Hedging strategies in discrete time

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The problem

The optimal strategy

The Delta strategy

Transaction costs

Applications

Conclusions

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Introduction

- Most mathematical models for financial markets assume continuous-time trading
- Strategies that are optimal in continuous time (e.g. BS delta hedging) need not to be optimal in discrete time
- Can we effectively compute a discrete time optimal strategy?
- Can we measure the error due to time discretization?

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The setting

- Let H be the payoff of a contingent claim
- Let $S = (S_k)_{k=0}^N$ be the asset price process and $\vartheta = (\vartheta_k)_{k=1}^N$ be the hedging ratio of a self-financing trading strategy
- Assume zero interest rate
- The gains process is

$$G_N(\vartheta) = \sum_{j=1}^{N} \vartheta_j \Delta S_j$$

where $\Delta S_j = S_j - S_{j-1}$

- Given an initial value c, follow the strategy ϑ. The hedging error is

$$\varepsilon(\vartheta, c) = H - c - G_N(\vartheta)$$

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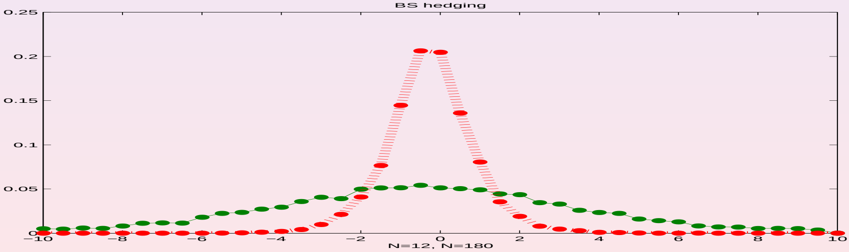
The discretization error in the BS model



The problem

- Measurement of hedging error
- Minimization Problem:

$$\min_{\vartheta \in \Theta} E\left[\varepsilon(\vartheta, c)^2\right]$$

  for fixed $c \in \mathbb{R}$

- Computation of

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The existence of the optimal strategy

- Assume that process X satisfies the following Non-Degeneracy (ND) condition:

$$\frac{(E_{k-1}\Delta S_k)^2}{\mathrm{var}_{k-1}\Delta S_k} < M$$

  for all ω and k.
- Then there exists a unique optimal trading strategy θ^c that solves the basic problem (Schweizer (1995))
- A counterexample shows that the ND condition is necessary for the existence of a solution
- If X is a (non degenerate) martingale then condition ND is obviously satisfied

The structure of the optimal strategy

- In general, the explicit computation of the optimal strategy is a very hard task
- Schweizer (1995): "The optimal hedge ratio can be decomposed in 3 pieces: locally optimal (pure hedging demand) ξ^H, demand for mean-variance purposes, demand for hedging against mean-variance ratio stochastic changes (not present if the m.v.r. is deterministic)".

The Laplace transform approach

- Hubalek, Kallsen and Krawczyk (2006), Černý (2007)
- Let $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n \in (0,1,\ldots,N)}, P)$ be a filtered probability space. Consider a one-dimensional process

$$S_n = S_0 \exp(X_n),$$

where the process $X = (X_n)$ for $n = 0, 1, \ldots, N$, satisfies

1. X is adapted to the filtration $\mathcal{F}_n$, $n \in (0, 1, \ldots, N)$,
2. $X_0 = 0$,
3. $\Delta X_n = X_n - X_{n-1}$ $n = 1, \ldots, N$ are i.i.d.

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- Can compute the moment generating function of ΔX

$$m(z) = E[e^{z\Delta X}],$$

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The idea of the Laplace transform approach

- An *exponential claim* is a contingent claim with payoff

$$H(z) = S_N^z = S_0^z e^{zX_N}$$

- The i.i.d. assumption for ΔX_n makes computations easy for exponential claims.
- Consider a contingent claim which is a "linear combination" of exponential claims (let us call it a *simple claim*)
- Use linearity properties and Fubini

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The Laplace transform approach

- More precisely consider contingent claims whose payoff is an inverse Laplace Transform

$$H = \int S_N^z \Pi(dz)$$

- A European call is a simple claim!

$$(S_N - K)^+ = \frac{1}{2\pi i} \int_{R-i\infty}^{R+i\infty} S_N^z \frac{K^{1-z}}{z(z-1)} dz,$$

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The results

- Locally optimal hedge ratio at 0

$$\xi_1 = \int S_0^{z-1} g(z) h(z)^{N-1} \Pi(dz)$$

- Value at 0 of the optimal portfolio

$$V_0 = \int S_0^{z} h(z)^{N-1} \Pi(dz)$$

- Optimal variance

$$\mathrm{Var}_0 = \int \int J_0(y,z) \Pi(dy) \Pi(dz)$$

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Delta hedging, previous results

