A MAXIMUM PRINCIPLE FOR STOCHASTIC DIFFERENTIAL GAMES WITH PARTIAL INFORMATION

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Abstract. In this paper we first deal with the problem of optimal control for zero-sum stochastic differential games. We give a necessary and sufficient maximum principle for that problem with partial information. Then we use the result to solve a problem in finance. Finally, we extend our approach to general stochastic games (nonzero-sum), and obtain an equilibrium point of such game.

1. Introduction

Game theory had been an active area of research and a useful tool in many applications, particularly in biology and economic. In the recent paper by Mataramvura and Øksendal [5], the stochastic differential game was solved with restriction to consider only Markov controls. Then the equilibrium point or other type of solution are constructed using Hamilton-Jacobi-Bellman (HJB) equations. In this paper, we require that the control process is adapted to a given sub-filtration of the filtration generated by the underlying Lévy processes. So we cannot use dynamic programming and HJB equations to solve the problem. Here we establish a maximum principle for such stochastic control problem. There is already a lot of literature on the maximum principle. See e.g. [1], [2], [4], [8] and the references therein.

Our paper is organized as follows: In section 2 we give a sufficient maximum principle for zero-sum stochastic differential games (Theorem 1). And a necessary type of this problem is given in the section 3. In section 4 we put a problem in finance into the framework of a stochastic differential game with partial information and use Theorem 1 to solve it. With complete information this problem is solved in [7] by using HJB equations. In section 4 we generalize our approach to the general case, not necessarily of zero-sum type, and also give an equilibrium point for nonzero-sum games.

2. The sufficient maximum principle for zero-sum games

Suppose the dynamics of a stochastic system is described by a functional differential equation on a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ of the form

$$dX(t) = b(t, X(t), u_0(t))\,dt + \sigma(t, X(t), u_0(t))\,dB(t)$$

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$$
\begin{aligned}
& + \int_{\mathbb{R}^n} \gamma(t, X(t^-), u_1(t^-, z), z)\widetilde{N}(\,dt,\,dz) \qquad t \in [0,T] \qquad (2.1)\\
X(0) &= x \in \mathbb{R}^n
\end{aligned}
$$

Here $b : [0,T] \times \mathbb{R}^n \times K \to \mathbb{R}^n$; $\sigma : [0,T] \times \mathbb{R}^n \times K \to \mathbb{R}^{n\times n}$ and $\gamma : [0,T] \times \mathbb{R}^n \times K \times \mathbb{R}_0 \to \mathbb{R}^{n\times m}$ are given continuous functions, and $B(t)$ is n-dimensional Brownian motion, $\widetilde{N}(.,.)$ are n independent compensated Poisson random measures and K is a given closed subset of $\mathbb{R}^n$. The processes $u_0(t) = u_0(t,\omega)$ and $u_1(t) = u_1(t,z,\omega)$, $\omega \in \Omega$ are our *control processes*. We assume that $u_0(t), u_1(t,z)$ have values in a given set K for a.a. t, z and that $u_0(t), u_1(t,z)$ are càdlàg and adapted to a given filtration $\{\mathcal{E}_t\}_{t\geq 0}$, where

$$\mathcal{E}_t \subseteq \mathcal{F}_t, \qquad t \geq 0$$

Let $f : [0,T] \times \mathbb{R}^n \times K \to \mathbb{R}$ be a continuous function, namely the *profit rate*, and let $g : \mathbb{R}^n \to \mathbb{R}$ be a concave function, namely the *bequest function*. We call $u = (u_0, u_1)$ an *admissible control* if (2.1) has a unique strong solution and

$$E^x\Big[\int_0^T \mid f(t, X(t), u_0(t)) \mid \, dt + \mid g(X(T)) \mid \Big] < \infty \qquad (2.2)$$

If u is an admissible control we define the *performance criterion* $J(u)$ by

$$J(u) = E^x\Big[\int_0^T f(t, X(t), u_0(t))\,dt + g(X(T))\Big] \qquad (2.3)$$

Now suppose that the controls $u_0(t)$ and $u_1(t,z)$ have the form

$$u_0(t) = (\theta_0(t), \pi_0(t)); \qquad t \geq 0 \qquad (2.4)$$

$$u_1(t,z) = (\theta_1(t,z), \pi_1(t,z)); \qquad (t,z) \in [0,\infty) \times \mathbb{R}^n \qquad (2.5)$$

