same functional form as $\sigma(x)$ we can apply Lemma 3.2 on $\tilde{\Phi}_i(x)$ and get that

$$\begin{aligned}\langle \mathbb{F}^\ell \tilde{\Phi}_i(x); i \in I; \ell = 0,1,\ldots\rangle \\ = \langle \mathbb{F}^\ell(x^{n_j+n_k} e^{-(\alpha_j+\alpha_k)x}); i \in I; j,k = 1,\ldots,N_i; \ell = 0,1,\ldots\rangle,\end{aligned}$$

which proves the second equality.

To prove the first equality we just note that $\tilde{\Phi}_i - \tilde{\Phi}_j$ also has the same functional form as σ so Lemma 3.2 can be applied again. □

The next lemma disentangles the generators $\mathbb{F}r + \tilde{\Phi}_i$ into $\mathbb{F}r$ and $\tilde{\Phi}_i$. This turns out to be be important for the study of the invariant manifold in the next section, as it separates the effect of the vector field $\mathbb{F}r$, making calculations a lot easier and producing a neater parametrization.

Lemma 3.4. *Assume that the volatility σ_i has the following form*

$$\sigma_i(x) = \sum_{j=1}^{N_i} p_{i,j}(x) e^{-\alpha_{i,j} x},$$

where $i \in I$, $p_{i,j}$ is a polynomial of degree $n_{i,j}$, $\alpha_{i,j} \in \mathbb{R}$, and $N \in \mathbb{N}$. Then

$$\langle \mathbb{F}r + \tilde{\Phi}_i; i \in I \rangle = \langle \mathbb{F}r, \tilde{\Phi}_i; i \in I \rangle$$

Proof. Lemma 3.3 shows that $\tilde{\Phi}_i \in \mathscr{L}$. Therefore we get that $\mu_i - \tilde{\Phi}_i = \mathbb{F}r \in \mathscr{L}$, meaning that we can add $\mathbb{F}r$ to the set of generators, thereby disentangling the generators $\mathbb{F}r + \tilde{\Phi}_i$. □

The next lemma gives us an upper bound for the dimension of the Lie algebra for a given finite dimensional deterministic volatility model.

Proposition 3.3. *The dimension of the Lie algebra in* (14) *is at most given by*

$$\dim(\mathscr{L}) = 1 + \sum_{i=1}^{m} \Big(\sum_{j} n_j^i + \sum_{(j,k)} (n_j^i + n_k^i) + N_i(N_i + 1) \Big),$$

where $j, k = 1, \dots, N_i$.

Proof. For a given state j of the Markov chain, $\langle \mathbb{F}^k \sigma \rangle$ is spanned by $\sum_{j=1}^{N} (n_j + 1) = \sum_{j=1}^{N} n_j + N$ monomials of the form $x^k e^{-\alpha_j x}$. The space $\langle \mathbb{F}^k \tilde{\Phi} \rangle$ is spanned by $\sum_{(j,k)} (n_j + n_k + 1) = \sum_{(j,k)} (n_j + n_k) + N^2$ monomials of the form $x^\ell e^{-(\alpha_j + \alpha_k)}$. Finally we sum over all states of the Markov chain and add the vector field $\mathbb{F}r$. As it can happen that some combinations of indices give rise to the same function $e^{-\alpha x}$ or that some coefficients in the polynomials are zero, the calculation only gives an upper bound on the dimension. □

Example 3.1 (A simple regime-switching model)**.** *In a simple, non-degenerate (i.e. where $\alpha_i \neq 0$), regime switching model with deterministic volatility, the Markov chain has two states and the volatility has the form $e^{-\alpha_i x}$. In terms of the above proposition this means that $m = 2$, $N_1 = N_2 = 1$, $n_1^1 = n_1^2 = 0$ so that the size of the realization is* $\dim(\mathscr{L}) = 5$.

Remark 3.1. *It follows from* (20) *that we can use Φ_i as a generator instead of $\tilde{\Phi}_i$ in* (16)*. But, in the next section, when computing the invariant manifold and constructing realizations we will use $\tilde{\Phi}_i$ as we know that it satisfies a particular ODE.*

3.2 The Invariant manifold

Having derived the Lie algebra we can now parametrize the manifold in which the forward rate curve evolves. This is done by applying step (ii) in the procedure outlined in Section 2.2. The parametrization will be used in the next section when the state variable dynamics of the FDR are derived.

