Option Pricing in Illiquid Markets and Nonlinear Black-Scholes Equations

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1. Introduction

- In an illiquid market the attempt to trade at a given point in time moves price against the trader; in extreme cases trading impossible.
- Market liquidity relevant in the risk management of derivatives since in an illiquid market the implementation of a (dynamic) hedging strategy affects the price process of the underlying. $\Rightarrow$ Different hedging strategies and suitable pricing adjustments needed.
- Not just an academic problem: performance of portfolio insurance in 1987-stock market crash, LTCM crises, anecdotal evidence in [Taleb, 1996].
- Hence derivative asset analysis in illiquid markets active area in recent research.

Modelling illiquid markets

Recent research on derivative asset analysis in illiquid markets uses three different modelling approaches. In all model types price impact of hedging strategies is (essentially) exogenously imposed.

- (Quadratic) transaction-cost models such as [Cetin et al., 2004].
- SDE models such as: [Frey, 2000], [Frey and Patie, 2002],
- Reaction-function or temporary-equilibrium models such as: [Jarrow, 1994], [Frey and Stremme, 1997], [Frey, 1998], [Sircar and Papanicolaou, 1998], [Platen and Schweizer, 1998] and the very general [Bank and Baum, 2004]

Nonlinear Black-Scholes equations

Assume that option traders use Markovian stock trading strategies of the form $\Phi_t = \phi(t, S_t)$. Then in all models above the hedge cost for derivatives $u(t, S)$ can be characterized by the following nonlinear version of the Black Scholes PDE.

$$u_t + \frac{1}{2}\sigma^2 S^2 v_\rho(S, u_{SS}) u_{SS} = 0, \quad u(T, S) = h(S)\,, \tag{1}$$

where

- $\rho \geq 0$ measures market liquidity: $\rho = 0 \Rightarrow$ standard Black-Scholes model; ρ large $\Rightarrow$ big market-impact of hedging.
- the function $v_\rho(S, q)$ is increasing in q with $v_0(S, q) \equiv 1$ and $\frac{\partial}{\partial \rho} v_\rho(S, q) \mid_{\rho=0} = 2Sq$ so that $v_\rho(S, q) \approx 1 + 2\rho S q$.
- Hedging strategy is given by $\Phi_t = u_S(t, S_t)$.

Nonlinear Black-Scholes PDEs ctd

Comments.

- Essentially, (1) gives the price of position with payoff $h(S_T)$ in a model where the "volatility" $\sigma v_\rho(S, u_{SS})$ is an increasing function of the "Gamma" u_{SS}.

- Models which are quite different at first sight lead to very similar solutions for pricing and hedging derivatives under market liquidity. $\Rightarrow$ the PDE (1) captures certain generic properties of hedge cost for derivatives in illiquid markets.

In this talk we study qualitative behavior of solutions (monotonicity wrt. ρ and h, convexity "in h", behavior for large ρ).

2. Derivation of the Nonlinear Black-Scholes PDE

Throughout we consider a market with one share, denoted S, and a risk-free money market account $B_t \equiv 1$; stock is 'illiquid' (its price is affected by trading); money market is assumed liquid.

a) Transaction-cost model. (special case of Cetin et. al. (2004))

- Fundamental price process S^0 with $dS_t^0 = \mu S_t^0 dt + \sigma S_t^0 dW_t$.
- Transaction price. An investor who wants to trade α shares at time t has to pay the transaction price

$$\bar{S}_t(\alpha) = \exp(\rho\alpha) S_t^0, \quad \text{for some } \rho \geq 0$$

for his purchase/sale. Essentially this models a bid-ask-spread whose size depends on α. Note that for $\alpha \to 0$, $\bar{S}_t(\alpha) \approx \rho S_t^0 \alpha$.

Wealth Dynamics in Cetin et. al. (2004)

Stock position Φ_t, bond position ψ_t $\Rightarrow$ paper value $V_t = \Phi_t S_t^0 + \psi_t$.