We denote Θ and Π to be given families of admissible controls $\theta = (\theta_0, \theta_1)$ and $\pi = (\pi_0, \pi_1)$, respectively. The *partial information zero-sum stochastic differential game problem* is to find $(\theta^*, \pi^*) \in \Theta \times \Pi$ such that

$$\Phi_{\mathcal{E}}(x) = J(\theta^*, \pi^*) = \sup_{\pi\in\Pi}\Big(\inf_{\theta\in\Theta} J(\theta,\pi)\Big) \qquad (2.6)$$

Such a control (θ^*, π^*) is called an *optimal control* (if it exists).

The intuitive idea is that there are two players, I and II. Player I controls $\theta := (\theta_0, \theta_1)$ and player II controls $\pi := (\pi_0, \pi_1)$. The actions of the players are antagonistic, which means that between I and II there is a payoff $J(\theta,\pi)$ which is a cost for I and a reward for II.

Let K_1, K_2 be two sets such that $\theta(t,z) \in K_1$ and $\pi(t,z) \in K_2$ for a.a t, z. Define the *Hamiltonian* $H : [0,T] \times \mathbb{R}^n \times K_1 \times K_2 \times \mathbb{R}^n \times \mathbb{R}^{n\times m} \times \mathcal{R} \mapsto \mathbb{R}$ by

$$
\begin{aligned}
H(t,x,\theta,\pi,p,q,r) =& f(t,x,\theta,\pi) + b^T(t,x,\theta,\pi)p + tr(\sigma^T((t,x,\theta,\pi)q)\\
&+ \sum_{i,j=1}^n \int_{\mathbb{R}_0} \gamma_{ij}(t,x,\theta,\pi,z) r_{ij}(t,z)\nu_j(dz_j) \qquad (2.7)
\end{aligned}
$$

where $\mathcal{R}$ is the set of functions $r : [0,T] \times \mathbb{R}_0 \mapsto \mathbb{R}^{n\times m}$ such that the integral in (2.7) converges. From now on we assume that H is differentiable with respect to x.

The *adjoint equation* in the unknown adapted processes $p(t) \in \mathbb{R}^n, q(t) \in \mathbb{R}^{n\times m}$ and $r(t,s) \in \mathbb{R}^{n\times m}$ is the backward stochastic differential equation (BSDE)

$$(2.8) \quad \begin{cases} dp(t) &= -\bigtriangledown_x H(t, X(t), \theta(t), \pi(t), p(t), q(t), r(t,.))\,dt \\ & \qquad +q(t)\,dB(t) + \int_{\mathbb{R}^n} r(t^-, z)\widetilde{N}(\,dt,\,dz), \qquad t < T \\ p(T) &= \bigtriangledown g(X(T)) \end{cases}$$

where $\bigtriangledown_y \varphi(.) = \left(\frac{\partial \varphi}{\partial y_1}, ..., \frac{\partial \varphi}{\partial y_n}\right)^T$ is the gradient of $\varphi : \mathbb{R}^n \to \mathbb{R}$ with respect to $y = (y_1, ..., y_n)$.

We can now state the following verification theorem for optimality:

Theorem 1. *Let $(\hat{\theta}, \hat{\pi}) \in \Theta \times \Pi$ with corresponding state process $\hat{X}(t) = X^{(\hat{\theta},\hat{\pi})}(t)$. Denoted by $X^{\pi}(t)$ and $X^{\theta}(t)$ to be $X^{(\hat{\theta},\pi)}(t)$ and $X^{(\theta,\hat{\pi})}(t)$, respectively. Suppose there exists a solution $(\hat{p}(t), \hat{q}(t), \hat{r}(t,z))$ of the corresponding adjoint equation (2.8) such that for all $\theta \in \Theta$ and $\pi \in \Pi$, we have*

(2.9)
$$E\Big[\int_0^T (\hat{X}(t) - X^{(\pi)}(t))^T \big\{\hat{q}\hat{q}^T(t) + \int_{\mathbb{R}^n} \hat{r}\hat{r}^T(t,z)\nu(dz)\big\}(\hat{X}(t) - X^{(\pi)}(t))dt\Big] < \infty$$