- Let $\vartheta = \Delta^N$ be the BS-delta and S is the BS-process.
- Hayashi and Mykland (2005) showed that

$$\varepsilon(\Delta^N, c) - \sqrt{\frac{T}{2N}} \int_0^T \Gamma_u \sigma^2 S_u^2 dW_u^* \to 0$$

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- Toft (1996) provides an asymptotic approximation for the variance

$$\frac{1}{2}\sigma^4\left(\frac{T}{N}\right)^2\sum_{k=0}^{N-1}E_0\left[(\Gamma_k S_k^2)^2\right]$$

- Kamal and Derman (1999) propose a more *trader-friendly* approximation

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Crucial observation

- The Delta of a simple claim is an inverse Laplace transform

$$\Delta_n = \int f(z)_n S_{n-1}^{z-1} \Pi(dz),$$

where $f(z)_n = z m_0(z)^{N-n+1}$ does not depend on S_{n-1}.

- The price at time t_{n-1} of a simple claim is

$$\begin{aligned} P_{n-1} &= E_{n-1}^{Q}\left[\int S_N^z \Pi(dz)\right] \\ &= \int E_{n-1}^{Q}[S_N^z]\Pi(dz) = \int S_{n-1}^z m_0(z)^{N-n+1}\Pi(dz), \end{aligned}$$

where E_{n-1}^{Q} is the pricing measure and $m_0(z)$ is the m.g.f. of corresponding ΔX.

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Delta hedging error

- The hedging error of a simple claim is an inverse Laplace transform

$$H - c - G_N(\Delta) = \int (H(z) - c - G_N(\Delta(z)))\Pi(dz)$$

- Can compute expected value and variance of the hedging error if I am able to compute things inside the integral

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Expected value

- $$E[H] = \int E[S_N^z]\Pi(dz) = \int S_0^z m(z)^N \Pi(dz)$$

- $$\begin{aligned} E[\Delta_n \Delta S_n] &= \int E[f(z)_n S_{n-1}^{z-1} \Delta S_n]\Pi(dz) \\ &= \int S_0^z f(z)_n (m(1)-1) m(z)^{n-1} \Pi(dz), \end{aligned}$$

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Variance of Delta hedging

- Need to compute all the covariances
- $$E[H(y)H(z)] = E[S_N^y S_N^z] = S_0^{y+z} m(y+z)^N$$
- $$E[H(y)S_{n-1}^z \Delta S_n] = E[S_N^y S_{n-1}^z \Delta S_n] = S_0^{y+z} v_2(y,z)_n,$$
  where $v_2(y,z)_n$ depends on m.g.f., N and $n = 1, \ldots, N$
- $$E[S_{n-1}^y S_{m-1}^z \Delta S_n \Delta S_m] = S_0^{y+z} v_3(y,z)_{n,m},$$
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The main result

- For a simple claim and any ϑ given by

$$\vartheta_n = \int f(z)_n S_{n-1}^{z-1} \Pi(dz),$$

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$$E[\varepsilon(\vartheta, 0)] = \int S_0^z \left[m(z)^N - (m(1) - 1) \sum_{k=1}^{N} f(z)_k m(z)^{k-1} \right] \Pi(dz)$$

$$E[\varepsilon(\vartheta, 0)^2] = \int \int S_0^{y+z} V(y, z) \Pi(dz) \Pi(dy),$$

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- With same technique may in principle compute moments of higher order
- Can choose a strategy ("$f(z)_n$") and a model ("$m(z)$")
- In particular, one can measure the hedging error of the Delta strategy for different data generating processes
- The result can be extended to the case of ΔX_n not identically distributed

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- For a simple (and increasing in the underlying as the Delta) strategy can be written too as an inverse Laplace transform because

$$|\Delta_{n+1} - \Delta_n| = \mathbf{1}_{\Delta X_n > 0} 2(\Delta_{n+1} - \Delta_n) - (\Delta_{n+1} - \Delta_n)$$

- Can compute the variance of transaction costs and the covariance with other terms by computing

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- How worse than optimal-variance is Delta hedging?
- Compare performances of various strategies for different models
- To measure performances use Sharpe ratio

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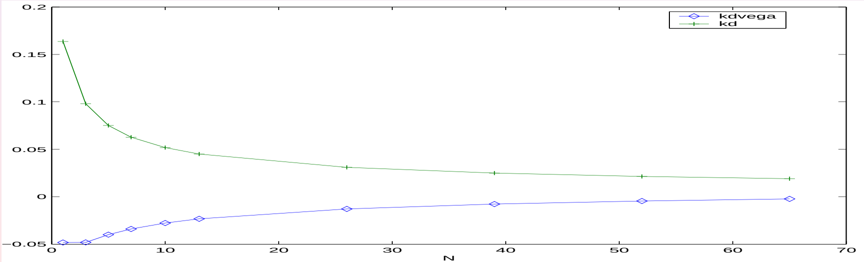
Relative errors of approximations of standard deviation