Selffinancing strategies. Consider simple predictable strategy $\Phi_t = \sum_i \phi_i 1_{(t_i, t_{i+1}]}$. Then selffinancing condition in t_i reads

$$-\Delta\psi_{t_i} = \Delta\Phi_{t_i}\bar{S}_{t_i}(\Delta\Phi_{t_i}) = \Delta\Phi_{t_i}S^0_{t_i} + \rho S^0_{t_i}\cdot(\Delta\Phi_{t_i})^2 + o((\Delta\Phi_{t_i})^2)$$

$$\Rightarrow \Delta V_{t_i} = \phi_i S^0_{t_{i+1}} - \phi_{i-1}S^0_{t_i} + \Delta\psi_{t_i}$$

$$= \phi_i(S^0_{t_{i+1}} - S^0_{t_i}) - \rho S^0_{t_i}\cdot(\Delta\Phi_{t_i})^2 + o((\Delta\Phi_{t_i})^2)$$

For a continuous semimartingale Φ with quadratic variation $[\Phi_t]$, in the limit the wealth dynamics of a selffinancing strategy becomes

$$dV_t = \Phi_t dS_t^0 - \rho S_t^0 d[\Phi_t];$$

the last term gives extra transaction cost due to market illiquidity. Similar results in the model of [Bank and Baum, 2004].

A nonlinear Black-Scholes PDE

Proposition 1. Given a selffinancing strategy with $\Phi_t = \phi(t, S_t^0)$, $V_t = u(t, S_t^0)$ for u and ϕ smooth. Then the strategy is selffinancing $\Leftrightarrow \phi = u_S$ and u solves the nonlinear PDE $u_t + \frac{1}{2}\sigma^2 S^2 (1 + 2\rho S u_{SS}) u_{SS} = 0$.

Proof. For $\Phi_t = u_S(t, S_t^0)$ we have $d[\Phi]_t = u_{SS}^2(t, S_t^0)\sigma^2 (S_t^0)^2 dt$. Itô's formula together with the nonlinear PDE for u therefore gives

$$\begin{aligned} du(t, S_t^0) &= u_S(t, S_t^0) dS_t^0 + \left(u_t(t, S_t^0) + \tfrac{1}{2}\sigma^2 (S_t^0)^2\right) dt \\ &\overset{!}{=} \Phi_t dS_t^0 - \underbrace{\rho u_{SS}^2(t, S_t^0)\sigma^2 (S_t^0)^2 \, dt}_{= \rho d[\Phi]_t} . \end{aligned}$$

Corollary 2. Hedge cost for a terminal-value claim $h(S_T^0)$ given by solution of $u_t + \frac{1}{2}\sigma^2 S^2 (1 + 2\rho S u_{SS}) u_{SS} = 0$, $u(T, S) = h(S)$

Note that PDE is of the form (1) with $v_\rho(S, q) = 1 + 2\rho S q$

b) SDE-model

Asset price dynamics. Given parameters σ, ρ, an exogenous Brownian motion W and a semimartingale trading strategy Φ, S is assumed to solve the SDE $dS_t = \sigma S_{t-} dW_t + \rho S_{t-}\, d\Phi_t$.

Proposition 3. a) If strategy is of the form $\Phi_t = \phi(t, S_t)$ such that $(1 - \rho S \phi_S(t, S) > 0$ for all (t, S), S_t follows a diffusion of the form

$$dS_t = \sigma (1 - \rho S_t \phi_S(t, S_t))^{-1} S_t W_t + \ldots dt\,.$$

b) A strategy with stock position $\Phi_t = \phi(t, S_t)$, paper value $V_t = u(t, S_t)$ with u and ϕ smooth and $1 - \rho S \phi_S > 0$ is selffinancing $\Leftrightarrow$ $\phi = u_S$ and u solves the nonlinear PDE

$$u_t + \tfrac{1}{2}\sigma^2 S^2 (1 - \rho S u_{SS})^{-2} u_{SS} = 0\,. \qquad (2)$$

Note that PDE (2) is of the form (1) with $v_\rho(S, q) = (1 - \rho S q)^{-2}$.