First of all we have to select which vector fields to use to span the Lie algebra. As the preceding section demonstrates, there are a lot of possible choices. It turns out, that for the task at hand, the most important choice is that the vector fields $\mu_i = \mathbb{F}r + \Phi_i$ is broken up into $\mathbb{F}r$ and Φ_i. This is because we are dealing with deterministic volatility, so the only non constant vector field in any parametrization will be one involving the term $\mathbb{F}r$. Here we use the following generator set for the Lie algebra $\mathscr{L}$.

$$\langle \mathbb{F}r, \mathbb{F}^k\sigma_i, \mathbb{F}^\ell\tilde{\Phi}_i; i \in I; k = 0, 1, \ldots, \mathfrak{n}_1^i; \ell = 0, 1, \ldots, \mathfrak{n}_2^i\rangle_{LA}.$$

To find a parametrization of the invariant manifold we apply Theorem 2.2. In practice this amounts to calculating the operator e^{tf_0}, $e^{tf_{1,k}^i}$, and $e^{tf_{2,\ell}^i}$ where

$$\begin{aligned} f_0 &= \mathbb{F}r \\ f_{1,k}^i &= \mathbb{F}^k\sigma_i \qquad k = 0, 1, \ldots, \mathfrak{n}_1^i \\ f_{2,\ell}^i &= \mathbb{F}^\ell\tilde{\Phi}_i \qquad \ell = 0, 1, \ldots, \mathfrak{n}_2^i, \end{aligned}$$

and then applying them in turn to the initial forward rate curve r_0.

Recall that according to Definition 2.4, to calculate the operator e^{tf_0} we need to solve the equation

$$\frac{dr}{dt} = \mathbb{F}r$$

which is a linear equation with solution

$$r_t = e^{t\mathbb{F}}r_0.$$

As $\mathbb{F}$ is the generator of the semigroup of left translations[7], given the initial forward rate curve r_0, the solution can be written as

$$(e^{t\mathbb{F}}r_0)(x) = r_0(x + t) \tag{22}$$

The vector fields $f_{1,k}^i$ and $f_{2,\ell}$ are constant so the solution to the corresponding ODEs are easily given by

$$\begin{aligned} (e^{tf_{1,k}^i}r_0)(x) &= r_0(x) + \mathbb{F}^k\sigma_i(x)\,t, \\ (e^{tf_{2,k}^i}r_0)(x) &= r_0(x) + \mathbb{F}^k\tilde{\Phi}_i(x)\,t. \end{aligned}$$

Applying all the operators in turn on the initial forward rate produces the following parametrization of the invariant manifold.

Proposition 3.4. *The invariant manifold generated by the initial forward rate curve r_0 is parametrized as*

$$G(\mathbf{z}) = r_0(x + z_0) + \sum_{i=1}^{m}\Big(\sum_{k=0}^{\mathfrak{n}_1^i}\mathbb{F}^k\sigma_i(x)z_{1,k}^i + \sum_{\ell=0}^{\mathfrak{n}_2^i}\mathbb{F}^\ell\Phi_i(x)z_{2,\ell}^i\Big) \tag{23}$$

where $\mathbf{z} = (z_0, z_{1,k}^i, z_{2,\ell}^i; i \in I; k, l = 0, 1, \ldots)$.

[7] For details see e.g. Section 2.10 in Engel and Nagel (2000).

Remark 3.2. *In the previous proposition we can replace the vector fields $\mathbb{F}^k\sigma_i$ and $F^\ell\Phi_i$ by the monomials of the quasi exponential functions that span the same space. This provides us with the following parametrization of the invariant manifold*

$$G(\mathbf{z}) = r_0(x+z_0) + \sum_{i=1}^{m}\Big(\sum_{j=1}^{N_i}\sum_{k=0}^{n_{i,j}} x^k e^{-\alpha_{i,j}x}\cdot z^1_{i,j,k} + \sum_{(j,k)}\sum_{\ell=0}^{n_{i,j}+n_{i,k}} x^\ell e^{-(\alpha_{i,j}+\alpha_{i,k})x}\cdot z^2_{i,j,k,\ell}\Big).$$

3.3 State variable dynamics

Having computed the manifold G we can now proceed to step (iii) in the procedure outlined in section 2.2.

