Variance of hedging strategies in discrete time

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Abstract

We compute the expected value and the variance of the discretization error of delta hedging and of other strategies. The results apply to a fairly general class of models, including Black-Scholes, Merton's jump-diffusion and Normal Inverse Gaussian. The most important contribution of our results is that they are not asymptotical approximations but exact and efficient formulas, valid for any number of trading dates. Another relevant improvement is that they can also be employed under model mispecification, hence measuring the influence of model risk on hedging strategies.

1 Introduction

The object of this paper is the measurement of the hedging error due to trading in discrete time, usually referred to as the "discretization error". Most of the financial models for pricing and hedging derivatives assume that trading is possible in continuous time. Of course, such an assumption does not hold for practical applications. For example, the widely used Black-Scholes delta hedging strategy produces a discretization error even if all other assumptions of the model are met. Quantifying the discretization error associated to an hedging strategy is a problem that is relevant both from a practical and a theoretical point of view and has been in fact addressed by many papers in the literature. Hayashi and Mykland [8] use a weak convergence argument to derive the asymptotic distribution of the hedging error as the number of trades goes to infinite. The discretization error depends on the path followed by the underlying asset until maturity and hence, even computing the variance of the error of replicating a claim as simple as a European call, can be a very hard task. The practical importance of such a computation should be self-evident, since it provides a way to measure the risk involved with discrete trading and, consequently, to quantify a compensation for it. Some approximating formulas for the variance have been obtained, under the assumption of small trading intervals and for the log-normal model, by Kamal and Derman [9], by Mello and Neuhaus [11] and, in presence of transaction costs, by Toft [17]. The fact that such approximations hold for vanishing time intervals constitutes an important limitation to their application, since in this case the error would also vanish. Moreover (to the best of our knowledge) the error associated with such approximations has never been measured. Our results may also be useful to assess this last point.

A related, very important, problem is that of determining a strategy that minimizes the variance of the hedging error in an incomplete market. An extremely rich branch of the financial literature flourished after the seminal papers of Föllmer and Sondermann [4]. Schweizer [15] contains a review of the main results and contributions. The general solution in a discrete setting was found by Schweizer [16]. He provided a characterization of the optimal strategies and a general formula for the optimal variance. However, an explicit computation for practical application is usually quite burdensome. For this reason, some algorithms useful for actual implementation have been proposed, for example by Bertsimas et al. [1] , with a dynamic programming approach. Wilmott [18], following an independent approach, specific to the

Black-Scholes model and based on second order approximation, proposed a very easily implementable trading strategy.

A breakthrough in the problem of determining an efficient way to compute optimal strategies and their associated variances was proposed by Hubalek et al. [6] and by Černý [2]. Their idea is to consider contingent claims whose payoff function can be written as an inverse Laplace transform. They show that, under quite general assumption on the dynamics of the underlying, it is possible to compute the optimal strategy and its variance as an inverse Laplace transform of a function that depends on the claim and on the underlying process. Inverse Laplace transform can be evaluated very efficiently with standard numerical algorithms. By following their approach we show that the expectation and the variance associated with other "non-optimal" strategies, such as the standard Black-Scholes delta or Wilmott hedge ratio, can be readily computed. This contributes to the literature by providing efficient formulas to the delta hedging discretization error when the number of hedging intervals is fixed (i.e. non asymptotically). We can therefore assess the precision of some useful approximating formulas, that hold under much more restrictive assumptions on the model and on the claim, like that of Kamal and Derman [9]. Moreover, our result can be applied to measure the performance of a hedging strategy under model misspecification. For example, in the case of a trader that detects a market implied volatility higher than what she expects and wishes to exploit it. In this case, we can measure the expected performances of the hedging strategies in terms of Sharpe ratios. From a practical point of view this could help a trader to choose which trading strategy to adopt.

The rest of the paper is composed as follows: Section 2 contains the general setting and defines the class of strategies, that we call "simple", whose hedging error will be measured. Section 3 contains the main result. We provide some details of the numerical implementation of our results in Section 4, showing some applications in Section 5.

2 Simple hedging strategies

Let $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n\in(0,1,\dots,N)}, P)$ be a filtered probability space. We consider a one-dimensional process

$$S_n = S_0 \exp(X_n),$$

where the process $X = (X_n)$ for $n = 0, 1, \dots, N$, satisfies

1. X is adapted to the filtration $(\mathcal{F}_n)_{n\in(0,1,\ldots,N)}$,
2. $X_0 = 0$,
3. $\Delta X_n = X_n - X_{n-1}$ has the same distribution for $n = 1, \ldots, N$,
4. ΔX_n is independent from $\mathcal{F}_{n-1}$ for $n = 1, \ldots, N$.

We denote the moment generating function of X_1 by $m(z)$. We assume that $E[S_1^2] < \infty$ so that the moment generating function $m(z)$ is defined at least for complex z with $0 \leq Re(z) \leq 2$. Moreover, we exclude the case when S is a deterministic process. We suppose, without loss of generality, that the risk-free rate r is zero or, equivalently, that S represents a discounted price.

Following the approach proposed by Hubalek et al. [6] we consider European contingent claims written on S with maturity T and payoff $H = f(S_N)$, where $f : (0, \infty) \to \mathbb{R}$ is of the form

$$f(s) = \int s^z \Pi(dz), \tag{2.1}$$

for some finite complex measure Π on a strip in the complex plane. Property (2.1) says that the payoff function is written as an inverse Laplace tranform. For instance, for a European call option with strike price $K > 0$, the function $(s - K)^+$ may be written as

$$(s - K)^+ = \frac{1}{2\pi i} \int_{R-i\infty}^{R+i\infty} s^z \frac{K^{1-z}}{z(z-1)} dz,$$

for an arbitrary $R > 1$ and for each $s > 0$. For more details on this and for other examples of integral representation of payoff functions we refer to [6].

Given an admissible trading strategy $\vartheta = (\vartheta_n)$, for $n = 1, \ldots, N$, with cumulative gains $G_n(\vartheta) = \sum_{k=1}^n \vartheta_k \Delta S_k$ [1], the hedging error of the strategy is

$$\varepsilon(\vartheta, c) = H - c - G_N(\vartheta).$$

The random variable ϑ_n may be interpreted as the number of shares of the underlying asset held from time $n-1$ up to time n. If there exists a riskless asset, the strategy ϑ determines a unique self-financing portfolio and the hedging error $\varepsilon(\vartheta, c)$ may be viewed as the net loss one can suffer at maturity

[1] An admissible strategy is a predictable process such that the cumulative gains are all square-integrable, see [6] or [16]. Because of the hypothesis of a null interest rate, the money market account does not contribute to the cumulative gain.

if one starts with the initial capital c and follows the strategy. The problem is to evaluate its expected value $E[\varepsilon]$ and its variance $\mathrm{var}(\varepsilon)$.