3. Properties of the Hedge Cost

We use tools from viscosity-solution theory to derive properties of the hedge cost for derivatives in illiquid markets, defined as solution to the following generic terminal-boundary-value problem:

Given maturity T, boundary points $K_1 > 0$ small, K_2 large and a feedback-function $v_\rho(S, q)$, the hedge cost for the claim with continuous payoff $h(S)$ is given by $u \in \mathcal{C}^0\left([0,T] \times [K_1, K_2]\right)$ with

$$\begin{aligned} u_t + \frac{1}{2}\sigma^2 S^2 v_\rho(S, u_{SS}) u_{SS} &= 0, \text{ for } (t,S) \in [0,T) \times (K_1, K_2)\,, \\ u(t, K_1) = h(K_1), u(t, K_2) &= h(K_2), \text{ for } t \in [0,T]\,, \\ u(T,S) &= h(S), \text{ for } S \in [K_1, K_2]\,. \end{aligned} \qquad \text{(BP)}$$

Reduction to a bounded domain mainly for technical reasons.

The feedback-function v_ρ.

Assumptions.

A1. The function $q \mapsto v_\rho(S,q)q$ is increasing for fixed $S \in [K_1, K_2]$. (ensures ellipticity resp. "properness")

A2. The function $\rho \mapsto v_\rho(S,q)q$ is increasing for fixed $(S,q) \in [K_1, K_2] \times \mathbf{R}$. (monotonicity in ρ).

A3. $\underline{\sigma} := \inf_{S,q} v_\rho(S,q) > 0$, $\overline{\sigma} := \sup_{S,q} v_\rho(S,q) < \infty$.

Remark. The functions $v_\rho(S,q)$ in the models of Cetin-et-al and Frey satisfy Assumptions A1, A2, A3 after minor modifications for $|q|$ large.

The main tool: a comparison result

Theorem 4. Suppose that v_ρ satisfies A1. Then the comparison principle holds for (BP), i.e. for two functions $u^{(1)}, u^{(2)}$ such that

- $u^{(1)} \leq h$ on the boundary, $u_t^{(1)} + \frac{1}{2}S^2 v_\rho(S, u_{SS}^{(1)}) u_{SS}^{(1)} \geq 0$ "in the viscosity sense" (viscosity subsolution of (BP))
- $u^{(2)} \geq h$ on the boundary, $u_t^{(2)} + \frac{1}{2}S^2 v_\rho(S, u_{SS}^{(2)}) u_{SS}^{(2)} \leq 0$ "in the viscosity sense" (viscosity supersolution of (BP))

we have $u^{(1)}(t,S) \leq u^{(2)}(t,S)$, $(t,S) \in [0,T] \times [K_1, K_2]$.

Corollary 5. Under A1 we have uniqueness (for viscosity solutions and hence in particular for classical solutions) for (BP).

Remark. Under Assumptions A1 and A3, existence of viscosity solutions for (BP) is immediate from Perron's method. Existence of classical solutions is discussed by [**?**], [Frey, 1998] and is not our focus here.

Monotonicity Properties

Lemma 6 (Monotonicity in ρ). Given $\rho_1 < \rho_2$, suppose that v_ρ satisfies A1, A2 for $\rho \in [\rho_1, \rho_2]$. Let $u^{(1)}$, $u^{(2)}$ be viscosity solutions of (BP) with identical payoff h and feedback functions v_{ρ_1} and v_{ρ_2}. Then $u^{(1)} \leq u^{(2)}$.

Proof. (for classical solutions) Since by A2, $v_{\rho_1}(S,q)q \leq v_{\rho_2}(S,q)q$,

$$u_t^{(2)} + v_{\rho_1}(S, u_{SS}^{(2)}) S^2 u_{SS}^{(2)} \leq u_t^{(2)} + v_{\rho_2}(S, u_{SS}^{(2)}) S^2 u_{SS}^{(2)} = 0,$$

and the result follows from the comparison principle (Theorem 4).

Lemma 7 (Monotonicity in h). Consider two payoff functions $h_1 \leq h_2$, and suppose that A1 holds. Then the corresponding solutions $u^{(1)}$, $u^{(2)}$ of (BP) satisfy $u^{(1)} \leq u^{(2)}$.