As $\langle \mathbb{F}^k\sigma_i, k=0,1,\ldots\rangle$ and $\langle \mathbb{F}^\ell\tilde{\Phi}_i, \ell=0,1,\ldots\rangle$ are finite dimensional linear spaces of dimension $\mathfrak{n}_1^i+1$ and $\mathfrak{n}_2^i+2$ respectively. Therefore we know that σ_i and $\tilde{\Phi}_i$ must satisfy linear relations of the form

$$\mathbb{F}^{\mathfrak{n}_1^i+1}\sigma_i = \sum_{k=0}^{\mathfrak{n}_1^i} c^1_{i,k}\mathbb{F}^k\sigma_i \tag{24}$$

$$\mathbb{F}^{\mathfrak{n}_2^i+1}\sigma_i = \sum_{\ell=0}^{\mathfrak{n}_2^i} c^2_{i,\ell}\mathbb{F}^\ell\tilde{\Phi}_i. \tag{25}$$

We start by computing the Fréchet derivative of G. Let $\mathbf{h}_1^i := (h^i_{1,0},\ldots,h^i_{1,\mathfrak{n}_1^i})$, and $\mathbf{h}_2^i := (h^i_{2,0},\ldots,h^i_{2,\mathfrak{n}_2^i})$ where $i\in I$. Then

$$G'(\mathbf{z})(h_0,h_1^i,h_2^i; i\in I) = \mathbb{F}r_0(x+z_0)\,h_0 + \sum_{i=1}^{m}\Big(\sum_{k=0}^{\mathfrak{n}_1^i}\mathbb{F}^k\sigma_i(x)h^i_{1,k} + \sum_{\ell=0}^{\mathfrak{n}_2^i}\mathbb{F}^\ell\tilde{\Phi}_i(x)h^i_{2,\ell}\Big).$$

Next, to solve the equation $G'(\mathbf{z})\cdot\mathbf{a}(\mathbf{z},y_t) = \mu(r_t,y_t)$ for $\mathbf{a}(\mathbf{z},y_t)$ we need to compute $\mu(r_t,y_t) = \mathbb{F}r_t + \tilde{\Phi}(x,y_t) = \mathbb{F}G(\mathbf{z}) + \tilde{\Phi}(x,y_t)$. Taking the derivative of

(23) we get

$$\begin{aligned}
\mu(r_t, y_t) &= \mathbb{F} r_0(x+z_0) + \sum_{i=1}^{m} \Big(\sum_{k=0}^{\mathfrak{n}_1^i} \mathbb{F}^{k+1} \sigma_i(x) z_{1,k}^i + \sum_{\ell=0}^{\mathfrak{n}_2^i} \mathbb{F}^{\ell+1} \tilde{\Phi}_i(x) z_{2,\ell}^i \Big) + \tilde{\Phi}(x, y_t) \\
&= \lceil \text{using (24) and (25) } \rceil \\
&= \mathbb{F} r_0(x+z_0) + \sum_{i=1}^{m} \Big(\sum_{k=0}^{\mathfrak{n}_1^i - 1} \mathbb{F}^{k+1} \sigma_i(x) z_{1,k}^i + \sum_{k=0}^{\mathfrak{n}_1^i} c_{i,k}^1 \mathbb{F}^k \sigma_i z_{1,\mathfrak{n}_1^i}^i \\
&+ \sum_{\ell=0}^{\mathfrak{n}_2^i - 1} \mathbb{F}^{\ell+1} \tilde{\Phi}_i(x) z_{2,\ell}^i + \sum_{\ell=0}^{\mathfrak{n}_2^i} c_{i,\ell}^2 \mathbb{F}^\ell \tilde{\Phi}_i(x) z_{2,\mathfrak{n}_2^i}^i \Big) + \tilde{\Phi}(x, y_t) \\
&= \lceil \text{after change of indices} \rceil \\
&= \mathbb{F} r_0(x+z_0) + \sum_{i=1}^{m} \Big(\sum_{k=1}^{\mathfrak{n}_1^i} \mathbb{F}^{k} \sigma_i(x) z_{1,k-1}^i + \sum_{k=0}^{\mathfrak{n}_1^i} c_{i,k}^1 \mathbb{F}^k \sigma_i z_{1,\mathfrak{n}_1^i}^i \\
&+ \sum_{\ell=1}^{\mathfrak{n}_2^i} \mathbb{F}^{\ell} \tilde{\Phi}_i(x) z_{2,\ell-1}^i + \sum_{\ell=0}^{\mathfrak{n}_2^i} c_{i,\ell}^2 \mathbb{F}^\ell \tilde{\Phi}_i(x) z_{2,\mathfrak{n}_2^i}^i \Big) + \tilde{\Phi}(x, y_t)
\end{aligned}$$