It is well known that, for each c, there exists a strategy $\xi^{(c)}$ which minimizes the expected square value of the hedging error (see for instance [16]). In [6], where the mean-variance trade-off is deterministic, the optimal strategy and its variance are computed. It is also possible to compute the best choice of initial capital to invest in the portfolio. However, one of the most common trading strategy in practice is the Black-Scholes delta strategy. Traders estimate the volatility of the process at each trading date to hold a number of shares equal to the Black-Scholes delta. In this paper we will compute expected value and variance of the loss that the trader faces following such a strategy and we will compare it with other possible strategies, in particular with the optimal strategy.

First of all notice that, for claims satisfying (2.1), the Black-Scholes price C_n^{bs} of the claim at time n can be computed as

$$C_n^{bs} = E_n[H] = E_n\left[\int S_N^z \Pi(dz)\right],$$

where E_n is the Black-Scholes risk neutral expectation conditional to $\mathcal{F}_n$. By Fubini's theorem, we can exchange the expected value with the integral in the complex variable to get

$$\begin{aligned} C_n^{bs} &= \int E_n\left[S_N^z\right]\Pi(dz) = \int S_n^z E_n\left[\exp(z(\Delta X_{n+1} + \ldots \Delta X_N))\right]\Pi(dz) \\ &= \int S_n^z m_0(z)^{N-n}\Pi(dz), \end{aligned}$$

where we used the fact that the random variables ΔX_i are i.i.d. and denoted their moment generating function $m_0(z)$. Therefore, the units of underlying held from time $n-1$ to time n, when following the Black-Scholes delta hedging strategy, are

$$\Delta_n = \frac{\partial C_{n-1}^{bs}}{\partial S_{n-1}} = \int m_0(z)^{N-n+1}\frac{\partial S_{n-1}^z}{\partial S_{n-1}}\Pi(dz) = \int z m_0(z)^{N-n+1} S_{n-1}^{z-1}\Pi(dz)$$

Motivated by this computation, we give the following

Definition 2.1 *A hedging strategy ϑ is* **simple** *for a contingent claim with a payoff function satisfying Property 2.1 if it is of the form*

$$\vartheta_n = \int \vartheta(z)_n \Pi(dz),$$

with $\vartheta(z)_n = f^{\vartheta}(z)_n S_{n-1}^{z-1}$, where $f^{\vartheta}(z)_n$ is a function of the complex variable z which does not contain S_k for any k.

Other interesting strategies which are simple are:

- The strategy which minimizes the variance of the local costs. This is

$$\xi_n = \int \xi(z)_n \Pi(dz) = \int f^{\xi}(z)_n S_{n-1}^{z-1} \Pi(dz),$$

  where $f^{\xi}(z)_n$ is given in Theorem 2.1 in [6].

- The strategy constructed by Wilmott ([18]) in the Black-Scholes case to be an approximation of the local optimal strategy above:

$$\Delta_n^w = \Delta_n + \frac{T}{N}(\mu - \frac{1}{2}\sigma^2)\Gamma_n =$$
$$= \int S_{n-1}^{z-1} \left(z m_0(z)^{N-n+1} + \frac{T}{N}(\mu - \frac{1}{2}\sigma^2) z(z-1) m_0(z)^{N-n+1} \right) \Pi(dz),$$

  where μ and σ are respectively the drift and the volatility of the process. The expression of Δ_n^w has been obtained in an analogous way as for the delta, since the gamma of the claim Γ_n is the second derivative of C_{n-1}^{bs} with respect to S_{n-1}.

The delta and the Wilmott's delta strategies are conceived when S is a log-normal process. Nevertheless they may be considered also when another model generates the data; in this case, a sensible choice for parameters μ and σ would be to fit mean and variance of the returns of the model.

3 Measuring the discretization error

To assess the risk of the discretization error of simple strategies one can compute its variance and expected value. Černý [2] and Hubalek et al. [6] computed the variances for global and local optimal strategies. We will generalize their results to non optimal strategies and in the presence of model risk, that is when the adopted strategy is not consistent with the data generating process.

The hedging error of a simple strategy for the contingent claim satisfying Property 2.1 has the following integral representation

$$\begin{aligned} \varepsilon(\vartheta, c) &= H - c - \sum_{k=1}^{N} \vartheta_k \Delta S_k \\ &= \int \left(H(z) - \sum_{k=1}^{N} \vartheta(z)_k \Delta S_k \right) \Pi(dz) - c, \end{aligned} \tag{3.2}$$

where $H(z) = S_N^z$.

The following theorem provides formulas to compute the expected value and the variance of the hedging error of a simple strategy for a given initial capital c.

Theorem 3.1 *Let ϑ be a simple strategy for a contingent claim H and let c be its initial value, then*

$$E[\varepsilon(\vartheta, c)] = \int S_0^z \left[m(z)^N - (m(1)-1) \sum_{k=1}^{N} f^{\vartheta}(z)_k m(z)^{k-1} \right] \Pi(dz) - c \tag{3.3}$$

and

$$E[\varepsilon(\vartheta, 0)^2] = \int \int S_0^{y+z} (v_1(y,z) - v_2(y,z) - v_3(y,z) + v_4(y,z)) \Pi(dz)\Pi(dy), \tag{3.4}$$

where

$$v_1(y,z) = m(y+z)^N,$$

$$v_2(y,z) = \sum_{k=1}^{N} f^{\vartheta}(y)_k m(y+z)^{k-1} m(z)^{N-k} (m(z+1) - m(z)),$$

$$v_3(y,z) = \sum_{k=1}^{N} f^{\vartheta}(z)_k m(y+z)^{k-1} m(y)^{N-k} (m(y+1) - m(y)),$$

$$\begin{aligned} v_4(y,z) &= (m(2) - 2m(1) + 1) \sum_{k=1}^{N} f^{\vartheta}(y)_k f^{\vartheta}(z)_k m(z+y)^{k-1} + \\ &+ (m(1)-1) \sum_{k<j} \sum_{j=2}^{N} f^{\vartheta}(y)_k f^{\vartheta}(z)_j m(y)^{j-1-k} m(z+y)^{k-1} (m(y+1) - m(y)) + \\ &+ (m(1)-1) \sum_{j<k} \sum_{k=2}^{N} f^{\vartheta}(y)_k f^{\vartheta}(z)_j m(z)^{k-1-j} m(z+y)^{j-1} (m(z+1) - m(z)). \end{aligned}$$