Model risk

- Data generating process: Merton jump-diffusion process with normally distributed jumps, with returns with annual mean $\mu \approx 0.14$ and volatility $\sigma \approx 0.44$
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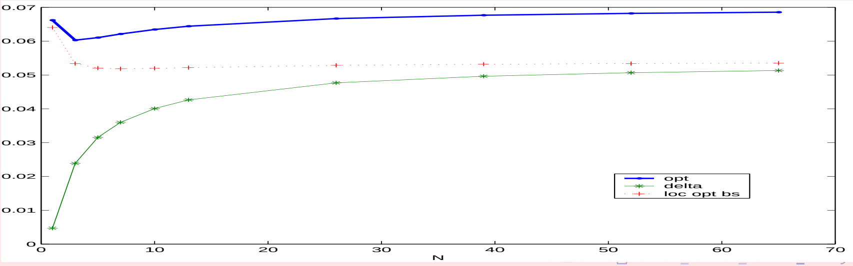
Model risk

Sharpe index of different strategies



Model mispecification

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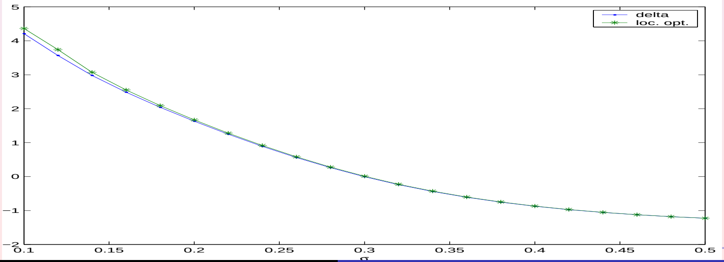
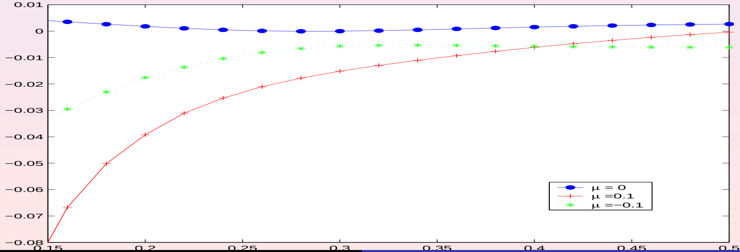
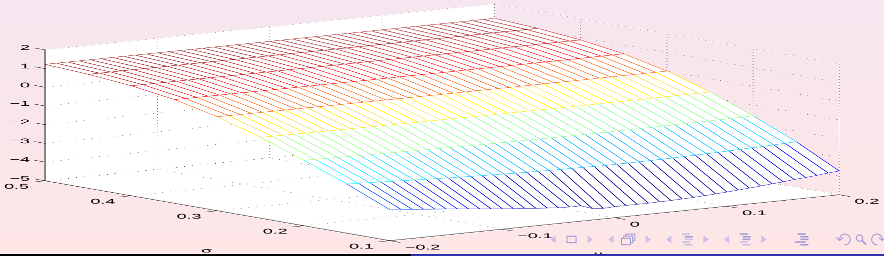
Sharpe index as a function of realized volatility σ, with $\mu_0 = \mu = 0.1$ and $N = 10$



$s(\Delta) - s(\xi^H)$ as a function of σ, for different μ ($\sigma_0 = 0.3$, $\mu_0 = 0.1$)



Sharpe index of local optimal strategy as a function of σ and μ ($\sigma_0 = 0.3$, $\mu_0 = 0$, $S = K = 100$, $N = 10$)



Conclusions

- We have an efficient way to compute moments of hedging errors of strategies in presence of transaction costs for simple claims and for a wide class of data generating process
- This allows to measure the performance of hedging strategies in different settings, for instance under model mispecification
- Need to study the effect of transaction costs
- Is it possible to find optimal strategy? Or best hedging volatility?
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Appendix: Inverse Laplace transform

- The (one-dimensional) Inverse Laplace transform is

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{1}{2\pi i}\int_{R-i\infty}^{R+i\infty} e^{st}F(s)ds$$

- There are several algorithms that perform numerical inversion of the Laplace transform in an efficient way
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Appendix: Numerical implementation

- The formulas we wish to compute involve one- and two-dimensional Laplace transforms.
- There are at least two possible approaches: numerical integration and inversion of Laplace transform.
- Second approach, implementing the algorithms in MATLAB.
- One-dimensional case: we used "invlap.m" constructed by Hollenbeck (1998), very accurate
- Bi-dimensional case: we wrote a code based on Choudhury, Lucantoni, Whitt (1994), quite accurate.