This is in stark contrast to models with linear transaction cost, where strict superhedging may be optimal ([Bensaid et al., 1992]).

Convexity in the payoff

Hedge cost in illiquid markets mainly driven by nonlinearities in the payoff profile $\Rightarrow$ holding a more equilibrated position should lead to "diversification benefit." Formally, one expects that hedge costs are convex in the payoff.

Proposition 8 (Convexity in h). Let $u^{(0)}$ and $u^{(1)}$ be classical solutions of (BP) corresponding to payoff functions h_0 and h_1. Suppose that $q \mapsto v_\rho(S,q)q$ is convex in some interval $I \subset \mathbf{R}$ containing the range of $u^{(0)}_{SS}$, $u^{(1)}_{SS}$, and that A1 holds. Denote for $\lambda \in [0,1]$ by $u^{(\lambda)}$ the solution of (BP) with payoff $(1-\lambda)h_0 + \lambda h_1$. Then $u^{(\lambda)} \leq (1-\lambda)u^{(0)} + \lambda u^{(1)}$.

Remarks. 1.) For the Cetin-et-al and Frey models, Proposition 8 applies for ρ not too large.

2. Hedge cost behaves like convex (but non-coherent) risk measure.

Concave envelope of h

Recall that for $S \in [K_1, K_2]$ the concave envelope of h is given by

$$\operatorname{conc} h\,(S) = \min\{f(S)\colon f : [K_1, K_2] \to \mathbf{R} \text{ concave}, f \geq h\}$$
$$= \min\{c \in \mathbf{R}\colon \exists \xi,\ c + \xi(\tilde{S} - S) \geq h(\tilde{S}), K_1 \leq \tilde{S} \leq K_2\}$$

Lemma 9 (Boundedness by $\operatorname{conc} h$). Under A1, the solution u of (BP) with payoff function h satisfies $u(t, S) \leq \operatorname{conc} h\,(S)$, $(t, S) \in [0, T] \times [K_1, K_2]$.

Financial motivation: The strategy with $V_0 = \operatorname{conc} h\,(S_0)$ and stock position ξ forms a static superhedge for h.
Formal argument: $U(t, S) := \operatorname{conc} h\,(S)$ is viscosity supersolution of (BP).

Behavior of u for $\rho \to \infty$

Theorem 10 (convergence to $\mathrm{conc}\, h$). Suppose that v_ρ satisfies A1, A2, and that moreover