By setting $G'(\mathbf{z})\,\mathbf{a}(\mathbf{z}, y_t) - \mu(r_t, y_t) = 0$ and rearranging, we get

$$\begin{aligned}
\mathbb{F} r_0(x+z_0)[a_0 - 1] + \sum_{i=1}^{m} \Big(\sum_{k=1}^{\mathfrak{n}_1^i} \mathbb{F}^k \sigma_i [a_{1,k}^i - z_{1,k-1}^i - c_{1,k}^i z_{1,\mathfrak{n}_1^i}] \\
\sigma_i(x)[a_{1,0}^i - c_{1,0}^i z_{1,\mathfrak{n}_1^i}] + \sum_{\ell=1}^{\mathfrak{n}_2^i} \mathbb{F}^\ell \tilde{\Phi}_i(x) [a_{2,\ell}^i - z_{2,\ell-1}^i - c_{2,\ell}^i z_{2,\mathfrak{n}_2^i}] \\
+ \tilde{\Phi}_i [a_{2,0}^i - c_{2,0} z_{2,\mathfrak{n}_2^i}^i - \mathbb{I}_t^i] \Big) = 0
\end{aligned}$$

Here $\mathbb{I}_t^i$ is the indicator function such that $\mathbb{I}_t^i = 1$ if $y_t = e_i$ and $\mathbb{I}_t^i = 0$ otherwise. Now as the preceding equation must hold for all x we can solve for a_0, $a_{1,k}^i$, and $a_{2,\ell}^i$ and get

$$\begin{aligned}
a_0 &= 1 \\
a_{1,0}^i &= c_{1,0}^i z_{1,\mathfrak{n}_1^i}^i \\
a_{1,k} &= z_{1,k-1}^i + c_{1,k}^i z_{1,\mathfrak{n}_1^i}^i \qquad k = 1, \ldots, \mathfrak{n}_1^i \\
a_{2,0}^i &= c_{2,0}^i z_{2,\mathfrak{n}_2^i}^i + \mathbb{I}_t^i \\
a_{2,\ell}^i &= z_{2,\ell-1}^i + c_{2,\ell}^i z_{2,\mathfrak{n}_2^i} \qquad \ell = 1, \ldots, \mathfrak{n}_2^i
\end{aligned}$$

Next we solve the equation $G'(\mathbf{z}) \cdot \mathbf{b}(\mathbf{z}, y_t) = \sigma(y_t)$ for $\mathbf{b}(\mathbf{z}, y_t)$

$$\mathbb{F} r_0(x+z_0)\, b_0 + \sum_{i=1}^{m} \Big(\sum_{k=1}^{\mathfrak{n}_1^i} \mathbb{F}^k \sigma(x) b_{1,k}^i + \sigma_i(x)[b_{1,0}^i - \mathbb{I}_t^i] + \sum_{\ell=0}^{\mathfrak{n}_2^i} \mathbb{F}^\ell \tilde{\Phi}_i(x) b_{2,\ell}^i \Big) = 0$$

The solution is

$$
\begin{aligned}
b_0 &= 0 \\
b^i_{1,0} &= \mathbb{I}^i_t \\
b^i_{1,k} &= 0 \qquad k = 1,\dots,\mathfrak{n}^i_1 \\
b^i_{2,\ell} &= 0 \qquad \ell = 0,\dots,\mathfrak{n}^i_2
\end{aligned}
$$

Proposition 3.5. *Given a deterministic volatility as in* (13) *of the quasi-exponential form and an initial forward rate curve* r_0*, the forward rate model in* (2) *has a finite dimensional realization in the sense of Definition 2.3 where the dynamics of the state variable* $\mathbf{z}$ *are given by*[8]

$$
\begin{aligned}
dz_0 &= dt \\
dz^i_{1,0} &= c^i_{1,0} z^i_{1,\mathfrak{n}^i_1}\, dt + \mathbb{I}^i_t\, dW_t \\
dz^i_{1,k} &= (z^i_{1,k-1} + c^i_{1,k} z^i_{1,\mathfrak{n}^i_1})\, dt \qquad k = 1,\dots,\mathfrak{n}^i_1. \\
dz^i_{2,0} &= (c^i_{2,0} z^i_{2,\mathfrak{n}^i_2} + \mathbb{I}^i_t)\, dt \\
dz^i_{2,\ell} &= (z^i_{2,\ell-1} + c^i_{2,\ell} z^i_{2,\mathfrak{n}^i_2})\, dt \qquad \ell = 1,\dots,\mathfrak{n}^i_2.
\end{aligned}
$$

3.4 Example: Ho-Lee

The simplest possible case of regime switching deterministic volatility one can think of is when the volatility jumps between constant states i.e. $\sigma(r,y,x) = \sigma(y_t)$.