(2.10)
$$E\Big[\int_0^T (\hat{X}(t) - X^{(\theta)}(t))^T \big\{\hat{q}\hat{q}^T(t) + \int_{\mathbb{R}^n} \hat{r}\hat{r}^T(t,z)\nu(dz)\big\}(\hat{X}(t) - X^{(\theta)}(t))dt\Big] < \infty,$$

and

$$\begin{aligned} &E\Big[\int_0^T \hat{p}(t)^T \big\{\sigma\sigma^T(t, X(t), \theta(t), \hat{\pi}(t)) \\ (2.11) \qquad &\qquad + \int_{\mathbb{R}_0} \gamma\gamma^T(t, X^{(\theta)}(t), \theta(t), \hat{\pi}(t), z)\nu(dz)\big\}p(t)dt\Big] < \infty, \end{aligned}$$

$$\begin{aligned} &E\Big[\int_0^T \hat{p}(t)^T \big\{\sigma\sigma^T(t, X(t), \hat{\theta}(t), \pi(t)) \\ (2.12) \qquad &\qquad + \int_{\mathbb{R}_0} \gamma\gamma^T(t, X^{(\pi)}(t), \theta(t), \hat{\pi}(t), z)\nu(dz)\big\}p(t)dt\Big] < \infty, \end{aligned}$$

Moreover, suppose that for all $t \in [0,T]$, the following partial information maximum principle *satisfies:*

$$\begin{aligned} &\inf_{\theta\in\Theta} E[H(t, X(t), \theta(t), \hat{\pi}(t), \hat{p}(t), \hat{q}(t), \hat{r}(t,.)) \mid \mathcal{E}_t] \\ &\qquad = E[H(t, X(t), \hat{\theta}(t), \hat{\pi}(t), \hat{p}(t), \hat{q}(t), \hat{r}(t,.)) \mid \mathcal{E}_t] \\ (2.13) \qquad &\qquad = \sup_{\pi\in\Pi} E[H(t, X(t), \hat{\theta}(t), \pi(t), \hat{p}(t), \hat{q}(t), \hat{r}(t,.)) \mid \mathcal{E}_t] \end{aligned}$$

i) *For all* $t \in [0,T]$, $g(x)$ *is concave and* $H(t,x,\theta,\pi,\hat{p}(t),\hat{q}(t),\hat{r}(t,.))$ *is concave for all* $\theta = \hat{\theta}$. *Then*

$$J(\hat{\theta},\hat{\pi}) \geq J(\hat{\theta},\pi) \qquad \text{for all } \pi \in \Pi$$

and

$$J(\hat{\theta},\hat{\pi}) = \sup_{\pi\in\Pi} J(\hat{\theta},\pi)$$

ii) *For all* $t \in [0,T]$, $g(x)$ *is convex and* $H(t,x,\theta,\pi,\hat{p}(t),\hat{q}(t),\hat{r}(t,.))$ *is convex for all* $\pi = \hat{\pi}$. *Then*

$$J(\hat{\theta},\hat{\pi}) \leq J(\theta,\hat{\pi}) \qquad \text{for all } \theta \in \Theta$$

and

$$J(\hat{\theta},\hat{\pi}) = \inf_{\theta\in\Theta} J(\theta,\hat{\pi})$$

iii) *If both cases (i) and (ii) hold then* $(\theta^*,\pi^*) := (\hat{\theta},\hat{\pi})$ *is an optimal control and*

$$\Phi_{\mathcal{E}}(x) = \sup_{\pi\in\Pi}\Big(\inf_{\theta\in\Theta} J(\theta,\pi)\Big) = \inf_{\theta\in\Theta}\Big(\sup_{\pi\in\Pi} J(\theta,\pi)\Big) \tag{2.14}$$

Proof. i) Suppose (i) holds. Choose $(\theta,\pi) \in \Theta\times\Pi$. Let us consider

$$J(\hat{\theta},\hat{\pi}) - J(\hat{\theta},\pi) = I_1 + I_2$$

where

$$I_1 = E\Big[\int_0^T \{f(t,\hat{X}(t),\hat{\theta}(t),\hat{\pi}(t)) - f(t,X^{(\pi)}(t),\hat{\theta}(t),\pi(t))\}dt\Big] \tag{2.15}$$

and

$$I_2 = E\Big[g(\hat{X}(T)) - g(X^{(\pi)}(T))\Big] \tag{2.16}$$

Since g is concave in x and from integration by parts formula for jump processes we get the following, where the L^2 conditions (2.9) and (2.12) ensure that the stochastic integrals with respect to the local martingales have zero expectation:

