On the Stopping of Diffusions with Discontinuous Coefficients

L. Rüschendorf[1] M. Urusov[2]

[1]University of Freiburg

[2]Technical University Berlin

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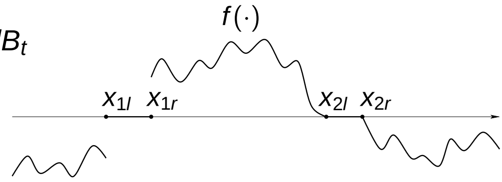
Formulation of the Problem

$$V^*(x) = \sup_{\tau} \mathrm{E}_x \int_0^{\tau} e^{-\lambda s} f(X_s)\, ds \qquad (*)$$

- $dX_t = b(X_t)\,dt + \sigma(X_t)\,dB_t$
- τ : stopping times of X
- E_x: $\mathrm{P}_x(X_0 = x) = 1$
- $\lambda \geq 0$



▸ Conditions on b, σ, and f

Outline

The Way of Solving $(*)$

For $\alpha < \beta$, $T_{\alpha,\beta} = \inf\{t : X_t \leq \alpha \text{ or } X_t \geq \beta\}$

- Conjecture the form of an optimal stopping time: $T_{x_1^*, x_2^*}$
- Let $V(x)$ be a candidate for $V^*(x)$.
  Formulate a certain FBP on (V, x_1^*, x_2^*)
- Verify that $V = V^*$ and $T_{x_1^*, x_2^*}$ is optimal in $(*)$

The Free Boundary Problem

$$\frac{\sigma^2(x)}{2}V''(x) + b(x)V'(x) - \lambda V(x) = -f(x), \quad x \in (x_1^*, x_2^*)$$

$$V(x) = 0, \quad x \in \mathbb{R} \setminus (x_1^*, x_2^*)$$

$$V'(x_1^*) = V'(x_2^*) = 0$$

Recall V^*

The Free Boundary Problem

$$\frac{\sigma^2(x)}{2}V''(x) + b(x)V'(x) - \lambda V(x) = -f(x)$$

$$\text{for } \mu_L\text{-a.a.} \quad x \in (x_1^*, x_2^*)$$

$$V(x) = 0, \quad x \in \mathbb{R} \setminus (x_1^*, x_2^*)$$

$$V'(x_1^*) = V'(x_2^*) = 0$$

The Free Boundary Problem

V' is absolutely continuous on finite intervals

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The Free Boundary Problem

(FB):

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$$\text{for } \mu_L\text{-a.a.} \quad x \in (x_1^*, x_2^*)$$

$$V(x) = 0, \quad x \in \mathbb{R} \setminus (x_1^*, x_2^*)$$

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Verification Theorem

Theorem (Verification Theorem)

Suppose (V, x_1^*, x_2^*) *is a solution of (FB). Then*

- *it is unique*
- $V = V^*$
- $T_{x_1^*, x_2^*}$ *is the unique optimal stopping time in* $(*)$

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Conjecture

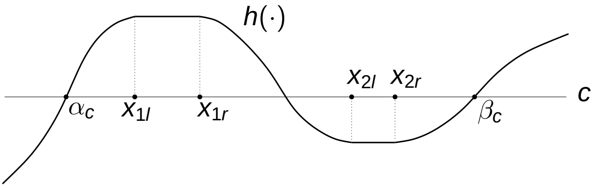
If $(*)$ *has an optimal stopping time of the form* $T_{x_1^*, x_2^*}$*, then (FB) has a solution*

Details

Notation

Throughout the sequel $b \equiv 0$, $\lambda = 0$

- $g(x) = -2f(x)/\sigma^2(x)$, $h(x) = \int_0^x g(y)\,dy$
- For $c \in \mathbb{R}$, $H(x, c) = h(x) - c$
- α_c (β_c) root of $h(x) = c$ with $\alpha_c < x_{1l}$ ($\beta_c > x_{2r}$)



Criterion for (FB) Has a Solution

Theorem (Case 0)