To derive the invariant manifold and the dynamics of the state variables we could now apply the general results of the previous section. This would correspond to the degenerate case where the polynomial has degree zero and $\alpha_i = 0$ for all $i \in I$. It is therefore more informative to obtain these results from scratch thereby providing a further illustration of simplicity of the use of the Lie algebraic machinery.

The Lie algebra $\mathscr{L}$ is generated by the vector fields

$$
\begin{aligned}
\sigma_i(r,x) &= \sigma_i \qquad\qquad i \in I \\
\mu_i(r,x) &= \mathbb{F}r + \Phi_i = \mathbb{F}r + \sigma_i^2 x
\end{aligned}
$$

First we notice that all σ_i are multiples of the same unit vector field $\mathbf{e}(x) \equiv 1 \in \mathscr{L}$. We also notice that, for $i,j \in I$, $\mu_i - \mu_j = (\sigma_i^2 - \sigma_j^2)x \in \mathscr{L}$. Therefore we can add e and x to the set of generators and after simplification we are left with the following three vector fields

$$
\begin{aligned}
f_0 &= \mathbb{F}r \\
f_1 &= e \\
f_2 &= x.
\end{aligned}
$$

[8]Recall that when the volatility is deterministic the Itō and Stratonovich dynamics are the same.

Remark 3.3. *The number of states of the Markov chain has no effect on the structure of the Lie algebra, and consequently neither on the invariant manifold nor on the state variable dynamics.*

To obtain the invariant manifold we apply Theorem 2.2 by first solving for the operators e^{tf_0}, e^{tf_1}, and e^{tf_2}, and then letting them act on the initial forward rate curve r_0. As in (22), using that $\mathbb{F}$ is the generator of the semi-group of left translation, we get

$$(e^{z_0\mathbb{F}}r_0)(x) = r_0(x+z_0).$$

The vector fields $\mathbf{e}$ and x being constant we get that

$$\begin{aligned}(e^{z_1 f_1}r_0)(x) &= r_0(x) + z_1\\ (e^{z_2 f_2}r_0)(x) &= r_0(x) + x\, z_2.\end{aligned}$$

Applying the operator in turn produces the following parametrization of the invariant manifold

$$\begin{aligned}G(z_0, z_1, z_2) &= e^{z_2 f_2}e^{z_1 f_1}e^{z_0 f_0}r_0(x)\\ &= r_0(x+z_0) + z_1 + xz_2.\end{aligned}$$

The dynamics of the state variables z_0, z_1, and z_3 are obtained by comparing the coefficients of the SDE in (2) with those obtained in (11). For the drift we obtain the equation $G'(\mathbf{z})\,\mathbf{a} = \mu$. Here

$$\begin{aligned}G'(\mathbf{z})\,\mathbf{a} &= (\mathbb{F}r_0(x+z_0), e, x)\cdot(a_1, a_2, a_3) = \mathbb{F}r_0(x+z_0)a_1 + a_2 + a_3 x\\ \mu &= \mathbb{F}r + \Phi = \mathbb{F}G(\mathbf{z}) + \Phi = \mathbb{F}r_0(x+z_0) + z_2 + \sigma^2(y)x.\end{aligned}$$

Setting $G'(\mathbf{z})\mathbf{a} - \mu = 0$ yields

$$[a_1 - 1]\mathbb{F}r_0(x+z_0) + [a_3 - \sigma^2(y)]x + a_2 - z_2 = 0.$$

As this equation must hold for all x we get

$$\begin{aligned}a_1 &= 1\\ a_2 &= z_2\\ a_3 &= \sigma^2(y)\end{aligned}$$

For the volatility we set $G'(\mathbf{z})\mathbf{b} - \sigma(y) = 0$ and obtain

$$\mathbb{F}r_0(x+z_0)b_1 + xb_3 + b_2 - \sigma(y) = 0.$$

Again, as this equation must hold for all x we get

$$\begin{aligned}b_1 &= 0\\ b_2 &= \sigma(y)\\ b_3 &= 0.\end{aligned}$$

We can now sum up our findings.