$$\begin{aligned}
I_2 &= E\Big[g(\hat{X}(T)) - g(X^{(\pi)}(T))\Big] \\
&\geq E\Big[(\hat{X}(T) - X^{(\pi)}(T))^T \bigtriangledown g(\hat{X}(T))\Big] \\
&= E\Big[(X^{(\hat{\theta},\hat{\pi})}(T) - X^{(\hat{\theta},\pi)}(T))^T \hat{p}(T)\Big] \\
&= E\Big[\int_0^T (\hat{X}(t^-) - X^{(\pi)}(t^-))^T d\hat{p}(t) + \int_0^T \hat{p}^T(t)(d\hat{X}(t) - dX^{(\pi)}(t)) \\
&\quad + \int_0^T tr\Big[\{\sigma(t,\hat{X}(t),\hat{\theta}(t),\hat{\pi}(t)) - \sigma(t,X^{(\pi)}(t),\hat{\theta}(t),\pi(t))\}^T \hat{q}(t)\Big]dt
\end{aligned}$$

$$
\begin{aligned}
&+\int_0^T \sum_{i,j=1}^n \int_{\mathbb{R}_0} \{\gamma_{ij}(t,\hat{X}(t),\hat{\theta}(t),\hat{\pi}(t),z) \\
&\qquad\qquad - \gamma_{ij}(t,X^{(\pi)}(t),\hat{\theta}(t),\pi(t),z)\}\hat{r}_{ij}(t,z)\nu(dz_j)dt\Big] \\
&= E\Big[\int_0^T (\hat{X}(t)-X^{(\pi)}(t))^T(-\bigtriangledown_x H(t,\hat{X}(t),\hat{\theta}(t),\hat{\pi}(t),\hat{p}(t),\hat{q}(t),\hat{r}(t,.)))dt \\
&+\int_0^T \hat{p}^T(t)\big\{b(t,\hat{X}(t),\hat{\theta}(t),\hat{\pi}(t)) - b(t,X^{(\pi)}(t),\hat{\theta}(t),\pi(t))\big\}dt \\
&+\int_0^T tr\Big[\{\sigma(t,\hat{X}(t),\hat{\theta}(t),\hat{\pi}(t)) - \sigma(t,X^{(\pi)}(t),\hat{\theta}(t),\pi(t))\}^T\hat{q}(t)\Big]dt \\
&+\int_0^T \sum_{i,j=1}^n \int_{\mathbb{R}_0} \{\gamma_{ij}(t,\hat{X}(t),\hat{\theta}(t),\hat{\pi}(t),z) \\
&\qquad\qquad - \gamma_{ij}(t,X^{(\pi)}(t),\hat{\theta}(t),\pi(t),z)\}\hat{r}_{ij}(t,z)\nu(dz_j)dt\Big]
\end{aligned}
\tag{2.17}
$$

By the definition (2.7) of H we have

$$
\begin{aligned}
I_1 &= E\Big[\int_0^T \{f(t,\hat{X}(t),\hat{\theta}(t),\hat{\pi}(t)) - f(t,X^{(\pi)}(t),\hat{\theta}(t),\pi(t))\}dt\Big] \\
&= E\Big[\int_0^T \{H(t,\hat{X}(t),\hat{\theta}(t),\hat{\pi}(t),\hat{p}(t),\hat{q}(t),\hat{r}(t,.)) \\
&\qquad\qquad - H(t,X^{(\pi)}(t),\hat{\theta}(t),\pi(t),\hat{p}(t),\hat{q}(t),\hat{r}(t,.))\}dt\Big] \\
&- E\Big[\int_0^T \hat{p}^T(t)\big\{b(t,\hat{X}(t),\hat{\theta}(t),\hat{\pi}(t)) - b(t,X^{(\pi)}(t),\hat{\theta}(t),\pi(t))\big\}dt\Big] \\
&- E\Big[\int_0^T tr\big[\{\sigma(t,\hat{X}(t),\hat{\theta}(t),\hat{\pi}(t)) - \sigma(t,X^{(\pi)}(t),\hat{\theta}(t),\pi(t))\}^T\hat{q}(t)\big]dt\Big] \\
&- E\Big[\int_0^T \sum_{i,j=1}^n \int_{\mathbb{R}_0} \{\gamma_{ij}(t,\hat{X}(t),\hat{\theta}(t),\hat{\pi}(t),z) \\
&\qquad\qquad - \gamma_{ij}(t,X^{(\pi)}(t),\hat{\theta}(t),\pi(t),z)\}\hat{r}_{ij}(t,z)\nu(dz_j)dt\Big]
\end{aligned}
\tag{2.18}
$$