Therefore, the variance of the hedging error is

$$\mathrm{var}(\varepsilon(\vartheta, c)) = \mathrm{var}(\varepsilon(\vartheta, 0)) = E[\varepsilon(\vartheta, 0)^2] - E[\varepsilon(\vartheta, 0)]^2.$$

Proof. Given (3.2), we have, by Fubini's Theorem,

$$\begin{aligned}
&E[H - \sum_{k=1}^{N} \vartheta_k \Delta S_k] = \int E[S_N^z - \sum_{k=1}^{N} f^{\vartheta}(z)_k S_{k-1}^{z-1} \Delta S_k] \Pi(dz) = \\
= \;& \int \left\{ E[S_0^z \exp(z(\Delta X_1 + \ldots \Delta X_N))] - \sum_{k=1}^{N} f^{\vartheta}(z)_k E[S_{k-1}^{z-1} \Delta S_k] \right\} \Pi(dz) = \\
= \;& \int S_0^z \left\{ m(z)^N - \sum_{k=1}^{N} f^{\vartheta}(z)_k E[\exp((z-1)(\Delta X_{k-1} + \ldots + \Delta X_1)) \right. \\
& \left. \times[\exp(\Delta X_k + \ldots + \Delta X_1) - \exp(\Delta X_{k-1} + \ldots + \Delta X_1)]] \right\} \Pi(dz) = \\
= \;& \int S_0^z \left\{ m(z)^N - \sum_{k=1}^{N} f^{\vartheta}(z)_k \right. \\
& \left. \times E[(\exp(z(\Delta X_{k-1} + \ldots + \Delta X_1) + \Delta X_k) - \exp(z(\Delta X_{k-1} + \ldots + \Delta X_1))] \right\} \Pi(dz) = \\
= \;& \int S_0^z \left[m(z)^N - \sum_{k=1}^{N} f^{\vartheta}(z)_k m(z)^{k-1}(m(1) - 1) \right] \Pi(dz).
\end{aligned}$$

which is (3.3). To prove (3.4) we need to compute

$$\begin{aligned}
&E[(H - \sum_{k=1}^{N} \vartheta_k \Delta S_k)^2] = \\
&= E[\int (H(z) - \sum_{k=1}^{N} \vartheta(z)_k \Delta S_k) \Pi(dz) \int (H(y) - \sum_{k=1}^{N} \vartheta(y)_k \Delta S_k) \Pi(dy)] = \\
&= E[\int \int (H(z) - \sum_{k=1}^{N} \vartheta(z)_k \Delta S_k)(H(y) - \sum_{k=1}^{N} \vartheta(y)_k \Delta S_k) \Pi(dz) \Pi(dy)] = \\
&= \int \int E[(H(z) - \sum_{k=1}^{N} \vartheta(z)_k \Delta S_k)(H(y) - \sum_{k=1}^{N} \vartheta(y)_k \Delta S_k)] \Pi(dz) \Pi(dy).
\end{aligned}$$

Let us compute all the expectations needed:

$$\begin{aligned}
E[H(z)H(y)] &= S_0^{y+z} \exp(z(\Delta X_N + \ldots + \Delta X_1) + y(\Delta X_N + \ldots + \Delta X_1)) = \\
&= S_0^{y+z} m(y+z)^N = S_0^{y+z} v_1(y, z).
\end{aligned}$$

$$
\begin{aligned}
&E[H(z)\sum_{k=1}^{N}\vartheta(y)_k\Delta S_k] = \\
&= \sum_{k=1}^{N} f^{\vartheta}(y)_k E[S_N^z S_{k-1}^{y-1}\Delta S_k] = \\
&= \sum_{k=1}^{N} f^{\vartheta}(y)_k S_0^{z+y} E[\exp(z(\Delta X_N + \ldots + \Delta X_1))\exp((y-1)(\Delta X_{k-1} + \ldots + \Delta X_1)) \\
&\times [\exp(\Delta X_k + \ldots + \Delta X_1) - \exp(\Delta X_{k-1} + \ldots + \Delta X_1)]] = \\
&= S_0^{z+y}\sum_{k=1}^{N} f^{\vartheta}(y)_k \\
&\times \{E[\exp((y+z)(\Delta X_{k-1} + \ldots + \Delta X_1))\exp(z(\Delta X_N + \ldots + \Delta X_k))\exp(\Delta X_k)] + \\
&- E[\exp((y+z)(\Delta X_{k-1} + \ldots + \Delta X_1))\exp(z(\Delta X_N + \ldots + \Delta X_k))]\} = \\
&= S_0^{z+y}\sum_{k=1}^{N} f^{\vartheta}(y)_k \left[m(y+z)^{k-1}m(z)^{N-k}m(z+1) - m(y+z)^{k-1}m(z)^{N-k+1}\right] = \\
&= S_0^{z+y} v_2(y,z).
\end{aligned}
$$

Analogously, computing the expectation

$$E[H(y)\sum_{k=1}^{N}\vartheta(z)_k\Delta S_k]$$

one gets $S_0^{y+z} v_3(y,z)$.

The last term is

$$
\begin{aligned}
&E[\sum_{k=1}^{N}\vartheta(z)_k\Delta S_k\sum_{j=1}^{N}\vartheta(y)_j\Delta S_j] = \\
&= \sum_{k=1}^{N}\sum_{j=1}^{N} f^{\vartheta}(z)_k f^{\vartheta}(y)_j E[S_{k-1}^{z-1}S_{j-1}^{y-1}\Delta S_k\Delta S_j] = \\
&= S_0^{y+z}\sum_{k=1}^{N}\sum_{j=1}^{N} f^{\vartheta}(z)_k f^{\vartheta}(y)_j \\
&\times E[\exp((z-1)(\Delta X_{k-1}+\ldots\Delta X_1))\exp((y-1)(\Delta X_{j-1}+\ldots\Delta X_1)) \\
&\times\exp(\Delta X_{k-1}+\ldots\Delta X_1)(\exp(\Delta X_k)-1) \\
&\times\exp(\Delta X_{j-1}+\ldots\Delta X_1)(\exp(\Delta X_j)-1)] = \\
&= S_0^{y+z}\sum_{k=1}^{N}\sum_{j=1}^{N} f^{\vartheta}(z)_k f^{\vartheta}(y)_j \\
&\times E[\exp(z(\Delta X_{k-1}+\ldots\Delta X_1))\exp(y(\Delta X_{j-1}+\ldots\Delta X_1)) \\
&\times(\exp(\Delta X_k)-1)(\exp(\Delta X_j)-1)].
\end{aligned}
$$