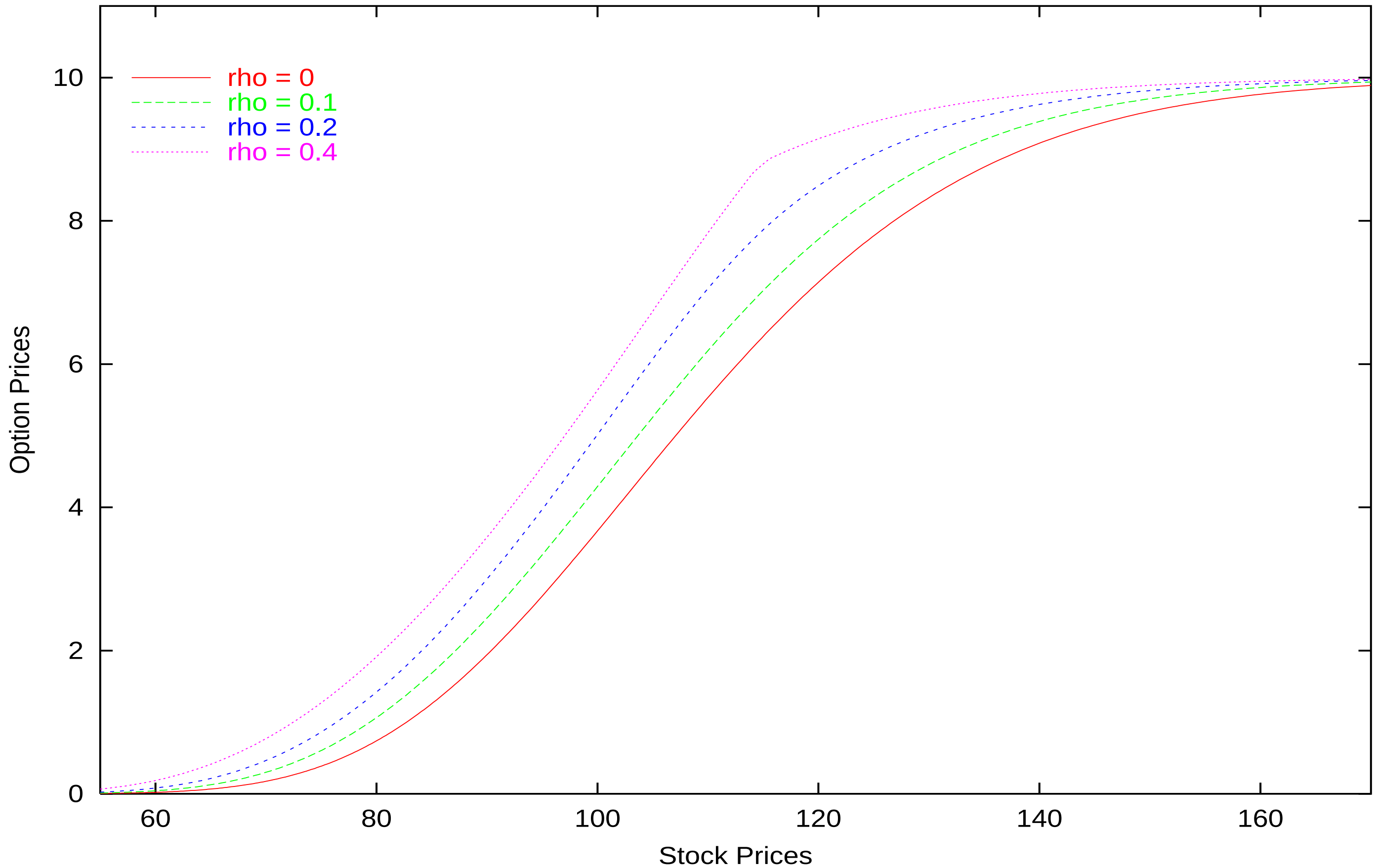
A4.
$$\lim_{\rho\to\infty} v_\rho(S,q) = \begin{cases} \infty\,, & q > 0, S \in [K_1, K_2]\,, \\ 0\,, & q < 0, S \in [K_1, K_2]\,. \end{cases}$$

Fix some $h : [K_1, K_2] \to \mathbf{R}$ and denote for $\rho \geq 0$ by u_ρ the solution of (BP) with payoff h and feedback function v_ρ. Then the sequence u_ρ is monotonically increasing and $\lim_{\rho\to\infty} u_\rho = \mathrm{conc}\, h$.

Remarks.

- Related to results by [Cvitanic et al., 1999] on superhedging under stochastic volatility.
- Key point of proof: By stability properties of viscosity solutions $u_*(t,S) := \liminf_{\rho\to\infty, \tilde{S}\to S} u_\rho(t,S)$ is supersolution of $u_{SS} = 0$ and hence concave.

The case of a call spread



Hedge cost for a call spread with strikes at 100, 120 for varying ρ

Why Markovian Strategies?

It has been shown that by using tame strategies (Φ continuous, FV), even in an illiquid market it is possible to find an ϵ-optimal strategy for any claim with hedge cost equal to the Black-Scholes price. (Bank and Baum, Cetin-et-al). But we believe that Markovian strategies are preferable for a number of reasons

- Robustness: Implementation of tame strategies requires very frequent trading of small amounts, which might be difficult in illiquid markets.
- Markovian strategies appear as limit of discrete-time models with feedback; see Gruber and Schweizer
- Markovian strategies natural in markets with many small hedgers, where individual price impact is negligible, but overall price impact of hedging is sizeable.

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