By concavity of H in x and π we have

$$
\begin{aligned}
H(t,\hat{X}(t),\hat{\theta}(t),&\hat{\pi}(t),\hat{p}(t),\hat{q}(t),\hat{r}(t,.)) - H(t,X^{(\pi)}(t),\hat{\theta}(t),\pi(t),\hat{p}(t),\hat{q}(t),\hat{r}(t,.)) \\
&\geq \bigtriangledown_x H(t,\hat{X}(t),\hat{\theta}(t),\hat{\pi}(t),\hat{p}(t),\hat{q}(t),\hat{r}(t,.))^T(\hat{X}(t)-X^{(\pi)}(t)) \\
&\quad + \bigtriangledown_\pi H(t,\hat{X}(t),\hat{\theta}(t),\hat{\pi}(t),\hat{p}(t),\hat{q}(t),\hat{r}(t,.))^T(\hat{\pi}(t)-\pi(t))
\end{aligned}
\tag{2.19}
$$

Since $E[H(t, X^\pi(t), \hat{\theta}(t), \pi(t), \hat{p}(t), \hat{q}(t), \hat{r}(t,.)) \mid \mathcal{E}_t]$ is maximum for $\pi = \hat{\pi}(t)$ and $\pi(t), \hat{\pi}(t)$ are $\mathcal{E}_t$-measurable, we get

$$\begin{aligned} &E\big[\bigtriangledown_\pi H(t, \hat{X}(t), \hat{\theta}(t), \hat{\pi}(t), \hat{p}(t), \hat{q}(t), \hat{r}(t,.))^T (\hat{\pi}(t) - \pi(t)) \mid \mathcal{E}_t\big] \\ &= \bigtriangledown_\pi E\big[H(t, X^\pi(t), \hat{\theta}(t), \pi(t), \hat{p}(t), \hat{q}(t), \hat{r}(t,.)) \mid \mathcal{E}_t\big]^T_{\pi=\hat{\pi}(t)} (\hat{\pi}(t) - \pi(t)) \geq 0 \end{aligned} \tag{2.20}$$

Combining (2.19) and (2.20) we obtain

$$\begin{aligned} &E\Big[\int_0^T \{H(t, \hat{X}(t), \hat{\theta}(t), \hat{\pi}(t), \hat{p}(t), \hat{q}(t), \hat{r}(t,.)) \\ &\qquad - H(t, X^{(\pi)}(t), \hat{\theta}(t), \pi(t), \hat{p}(t), \hat{q}(t), \hat{r}(t,.))\} dt\Big] \\ &\geq E\Big[\int_0^T \bigtriangledown_x H(t, \hat{X}(t), \hat{\theta}(t), \hat{\pi}(t), \hat{p}(t), \hat{q}(t), \hat{r}(t,.))^T (\hat{X}(t) - X^{(\pi)}(t))\Big] \end{aligned} \tag{2.21}$$

Hence

$$\begin{aligned} I_1 \geq &E\Big[\int_0^T \bigtriangledown_x H(t, \hat{X}(t), \hat{\theta}(t), \hat{\pi}(t), \hat{p}(t), \hat{q}(t), \hat{r}(t,.))^T (\hat{X}(t) - X^{(\pi)}(t))\Big] \\ &- E\Big[\int_0^T \hat{p}^T(t) \big\{b(t, \hat{X}(t), \hat{\theta}(t), \hat{\pi}(t)) - b(t, X^{(\pi)}(t), \hat{\theta}(t), \pi(t))\big\} dt\Big] \\ &- E\Big[\int_0^T tr\big[\{\sigma(t, \hat{X}(t), \hat{\theta}(t), \hat{\pi}(t)) - \sigma(t, X^{(\pi)}(t), \hat{\theta}(t), \pi(t))\}^T \hat{q}(t)\big] dt\Big] \\ &- E\Big[\int_0^T \sum_{i,j=1}^n \int_{\mathbb{R}_0} \{\gamma_{ij}(t, \hat{X}(t), \hat{\theta}(t), \hat{\pi}(t), z) \\ &\qquad - \gamma_{ij}(t, X^{(\pi)}(t), \hat{\theta}(t), \pi(t), z)\} \hat{r}_{ij}(t,z) \nu(dz_j) dt\Big] \end{aligned} \tag{2.22}$$