(FB) has a solution iff (A_1)–(A_3) *hold*

Details

(A_1) $h(\infty) > h(-\infty)$

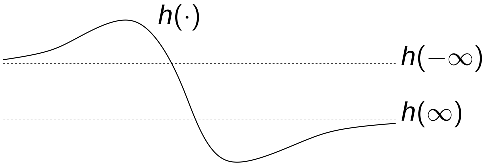
(A_2) If $h(\infty) < h(x_{1l})$, then

$$\int_{\alpha_{h(\infty)}}^{\infty} H(y, h(\infty))\, dy < 0$$

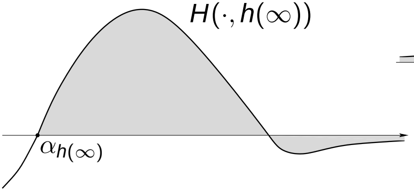
(A_3) If $h(-\infty) > h(x_{2r})$, then

$$\int_{-\infty}^{\beta_{h(-\infty)}} H(y, h(-\infty))\, dy > 0$$

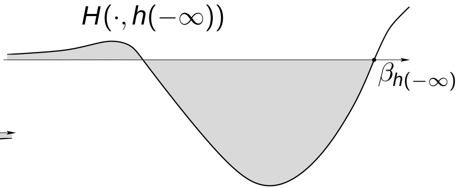
Study of $(*)$ When (A_1)–(A_3) Do Not Hold



Case 1: Not (A_1)



Case 2: (A_1) but not (A_2)



Case 3: (A_1) but not (A_3)

Solution of $(*)$ in Case 1

Theorem (Case 1)

- *There is no optimal stopping time in* $(*)$
- *There exist* $a_n \downarrow -\infty$ *and* $b_n \uparrow \infty$ *such that*

$$\mathsf{E}_x \int_0^{T_{a_n,b_n}} f(X_s)\, ds \longrightarrow V^*(x),$$

i.e., the sequence $\{T_{a_n,b_n}\}$ *is asymptotically optimal in* $(*)$

- *Set* $m = \frac{h(\infty)+h(-\infty)}{2}$, $K^{\pm} = \int_{\mathbb{R}} H^{\pm}(y,m)\, dy$. *We have*

$$V^*(x) < \infty \quad \forall x \in \mathbb{R} \quad \Longleftrightarrow \quad K^+ \vee K^- < \infty,$$
$$V^*(x) = \infty \quad \forall x \in \mathbb{R} \quad \Longleftrightarrow \quad K^+ \vee K^- = \infty$$

Remarks to Case 1

- $T \equiv \infty$ is not optimal because it is not admissible:

$$\int_0^\infty f^+(X_s)\,ds = \int_0^\infty f^-(X_s)\,ds = \infty \quad \mathsf{P}_x\text{-a.s.}$$

- There may exist $a_n \downarrow -\infty$ and $b_n \uparrow \infty$ such that $\{T_{a_n,b_n}\}$ is not asymptotically optimal in $(*)$

Solution of $(*)$ in Cases 2 and 3

$$T_\alpha^- = \inf\{t : X_t \leq \alpha\},\ T_\beta^+ = \inf\{t : X_t \geq \beta\}$$

Theorem (Cases 2 and 3)

In Case 2, $T^-_{\alpha_{h(\infty)}}$ is the unique optimal stopping time

In Case 3, $T^+_{\beta_{h(-\infty)}}$ is the unique optimal stopping time

Conditions on b, σ, and f

- $\sigma(x) \neq 0 \quad \forall x \in \mathbb{R}$
- $1/\sigma^2 \in L^1_{loc}(\mathbb{R})$
- $b \in B_{loc}(\mathbb{R})$

- $f/\sigma^2 \in L^1_{loc}(\mathbb{R})$
- $1/f \in B_{loc}(x) \quad \forall x \in \mathbb{R} \setminus ([x_{1l}, x_{1r}] \cup [x_{2l}, x_{2r}])$

Return

The Free Boundary Problem

$$\frac{\sigma^2(x)}{2} V''(x) + b(x)V'(x) - \lambda V(x) = -f(x),$$

$$x \in (x_1^*, x_2^*)$$

$$V(x) = 0, \quad x \in \mathbb{R} \setminus (x_1^*, x_2^*)$$

$$V'(x_1^*) = V'(x_2^*) = 0$$

$$V^*(x) = \sup_\tau \mathrm{E}_x \int_0^\tau e^{-\lambda s} f(X_s)\, ds$$

Return

Verification Theorem

Theorem (Verification Theorem)

Suppose (V, x_1^*, x_2^*) *is a non-trivial solution of (FB). Then*

- *it is unique*
- $V = V^*$
- $T_{x_1^*, x_2^*}$ *is the unique optimal stopping time in* $(*)$

Conjecture

If $(*)$ *has an optimal stopping time of the form* $T_{x_1^*, x_2^*}$*,*
then (FB) has a non-trivial solution

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Criterion for (FB) Has a Solution

Theorem (Case 0)

(FB) has a non-trivial solution iff (A_1)–(A_3) *hold*

Return

(A_1) $h(\infty) > h(-\infty)$

(A_2) If $h(\infty) < h(x_{1l})$, then

$$\int_{\alpha_{h(\infty)}}^{\infty} H(y, h(\infty))\, dy < 0$$

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