The last sum may be computed separating the cases $k=j$, $k<j$ and $k>j$ as

$$
\begin{aligned}
&\sum_{k=j}\sum_{k=1}^{N} f^{\vartheta}(z)_k f^{\vartheta}(y)_k m(y+z)^{k-1}(m(2)-2m(1)+1)+ \\
&+\sum_{k<j}\sum_{j=2}^{N} f^{\vartheta}(z)_k f^{\vartheta}(y)_j m(y+z)^{k-1}(m(y+1)-m(y))(m(1)-1)+ \\
&+\sum_{k>j}\sum_{k=2}^{N} f^{\vartheta}(z)_k f^{\vartheta}(y)_j m(y+z)^{j-1}(m(z+1)-m(z))(m(1)-1),
\end{aligned}
$$

which is $v_4(y,z)$. □

Theorem 3.1 states that the expected value and the variance of the hedging error may be represented respectively as one- and two-dimensional inverse Laplace transforms. Although the formulas seem rather involved, they are quite easy to evaluate numerically. In Section 4 we will give some details on their implementation and we will discuss the precision of the algorithm used.

A similar argument can be applied to compute higher order moments of the hedging errors, that can be useful to get more information on the probability distribution.

Notice that for the Theorem to hold it is not necessary that the simple strategy is consistent with the model. We can, for instance, consider the case where the data generating process is the Black-Scholes process with a certain drift and volatility, while the strategy is based on different estimates. Or it may be the case that the data generating model is the Merton jump-diffusion model ([12]), while the strategy is conceived according to the Black-Scholes world, perhaps fitting mean and variance of the returns.

A more general form of the Theorem holds if Hypothesis 3 on the process X is relaxed, namely if the increments ΔX_n are not identically distributed. All the computations would go through to obtain similar results basically by substituting all the powers of $m(\cdot)$ in the formulas by suitable products of $m_n(\cdot)$, where $m_n(\cdot)$ clearly stands for the moment generating function of ΔX_n, for $n = 1, \ldots, N$. Of course the formulas would get even more involved. However, such a generalization may have interesting application for interest rate sensitive derivatives, e.g. when the model belongs to the class of Gaussian Heath Jarrow and Morton models. This is because the volatility of the bond prices decreases approaching to maturity.

Since a simple strategy does not depend on the invested initial value, the expected value of the error produced by a simple strategy with initial value c can be obtained by subtracting c from the expected value of the same strategy with zero starting value. For the same reason, the variance of the error produced by a simple strategy does not depend on c. On the other hand, the optimal strategy does depend on the initial capital. Next result measures the influence of the initial capital c on the expectation and the variance of the optimal strategy. We indicate with V_0 the optimal initial capital, that is the value of c that minimizes the expectation of the square of the discretization error (namely, the solution of Problem (3.1) in [16]).

Proposition 3.1 *Let ξ^c be the optimal, N-step strategy for a contingent claim H with an initial capital c and let $\varepsilon(\xi^c, c)$ be its hedging error. Then*

$$E[\varepsilon(\xi^c, c)] = (V_0 - c)\left(\frac{m(2) - m(1)^2}{m(2) - 2m(1) + 1}\right)^N \tag{3.5}$$

where V_0 is the optimal initial capital. Moreover, the variance of $\varepsilon(\xi^c, c)$ does not depend on c.

Proof. From Corollary 2.5 in [16] it follows that

$$\begin{aligned} E[\varepsilon(\xi^c, c)] &= E[H\tilde{Z}^0] - cE[\tilde{Z}^0] \\ &= (V_0 - c)E[\tilde{Z}^0] \end{aligned}$$

where (using the fact that we are in the case of a deterministic mean-variance payoff),

$$\tilde{Z}_0 = \prod_{k=1}^{N}(1 - \alpha_k \Delta S_k)$$

with

$$\alpha_k = \frac{E[\Delta S_k|\mathcal{F}_{k-1}]}{E[\Delta S_k^2|\mathcal{F}_{k-1}]}.$$

Setting

$$\lambda = \frac{(m(1)-1)}{m(2) - 2m(1) + 1},$$

we have

$$E[\tilde{Z}^0] = \prod_{k=1}^{N}(1 - \lambda(m(1)-1)) = \left(\frac{m(2) - m(1)^2}{m(2) - 2m(1) + 1}\right)^N,$$

from which we get (3.5).

To prove the statement on the variance, recall that, from Theorem 4.4 in [16],

$$\begin{aligned} &E[(H - c - G_N(\xi^{(c)})^2] = (V_0 - c)^2 \prod_{k=1}^{N}(1 - \alpha_k E[\Delta S_k|\mathcal{F}_{k-1}]) + \mathrm{var}(H - G_N(\xi^{V_0})) \\ &= (V_0 - c)^2 \left(\frac{m(2) - m(1)^2}{m(2) - 2m(1) + 1}\right)^N + \mathrm{var}(H - G_N(\xi^{V_0})) \\ &= E[H - c - G_N(\xi^{(c)}]^2 + \mathrm{var}(H - G_N(\xi^{V_0})), \end{aligned}$$

where we used (see (2.8) in [16]),

$$E[\tilde{Z}^0] = E[(\tilde{Z}^0)^2]$$

□

4 Numerical implementation

There are at least two possible approaches to compute Equations (3.3) and (3.4): numerical integration and inversion of Laplace transform. We followed the second approach, implementing the algorithms in MATLAB. The formulas we wish to compute involve one- and two-dimensional Laplace transforms. A list of the available MATLAB codes for the one-dimensional case can be found in [10]. For the one dimensional case, that is to compute expected values, we used "invlap.m" constructed by [5], based on the method by [7], which is accurate and fast. We wrote a code based on Formula (2.11) in [3] for the bi-dimensional case (to compute second moments).

The parameters of the algorithm in the code invlap.m for the one-dimensional inversion are two: the pole with largest real part of the function to be inverted and a tolerance parameter which essentially gives the distance from the largest pole of the vertical line of integration. The largest pole has to be given correctly: for instance, in the case of a call option, the largest pole is 1. The default value for the tolerance parameter (10^{-9}) gives, in our experience, rather accurate results.

As explained in [3], the two-dimensional algorithm depends on six parameters, A_1, A_2, l_1,l_2, n and m, and has three possible sources of error: the aliasing error, the roundoff error and the truncation error. Each error is controlled by two parameters. We found that $A_1 = A_2 = 30$ and $l_1 = l_2 = 1$ are good choices to get small, respectively, aliasing and roundoff errors. The parameters n and m are the most relevant for our computations; in fact, these are the parameters that control the Euler approximation to the infinite sums.