Adding (2.17), (2.22) above, we get

$$J(\hat{\theta}, \hat{\pi}) - J(\hat{\theta}, \pi) = I_1 + I_2 \geq 0 \tag{2.23}$$

We therefore conclude that $J(\hat{\theta}, \hat{\pi}) \geq J(\hat{\theta}, \pi)$ for all $\pi \in \Pi$.

ii) Prove in the same way as in (i) we can show that $J(\hat{\theta}, \hat{\pi}) \leq J(\theta, \hat{\pi})$ for all $\theta \in \Theta$ if (ii) holds.

ii) If both (i) and (ii) hold then

$$J(\hat{\theta}, \pi) \leq J(\hat{\theta}, \hat{\pi}) \leq J(\theta, \hat{\pi})$$

for any $(\theta, \pi) \in \Theta \times \Pi$. Thereby

$$J(\hat{\theta}, \hat{\pi}) \leq \inf_{\theta \in \Theta} J(\theta, \hat{\pi}) \leq \sup_{\pi \in \Pi} \big(\inf_{\theta \in \Theta} J(\theta, \pi)\big)$$

On the other hand

$$J(\hat{\theta}, \hat{\pi}) \geq \sup_{\pi \in \Pi} J(\hat{\theta}, \pi) \geq \inf_{\theta \in \Theta} \big(\sup_{\pi \in \Pi} J(\theta, \pi)\big)$$

Now due to the inequality

$$\inf_{\theta\in\Theta}\big(\sup_{\pi\in\Pi} J(\theta,\pi)\big) \geq \sup_{\pi\in\Pi}\big(\inf_{\theta\in\Theta} J(\theta,\pi)\big)$$

we have

$$\Phi_{\mathcal{E}}(x) = \sup_{\pi\in\Pi}\big(\inf_{\theta\in\Theta} J(\theta,\pi)\big) = \inf_{\theta\in\Theta}\big(\sup_{\pi\in\Pi} J(\theta,\pi)\big)$$

□

If the control process (θ,π) is admissible adapted to the filtration $\mathcal{F}_t$ we have the following Corollary

Corollary 2. *Suppose $\mathcal{E}_t = \mathcal{F}_t$. With assumptions as in Theorem 1 except the partial information maximum condition is exchanged to*

$$\begin{aligned}
&\inf_{\theta\in\Theta} H(t,X(t),\theta(t),\hat{\pi}(t),\hat{p}(t),\hat{q}(t),\hat{r}(t,.))\\
&\qquad = H(t,X(t),\hat{\theta}(t),\hat{\pi}(t),\hat{p}(t),\hat{q}(t),\hat{r}(t,.))\\
&\qquad = \sup_{\pi\in\Pi} H(t,X(t),\hat{\theta}(t),\pi(t),\hat{p}(t),\hat{q}(t),\hat{r}(t,.)) \qquad (2.24)
\end{aligned}$$

For all $t\in[0,T]$ we have

i) *If $g(x)$ is concave and $H(t,x,\theta,\pi,\hat{p}(t),\hat{q}(t),\hat{r}(t,.))$ is concave for all $\theta=\hat{\theta}$ then*

$$J(\hat{\theta},\hat{\pi}) \geq J(\hat{\theta},\pi) \qquad \text{for all } \pi\in\Pi$$

and

$$J(\hat{\theta},\hat{\pi}) = \sup_{\pi\in\Pi} J(\hat{\theta},\pi)$$

ii) *If $g(x)$ is convex and $H(t,x,\theta,\pi,\hat{p}(t),\hat{q}(t),\hat{r}(t,.))$ is convex for all $\pi=\hat{\pi}$ then*

$$J(\hat{\theta},\hat{\pi}) \leq J(\theta,\hat{\pi}) \qquad \text{for all } \theta\in\Theta$$

and

$$J(\hat{\theta},\hat{\pi}) = \inf_{\theta\in\Theta} J(\theta,\hat{\pi})$$

iii) *If both cases (i) and (ii) hold, then $(\theta^*,\pi^*) := (\hat{\theta},\hat{\pi})$ is an optimal control based on the information flow $\mathcal{F}_t$ and*

$$\Phi_{\mathcal{E}}(x) = \sup_{\pi\in\Pi}\Big(\inf_{\theta\in\Theta} J(\theta,\pi)\Big) = \inf_{\theta\in\Theta}\Big(\sup_{\pi\in\Pi} J(\theta,\pi)\Big) \qquad (2.25)$$