Proposition 3.6 (Ho-Lee example)**.** *Assume that the volatility is given by* $\sigma(r_t, y_t, x) = \sigma(y_t)$*. Then the invariant manifold generated by the initial forward rate curve is given by*

$$G(z_0, z_1, z_2) = r_0(x + z_0) + z_1 + xz_2.$$

where the state variables have the following dynamics

$$\begin{aligned} dz_0 &= dt \\ dz_1 &= z_2\, dt + \sigma(y_t) dW_t \\ dz_2 &= \sigma^2(y_t) dt \end{aligned}$$

3.5 Example: Hull-White

As our second example we look at the Hull and White (1990) extension of the Vasiček (1977) model where the volatility takes the form $\sigma(r, x) = \beta e^{-\alpha x}$. If we let both the parameters α and β be driven by the Markov chain y we get following volatility

$$\sigma(y, x) = \beta(y) e^{-\alpha(y)x}.$$

To calculate the invariant manifold we apply Proposition 3.5. The volatility solves the following ODE

$$\mathbb{F}\sigma_i(x) = -\alpha_i\, \sigma_i(x),$$

meaning that $c^i_{1,0} = -\alpha_i$. Next

$$\Phi_i = \frac{\beta_i^2}{\alpha_i}(e^{-\alpha_i x} - e^{-2\alpha_i x}),$$

so

$$\tilde{\Phi}_i = \Phi_i - \frac{\beta_i}{\alpha_i}\sigma_i = -\frac{\beta_i^2}{\alpha_i} e^{-2\alpha_i x},$$

and $\tilde{\Phi}$ solves the following ODE

$$\mathbb{F}\tilde{\Phi}_i(x) = -2\alpha_i \tilde{\Phi}_i(x)$$

giving $c^i_{2,0} = -2\alpha_i$. By Proposition 3.5 the state variables $\mathbf{z}$ will have the following dynamics

$$\begin{aligned} dz_0 &= dt \\ dz^i_{1,0} &= -\alpha_i z^i_{1,\mathfrak{n}^i_1}\, dt + \mathbb{I}^i_t\, dW_t \\ dz^i_{2,0} &= (-2\alpha_i z^i_{2,0} + \mathbb{I}^i_t)\, dt \end{aligned}$$

4 Separable volatility

After deterministic volatility, at the next (tractable) level of complexity we consider separable volatility, i.e. volatility of the following form

$$\sigma(r,x) = \varphi(r)\lambda(x).$$

Here the vector field σ has constant direction $\lambda \in \mathcal{H}$ but has varying length determined by the smooth scalar field φ. When introducing regime-switching in this type of volatility we can make either φ or λ or both dependent on the Markov chain y. We will consider these three cases in turn.

4.1 Stochastic length

Assume that the volatility is given by

$$\sigma(r,x,y) = \varphi(r,y)\lambda(x).$$

The Lie algebra is generated by the vector fields (recall equation (4))

$$\mu_i = \mathbb{F}r + \Phi_i - \frac{1}{2}\Delta_i = \mathbb{F}r + \underbrace{\varphi_i^2}_{\phi_i} \underbrace{\lambda(x)\int_0^x \lambda(s)\,ds}_{\Lambda} - \frac{1}{2}\varphi_i'[\lambda]\varphi_i\lambda$$

$$\sigma_i = \varphi_i\lambda.$$

To simplify notation we define

$$\Lambda(x) := \lambda(x)\int_0^x \lambda(s)\,ds,$$
$$\phi(r,y) := \varphi^2(r,y).$$

To avoid treating trivial cases we make the following assumption

Assumption 4.1. *Assume that $\varphi(r,y) \neq 0$ for all $r \in \mathcal{H}$.*

We observe that all the generators σ are multiples of the single vector field λ. Next we can remove suitable multiples of λ from all generators μ_i. After this operation we are left with the following vector fields

$$g_1^i := \mathbb{F}r + \phi_i(r)\Lambda \qquad i \in I \tag{26}$$
$$g_2 := \lambda. \tag{27}$$