To give an idea of the results produced by the algorithm, we consider an at-the-money European call option with maturity $T = 0.25$ years, hedged only once, at time 0, using the Black-Scholes delta. We assume that the underlying process is a Geometric Brownian motion with drift $\mu = 0.1$, volatility $\sigma = 0.4$ and that the initial price is $S_0 = 100$. In this case, the expected value and the variance of the hedging error produced by this static strategy (the units of underlying are 0.5398) can be explicitly computed. The expected value, for an initial capital equal to the Black-Scholes price of the option (7.9655), is 0.062723168, the variance is 39.10233. The one-dimensional algorithm produces an expected value equal to 0.062723143 with largest pole set to 1 and default tolerance parameter.

To compute the variance we use our code to invert a double dimensional

m	n	err
20	40	494.21
50	50	156.21
50	100	33.91
100	100	20.01
100	200	4.31
200	200	2.51
20	400	1.01
50	400	0.91
100	400	0.71
20	600	0.31

Table 1: Absolute error (multiplied by 10^4) of variance of the hedging error of the delta strategy computed with the two-dimensional inversion algorithm for an at-the money call option with number of trading dates $N = 1$ as a function of m and n. The other parameters used by the algorithm are $A_1 = A_2 = 30$, $l_1 = l_2 = 1$.

Laplace transform. In Table 1 we show the results produced by the algorithm as a function of m and n, the most important couple of parameters, keeping $A_1 = A_2 = 30$ and $l_1 = l_2 = 1$. We report the difference (multiplied by 10^4) between the computed variance and the exact one. Note that the number of terms in the approximating sum computed by the algorithm is $n + m$, hence the computational burden increases with n and m. From this example (but it is our general impression), it appears that a higher accuracy is reached by increasing n. We also note that the error is only marginally reduced by increasing the other parameter m.

5 Applications

First we shall assess the precision of some approximating formula. Kamal and Derman [9] provided a useful formula for computing an approximate value of the variance of the discretization error produced in the Black-Scholes model when hedging a European call option using the standard delta strategy. The formula, an approximation as the number of trading dates N goes to infinite, reads as follows

$$\begin{aligned} \operatorname{var}(\varepsilon(\Delta, c)) &\approx \frac{1}{2}\sigma^4 \left(\frac{T}{N}\right)^2 S_0^4 \Gamma_0^4 \sum_{i=0}^{N-1} g(t_i), \qquad (5.6) \\ g(t) &= \sqrt{\frac{T^2}{T^2 - t^2}} \exp\left(2\mu t - 2d_1 \frac{(\mu - r)t}{\sigma\sqrt{T}} - \frac{(\mu - r)^2 t^2}{\sigma^2 T}\right) \times \\ & \quad \exp\left(\left[d_2^2 + 2d_2 \frac{(\mu - r)t}{\sigma\sqrt{T}} - \frac{(\mu - r)^2 t^2}{\sigma^2 T}\right] \frac{t}{T + t}\right) \end{aligned}$$

where Γ_0 is the option's gamma computed at time 0, and d_1 and d_2, the usual quantities in the Black-Scholes formula, are also computed at time 0. Formula (5.6) is the same as the approximation in [17] in absence of transaction costs. They also propose a very easy-to-read approximation involving the option's vega κ_0 at time 0 as follows

$$\operatorname{var}(\varepsilon(\Delta, c)) \approx \frac{\pi}{4N}\sigma^2 \kappa_0^2 \qquad (5.7)$$

Figure 1 represents the relative error on standard deviations produced by the Kamal-Derman formulas for $N = [1, 3, 5, 7, 10, 13, 26, 39, 52, 65]$. The parameters used are $S_0 = 100$, $r = 0$, $\mu = 0.05$, $\sigma = 0.5$, $T = 1$, $K = 100$. We see that approximation (5.6) underestimate the standard deviation while that with the vega in (5.7) overestimate it. The error of the first is above 4% when the trading intervals are fewer than 10 and that it goes under 2% as N increases, in particular when greater than 26. The second is similar but works slightly better, especially for small values of N.

In the rest of the section we compute the mean and the variance of the errors produced by different strategies to hedge a European call option. We suppose that the initial value of the strategy c is equal to the Black-Scholes value. To measure the performances of the strategies we compute the expected values and the standard deviations of their final shortfalls. As it was shown above, the initial capital c only influences the expected values, leaving the standard deviations unaffected, also for the optimal strategy.

The natural goal for the hedger would be to have a negative expected loss (i.e. a gain) with a small variance. A possible way to take both such measures into account is to compute the Sharpe index of the strategy $s(\vartheta, c) = \frac{-E[\varepsilon(\vartheta,c)]}{\sqrt{\operatorname{var}(\varepsilon(\vartheta,c))}}$.

We considered the following strategies:

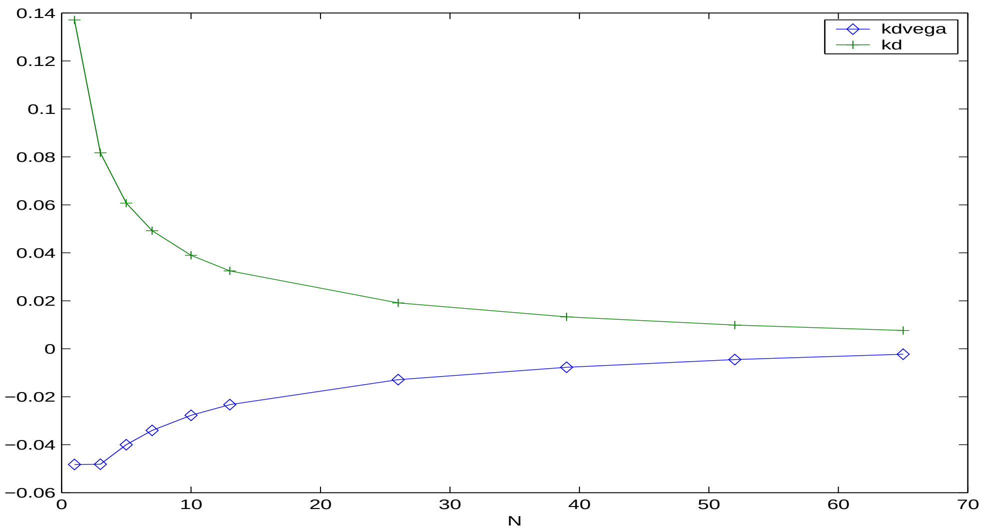

Figure 1 *Relative error (1-approx/exact) of Kamal and Derman approximations of the standard deviation of the hedging error as a function of the number of trading intervals. Black-Scholes model with* $S_0 = 100$*,* $r = 0$*,* $\mu = 0.05$*,* $\sigma = 0.5$*, European call option with* $K = 100$*,* $T = 1$*.*

1. Black-Scholes delta strategy Δ;
2. Wilmott strategy;
3. locally minimizing strategy ξ;
4. optimal strategy ξ^c.

However, in the figures below, for the sake of simplicity, we will not show the results given by the Wilmott strategy, as it is, for the cases we examined, very close to the locally minimizing strategy.