3. A necessary maximum principle for Zero-sum games

In addition to the assumptions in Section 2 we are now assume the following:

(A1) For all t, h such that $0 \leq t < t+h \leq T$, all bounded $\mathcal{E}_t$-measurable α, ρ, and for $s \in [0,T]$ the controls $\beta(s) := (0, ..., \beta_i(s), ..., 0)$ and $\eta(s) := (0, ..., \eta_i(s), ..., 0), i = 1, ..., n$ with

$$\beta_i(s) := \alpha_i \chi_{[t,t+h]}(s), \qquad \text{and} \qquad \eta_i(s) := \rho_i \chi_{[t,t+h]}(s)$$

belong to Θ and Π, respectively.

(A2) For given $\theta, \beta \in \Theta$ and $\pi, \eta \in \Pi$ with β, η are bounded, there exists $\delta > 0$ such that

$$\theta + y\beta \in \Theta, \qquad \text{and} \qquad \pi + v\eta \in \Pi$$

where $y, v \in (-\delta, \delta)$.

Denote $X^{\theta+y\beta}(t) = X^{(\theta+y\beta,\pi)}(t)$ and $X^{\pi+v\eta}(t) = X^{(\theta,\pi+v\eta)}$. For a given $\theta, \beta \in \Theta$ and $\pi, \eta \in \Pi$ with β, η bounded, we define the processes $Y^\theta(t)$ and $Y^\pi(t)$ by

$$Y^\theta(t) = \frac{d}{dy} X^{\theta+y\beta}(t) \mid_{y=0} = (Y_1^\theta(t), ..., Y_n^\theta(t))^T \tag{3.1}$$

$$Y^\pi(t) = \frac{d}{dv} X^{\pi+v\eta}(t) \mid_{v=0} = (Y_1^\pi(t), ..., Y_n^\pi(t))^T \tag{3.2}$$

We have that

$$dY_i^\theta(t) = \lambda_i^\theta(t)dt + \sum_{j=1}^n \xi_{ij}^\theta(t)dB_j(t) + \sum_{j=1}^n \int_{\mathbb{R}} \zeta_{ij}^\theta(t,z)\widetilde{N}_j(dt,dz), \tag{3.3}$$

and

$$dY_i^\pi(t) = \lambda_i^\pi(t)dt + \sum_{j=1}^n \xi_{ij}^\pi(t)dB_j(t) + \sum_{j=1}^n \int_{\mathbb{R}} \zeta_{ij}^\pi(t,z)\widetilde{N}_j(dt,dz), \tag{3.4}$$

where

$$\begin{cases} \lambda_i^\theta(t) &= \bigtriangledown_x b_i(t, X(t), \theta(t), \pi(t))^T Y^\theta(t) \\ & \qquad + \bigtriangledown_\theta\, b_i(t, X(t), \theta(t), \pi(t))^T \beta(t) \\ \xi_{ij}^\theta(t) &= \bigtriangledown_x \sigma_{ij}(t, X(t), \theta(t), \pi(t))^T Y^\theta(t) \\ & \qquad + \bigtriangledown_\theta\, \sigma ij(t, X(t), \theta(t), \pi(t))^T \beta(t) \\ \zeta_{ij}^\theta(t) &= \bigtriangledown_x \zeta_{ij}(t, X(t), \theta(t), \pi(t))^T Y^\theta(t) \\ & \qquad + \bigtriangledown_\theta\, \zeta_{ij}(t, X(t), \theta(t), \pi(t))^T \beta(t) \end{cases} \tag{3.5}$$