As $g_1^i - g_1^j = (\phi_i - \phi_j)\Lambda$ and $\phi_i - \phi_j \neq 0$, we can add Λ to the set of generators which now can be simplified to

$$f_1 := \mathbb{F}r$$
$$f_2 := \lambda$$
$$f_3 := \Lambda.$$

Notice that all the generators are now independent of the Markov chain y. As both f_2 and f_3 are constant, taking Lie brackets is easy and we obtain the following system of generators for the Lie algebra

$$\langle \mathbb{F}r, \mathbb{F}^k\lambda, \mathbb{F}^k\Lambda; k = 0,1,\ldots\rangle_{LA}$$

Remark 4.1. *We obtain the same Lie algebra as when $\varphi(r)$ does not depend on y and we make the assumption that $\phi''[\lambda; \lambda] \neq 0$. This case is studied in Björk and Svensson (2001) (on page 228).*

We can now apply the same reasoning as in section 3.1. The space $\langle \mathbb{F}^k \lambda \rangle_{LA}$ is finite dimensional if and only if λ is a quasi-exponential (QE) function of x. If λ is QE then Λ will also be QE according to point (iv) and (v) of Lemma 3.1 as it is an integral of λ multiplied by λ. Consequently $\langle \mathbb{F}^k \Lambda \rangle_{LA}$ also spans a finite dimensional space, and we have proved the following proposition.

Proposition 4.1. *Assume that the volatility has the following form*

$$\sigma(r, y, x) = \varphi(r, y)\lambda(x).$$

Assume furthermore that Assumption 4.1 is satisfied. Then the model in (2) *possesses a FDR if and only if $\lambda(x)$ is a quasi-exponential function in the variable x. The smooth vector field $\varphi(r, y)$ can be chosen freely.*

Remark 4.2. *As the Lie algebra has the same structure as in in Proposition 3.2, with vector fields of the quasi-exponential form we get the same result concerning the representation by monomials and the dimension as in section 3.1.*

4.2 Stochastic direction

Now assume that the volatility is given by

$$\sigma(r, x, y) = \varphi(r)\lambda(x, y).$$

Using the same notation as in the previous section, the Lie algebra is generated by the vector fields

$$\begin{aligned} \mu_i &= \mathbb{F}r + \phi\Lambda_i - \frac{1}{2}\varphi'[\lambda_i]\varphi\lambda_i \\ \sigma_i &= \varphi\lambda_i \end{aligned}$$

Also here we have to make an assumption concerning φ.

Assumption 4.2. *Assume that $\varphi(r) \neq 0$ for all $r \in \mathcal{H}$.*

This assumption allows us to remove suitable multiples of λ_i from μ_i. After this operation we are left with the following generators

$$\begin{aligned} g_1^i &= \mathbb{F}r + \phi\Lambda_i \qquad i \in I \\ g_2^i &= \lambda_i. \end{aligned}$$

To achieve more simplifications we take Lie brackets between the vector fields

$$[g_1^i, g_2^j] = \mathbb{F}\lambda_j + \phi'[\lambda_j]\Lambda_i.$$

Taking one bracket produces

$$[[g_1^i, g_2^j], g_2^k] = \phi''[\lambda_j, \lambda_k]\Lambda_i.$$

To proceed we make the following assumption.

Assumption 4.3. *Assume that $\phi''[\lambda_j, \lambda_k] \neq 0$ for all $r \in \mathcal{H}$ for at least two states j and k of the Markov chain y.*

This assumption allows us to remove suitable multiples of Λ_i from g_1^i. After these simplifications we are left with the following generators

$$f_0 = \mathbb{F}r$$
$$f_1 = \lambda_i$$
$$f_2 = \Lambda_i$$

so the set of generators of the Lie algebra now becomes

$$\langle \mathbb{F}r, \mathbb{F}^k\lambda_i, \mathbb{F}^k\Lambda_i; i \in I; k = 0, 1, \ldots\rangle_{LA}.$$

Applying the same reasoning as in the previous section we get the following result.