5.1 Comparing the strategies

The contingent claim we consider is an at-the-money European call option with maturity $T = 0.25$ years. We suppose that the initial price of the underlying asset is $S_0 = 100$.

We consider a trader who observes asset returns with annual mean $\mu \approx 0.1451$ and volatility $\sigma \approx 0.4379$, and wishes to compare different hedging strategies. She or he considers different possible numbers of trading dates, namely $N = (1, 3, 5, 7, 10, 13, 26, 39, 52, 65)$. We suppose that the price of the option is $c = 8.7176$, according to the observed σ. The trader sells the option and invest all the money in the hedging strategy.

We consider two cases for the data generating process:

1. Geometric Brownian motion with parameters μ and σ;
2. Merton jump-diffusion process with normally distributed jumps, that yields returns with same mean and variance as above. This it is the case for instance when the model is specified by parameters $\mu' = 0.05$ and $\sigma' = 0.3$ of the Geometric Brownian motion, intensity of the jumps $\lambda = 10$, and mean and standard deviation of the jumps respectively $\nu = 0$ and $\tau = 0.1$.

In the first case of course, as the number of trading dates increases, expected value and variance of all the strategies go to zero, as the model becomes complete in the limit. Also, differences among various strategies get soon quite small. In order to better quantify differences in standard deviation, we consider the ratio between the standard deviation of each strategy and the minimal that can be achieved within the considered model.

Figure 2 represents expected values and Figure 4 standard deviations ratio of the total loss for the strategies considered as functions of the number of trading dates N. We notice that the expected value of the loss for the standard Black-Scholes delta is positive and quite different from other strategies. This is not a surprise because we are considering a positive annual drift that is ignored by the delta hedging strategy but taken into account by all the others. The Sharpe index in Figure 6 shows that the delta strategy gives worse results than the optimal ones.

This first result seems to suggest that, at least as far as standard deviation is concerned, the Black-Scholes delta performs reasonably well.

The idea of the second possibility is as follows: the trader follows a strategy based on the Black-Scholes model, while the data generating process is the Merton one. As already observed, Theorem 3.1 holds even if the strategy is not consistent with the model. In particular, she or he may adopt the delta strategy or the locally minimizing strategy, both based on the Black-Scholes model with the observed parameters. Notice that we cannot analyze the performance of the optimal strategy based on a different model than the data generating one, because that is not simple. The local optimal, having similar behavior, would give a strong idea also on that. Alternatively, if she or he had perfect knowledge of the data generating process could use either the local or the global optimal strategy. The latter will be used as a benchmark, being the best one could do in this model. Therefore, in this case, the strategies analyzed are four. This analysis provides an insight on the influence of model risk on the performances of different hedging strategies. Since the model is not complete in the limit, the standard deviation of any strategy does not go to zero, the smallest value (2.87) achieved at $N = 65$ by the optimal one. It is interesting to notice that the local optimal strategy, constructed in the Black-Scholes setting, performs better, in terms of Sharpe index, than the delta strategy either when the model is correctly catched (Figure 6) or when it is not (Figure 7). In particular, the standard deviation of the delta strategy is consistently 2% higher than that of the minimal one, while that of the log-normal local optimal stays under 2%, as it is shown in Figure 5.

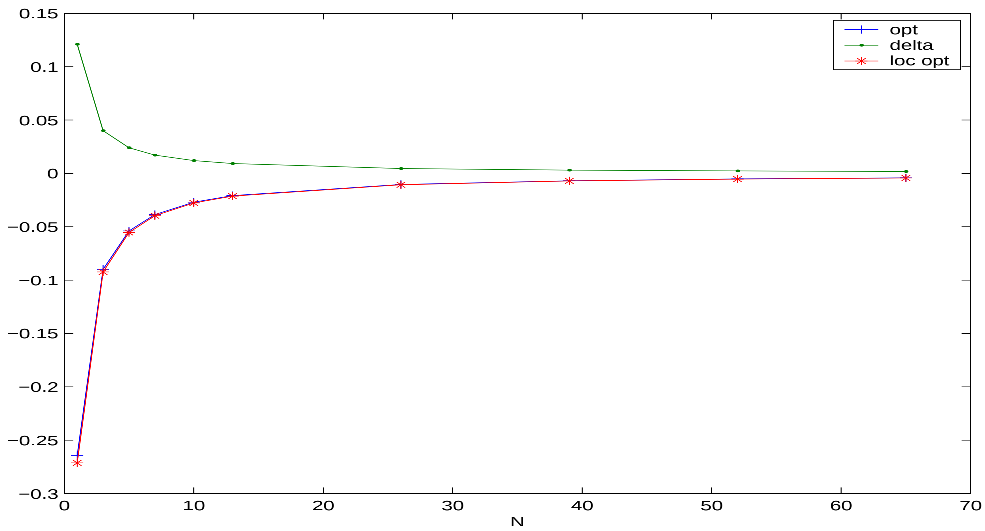

Figure 2 *Black-Scholes model: expected value of hedging error of different strategies as the number of trading dates increases.*

5.2 Exploiting personal views

Here we consider the case of a trader who wishes to exploit the difference between her or his view and the market on the volatility of the underlying in

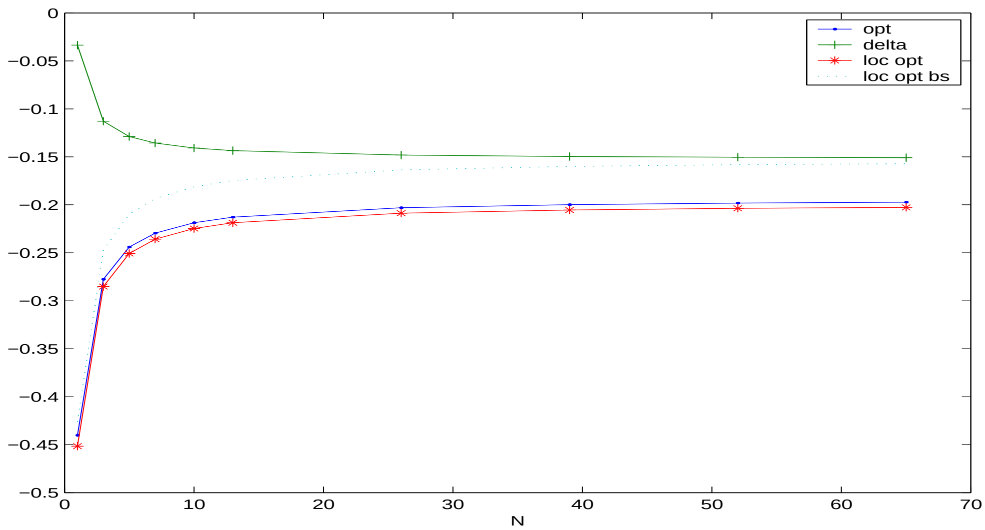

Figure 3 *Merton model: expected value of hedging error of different strategies as the number of trading dates increases.*