Proposition 4.2. *Assume that the volatility has the form*

$$\sigma(r, y, x) = \varphi(r)\lambda(x, y).$$

Assume furthermore that assumption 4.2 and assumption 4.3 are satisfied. Then the model in (2) *possesses a FDR if and only if $\lambda(x, y)$ is a quasi-exponential function in the variable x. The smooth vector field $\varphi(r)$ can be chosen freely.*

What if Assumption 4.3 is not satisfied? [Under Bearbetning]

4.3 Example: Cox-Ingersoll-Ross

[Under Bearbetning]

To remove the possibility of negative interest rates present in Gaussian models, Cox, Ingersoll, and Ross (1985) introduced the following process to model the short rate $R = r(0)$.

$$dR_t = a(b - R_t)\,dt + \rho\sqrt{R_t}\,dW_t.$$

This model has an equivalent HJM formulation where the volatility takes the form

$$\sigma(r, x) = \sqrt{r(0)}\,\lambda(x, \rho, a), \tag{28}$$

where

$$\lambda(x, \rho, a) = -\frac{\partial}{\partial x}\Big(\frac{2\rho(e^{\gamma x} - 1)}{(\gamma + a)(e^{\gamma x} - 1) + 2\gamma}\Big)$$

and where

$$\gamma = \sqrt{a^2 + 2\rho^2}.$$

A natural question to ask now is if the parameters a and ρ can be made stochastic, i.e. functions of the Markov chain y. From (28) it is clear that such a model will be of the stochastic direction type, studied in the previous section. Here $\varphi = \sqrt{r(o)}$ so $\phi'(r)[\lambda_i] = \lambda_i(0)$ and $\phi'[\lambda_i; \lambda_j] = 0$ so Assumption 4.3 is not valid.

4.4 Stochastic length and direction

In Section 4.1 the length φ of the volatility $\sigma = \varphi\lambda$ was made stochastic. We showed that this did only have a minor impact on the structure of the Lie algebra. The same Lie algebra as in the case without a Markov chain emerged but without having to assume that $\varphi''[\lambda;\lambda] \neq 0$. It turns out that we get a similar result when we now let both length and direction depend on y,

$$\varphi(r, y, x) = \varphi(r, y)\lambda(x, y).$$

We apply the same reasoning as in the previous section, and we make make assumption 4.1 as well as the following one

Assumption 4.4. *Assume that $\phi_i''[\lambda_j, \lambda_k] \neq 0$ for all $r \in \mathcal{H}$ for at least two states j and k of the Markov chain y.*

We simplify the generators and end up with the following system

$$\langle \mathbb{F}r, \mathbb{F}^k\lambda_i, \mathbb{F}^k\Lambda_i; i \in I; k = 0, 1, \ldots \rangle_{LA}. \tag{29}$$

And as expected we get the following result.

Proposition 4.3. *Assume that the volatility has the following form*

$$\sigma(r, y, x) = \varphi(r, y)\lambda(x, y).$$

Assume furthermore that assumption 4.2 and assumption 4.4 are satisfied. Then the model in (2) *possesses a FDR if and only if $\lambda(x, y)$ is a quasi-exponential function in the variable x. The smooth vector field $\varphi(r, y)$ can be chosen freely.*

It turns out that by letting φ depend on y we can replace assumption 4.4 by another assumption. As in the previous section, the first stage of simplification yields the following generators

$$\begin{aligned} g_1^i &= \mathbb{F}r + \phi_i\Lambda_i \\ g_2^i &= \lambda_i. \end{aligned}$$

Now observe that

$$\begin{aligned} g_1^i - g_1^j &= \phi_i\Lambda_i - \phi_j\Lambda_j \\ [g_1^i, \lambda_k] - [g_1^j, \lambda_k] &= \phi_i'[\lambda_k]\Lambda_i - \phi_j'[\lambda_k]\Lambda_j. \end{aligned}$$

We can now make the following assumption

Assumption 4.5. *Assume that $\phi_j\phi_i'[\lambda_k] - \phi_i\phi_j'[\lambda_k] \neq 0$ for all $i, j \in i$ and for some k.*

This allows us to solve for Λ_i and Λ_j and get the system of generators in (29).

Example 4.1. *In a two state model, setting $\varphi_1 = \sqrt{a_1 + br(0)}$, $\varphi_2 = \sqrt{a_2 + br(0)}$, where $a_1, a_2, b \in \mathbb{R}$ satisfies Assumption 4.5 but not Assumption 4.4.*

Notation

- m: Number of states in the Markov chain.
- $I = \{1, 2, \ldots, m\}$: Set to index the states of the Markov chain.
- Number of Wiener processes: 1 Wiener process.
- d number of vector fields spanning the Lie algebra i.e. dimension of realization.
- N number of exponential terms in σ
- n_j degree of the polynomial in front of the exponential term $e^{-\alpha_j x}$.

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