order to adopt a profitable strategy. In particular, we suppose that the price of the option considered in the previous section is $c = 5.9785$, according to an implied volatility $\sigma_0 = 0.3$. The trader believes that the underlying asset will follow a Geometric Brownian motion with lower volatility. To illustrate the results, we set $\mu_0 = 0.1$ as the drift of the process and $S_0 = 100$. Changing μ_0 would not substantially change the results, even if it is negative. She or he chooses to adopt a strategy according to the market implied volatility σ_0. We will analyze the behavior of both the delta and the local optimal strategies as a function of the actual volatility of the process σ, which we let vary from 0.1 to 0.5. The performance of the strategy is measured with the Sharpe index. The number of trading considered is $N = 10$.

The results are depicted in Figure 8. As the trader expected, the Sharpe index is positive when σ is lower than σ_0. The picture shows also that the local optimal strategy gives consistently a slightly better performance. When $\sigma = \sigma_0$ the Sharpe index of the delta is negative (-0.0052), while that of the local optimal strategy is positive (0.0099). In this case we can compute the Sharpe index of the optimal, which is 0.01.

Figure 9 shows the influence of the drift on the results of the strategies by plotting the differences between the Sharpe ratios. It appears that the

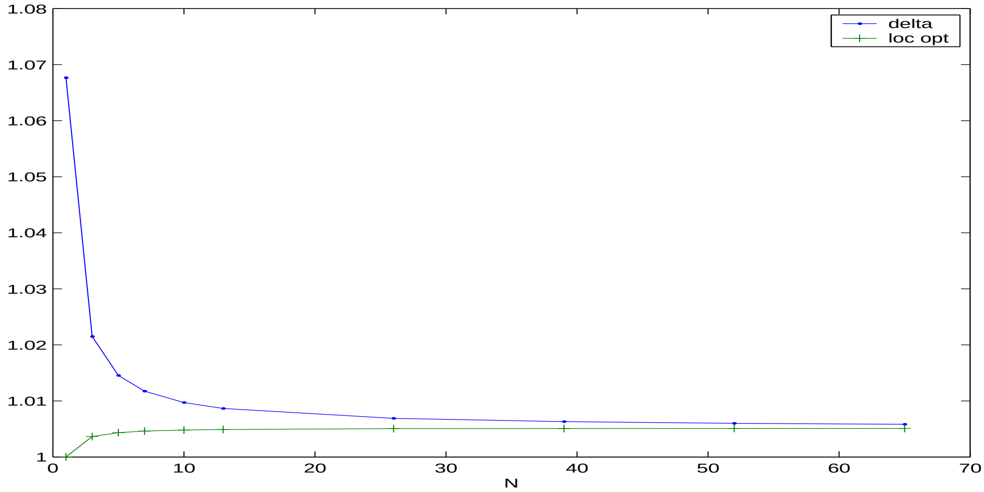

Figure 4 *Black-Scholes model: ratio of standard deviation over the minimal one of hedging error of different strategies as the number of trading dates increases.*

Sharpe ratio of the optimal strategy is consistently better than that of the delta hedging when μ is not zero. In the martingale case the performance of the two strategies are very similar.

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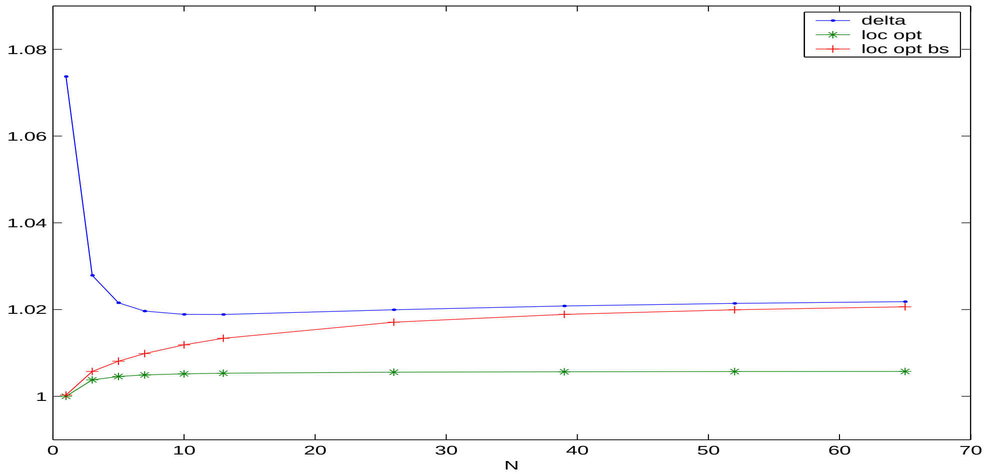

Figure 5 *Merton model: ratio of standard deviation over the minimal one of hedging error of different strategies as the number of trading dates increases.*

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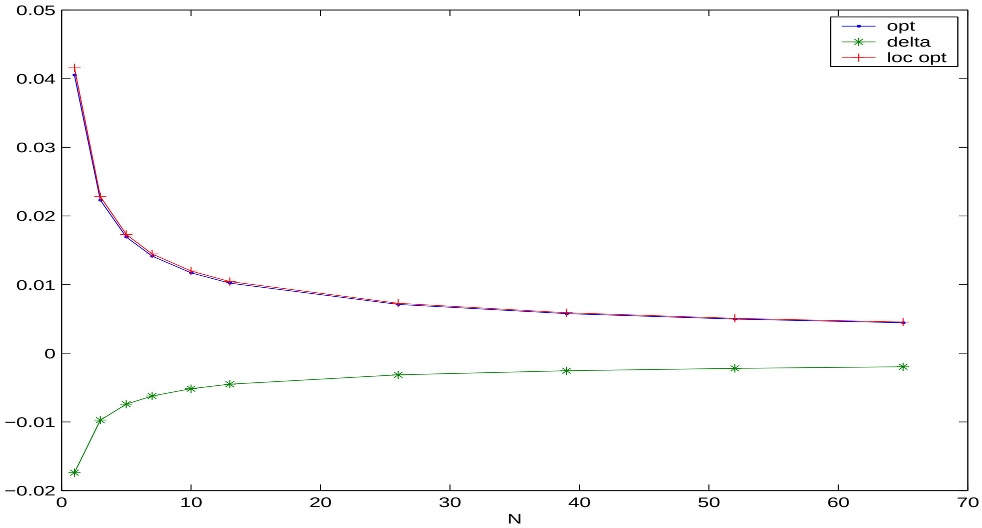

Figure 6 *Black-Scholes model: Sharpe index of hedging error of different strategies as the number of trading dates increases.*

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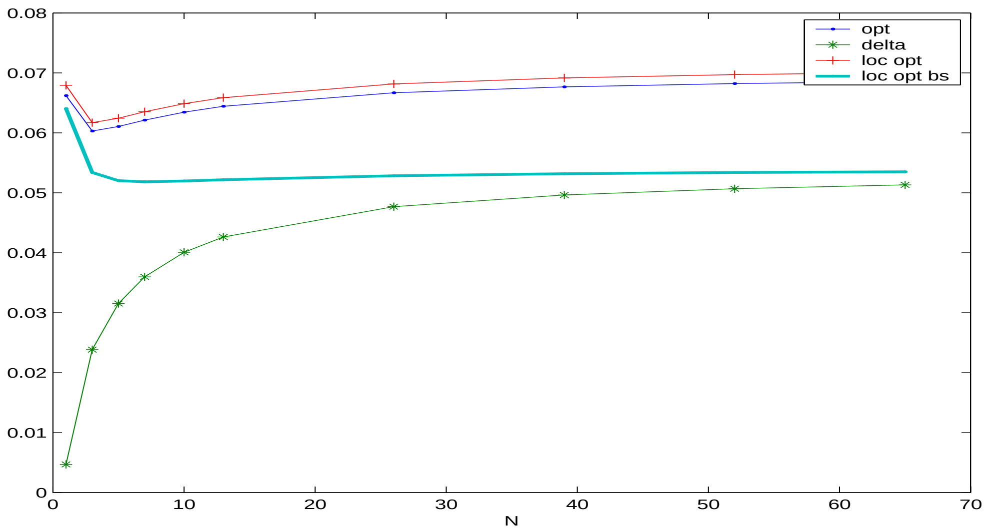

Figure 7 *Merton model: Sharpe index of hedging error of different strategies as the number of trading dates increases.*

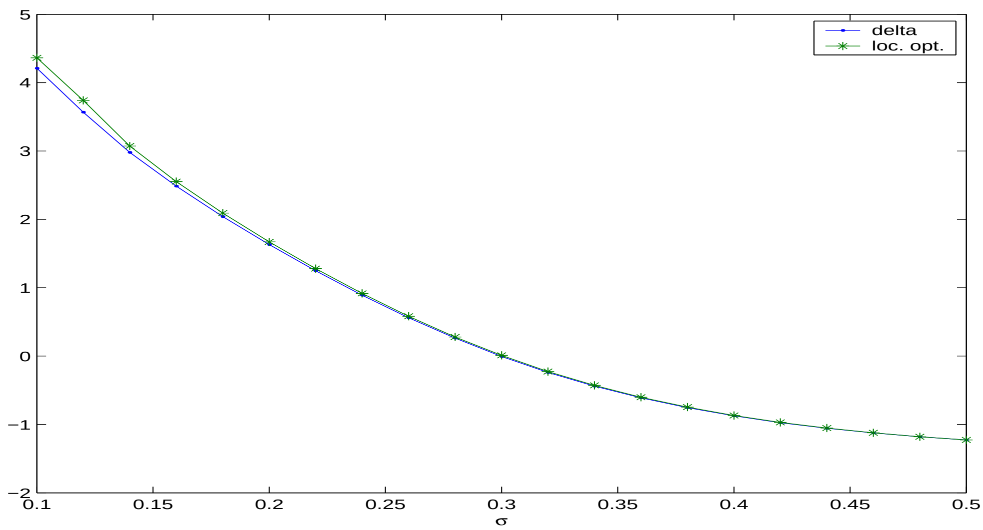

Figure 8 *Sharpe index of hedging error of delta and local optimal strategies as a function of* σ*, the actual volatility of the Black-Scholes process. The number of trading dates is* $N = 10$*. The strategies are constructed assuming* $\sigma_0 = 0.3$ *and that the drift* $\mu_0 = 0.1$ *is correctly estimated.*

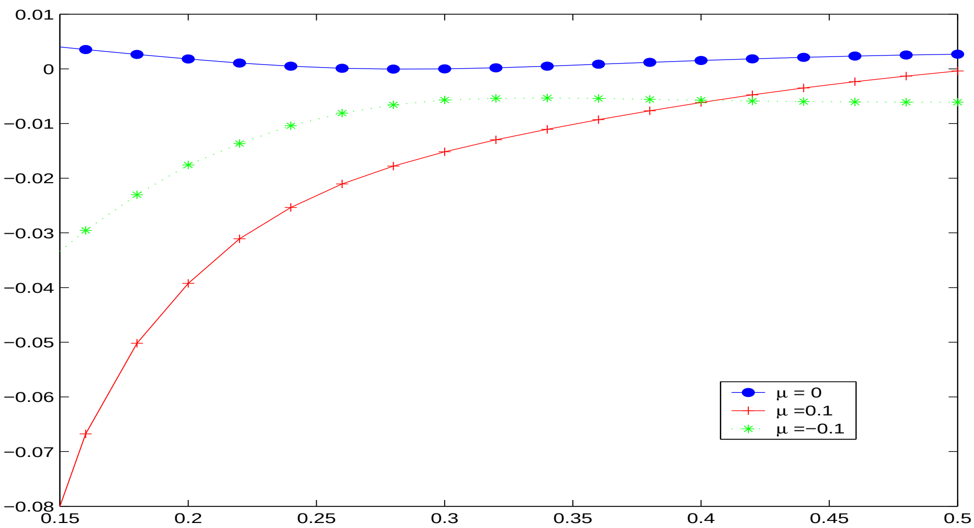

Figure 9 *Differences between Sharpe indexes of hedging error of delta and of local optimal strategies as a function of* σ*, the actual volatility of the Black-Scholes process. The number of trading dates is* $N = 10$*. The strategies are constructed assuming* $\sigma_0 = 0.3$ *and the correct value of* μ*. The three curves correspond to* $\mu = 0, 0.1, -0.1$