and

$$\begin{cases} \lambda_i^\pi(t) &= \bigtriangledown_x b_i(t, X(t), \theta(t), \pi(t))^T Y^\pi(t) \\ & \qquad + \bigtriangledown_\pi\, b_i(t, X(t), \theta(t), \pi(t))^T \eta(t) \\ \xi_{ij}^\pi(t) &= \bigtriangledown_x \sigma_{ij}(t, X(t), \theta(t), \pi(t))^T Y^\pi(t) \\ & \qquad + \bigtriangledown_\pi\, \sigma ij(t, X(t), \theta(t), \pi(t))^T \eta(t) \\ \zeta_{ij}^\pi(t) &= \bigtriangledown_x \zeta_{ij}(t, X(t), \theta(t), \pi(t))^T Y^\pi(t) \\ & \qquad + \bigtriangledown_\pi\, \zeta_{ij}(t, X(t), \theta(t), \pi(t))^T \eta(t) \end{cases} \tag{3.6}$$

Theorem 3. *Suppose $\hat{\theta} \in \Theta$ and $\hat{\pi} \in \Pi$ are respectively local minimum and maximum for $J(\theta, \pi)$, in the sense that for all bounded $\beta \in \Theta$ and $\eta \in \Pi$ there exist $\delta > 0$ such that $\hat{\theta} + y\beta \in \Theta$ and $\hat{\pi} + v\eta \in \Pi$ for all $y \in (-\delta, \delta)$ and $v \in (-\delta, \delta)$ and*

$$h(y, v) := J(\hat{\theta} + y\beta, \hat{\pi} + v\eta) \qquad y, v \in (-\delta, \delta)$$

is minimum at $y = 0$ and maximum at $v = 0$.

Suppose there exists a solution $\hat{p}(t), \hat{q}(t), \hat{r}(t, .)$ of the associated adjoint equation

$$\text{(3.7)} \quad \begin{cases} d\hat{p}(t) &= -\nabla_x H(t, \hat{X}(t), \hat{\theta}(t), \hat{\pi}(t), \hat{p}(t), \hat{q}(t), \hat{r}(t, .))\, dt \\ & \quad +\hat{q}(t)\, dB(t) + \int_{\mathbb{R}^n} \hat{r}(t^-, z)\widetilde{N}(\, dt, \, dz), \qquad t < T \\ \hat{p}(T) &= \nabla g(\hat{X}(T)) \end{cases}$$

Moreover suppose that with $Y^{\hat{\theta}}, Y^{\hat{\pi}}$ and $(\lambda_i^{\hat{\theta}}, \xi_{ij}^{\hat{\theta}}, \zeta_{ij}^{\hat{\theta}})$ and $(\lambda_i^{\hat{\pi}}, \xi_{ij}^{\hat{\pi}}, \zeta_{ij}^{\hat{\pi}})$ are the corresponding coefficients (see (3.5)-(3.6))

$$\text{(3.8)} \qquad E\Big[Y^{\hat{\theta}^T}(t)\Big\{\hat{q}\hat{q}^T + \int_{\mathbb{R}} \hat{r}\hat{r}^T(t, z)\nu(dz)\Big\}Y^{\hat{\theta}}(t)dt\Big] < \infty,$$

$$\text{(3.9)} \qquad E\Big[Y^{\hat{\pi}^T}\Big\{\hat{q}\hat{q}^T + \int_{\mathbb{R}} \hat{r}\hat{r}^T(t, z)\nu(dz)\Big\}Y^{\hat{\pi}}(t)dt\Big] < \infty,$$

and

$$\text{(3.10)} \qquad E\Big[\int_0^T \hat{p}^T(t)\Big\{\xi^{\hat{\theta}}\xi^{\hat{\theta}^T}(t, \hat{X}(t), \hat{\theta}(t), \hat{\pi}(t)) + \int_{\mathbb{R}} \gamma\gamma^T(t, \hat{X}(t), \hat{\theta}(t), \hat{\pi}(t))\nu(dz)\Big\}\hat{p}(t)dt\Big] < \infty,$$

$$\text{(3.11)} \qquad E\Big[\int_0^T \hat{p}^T(t)\Big\{\xi^{\hat{\pi}}\xi^{\hat{\pi}^T}(t, \hat{X}(t), \hat{\theta}(t), \hat{\pi}(t)) + \int_{\mathbb{R}} \gamma\gamma^T(t, \hat{X}(t), \hat{\theta}(t), \hat{\pi}(t))\nu(dz)\Big\}\hat{p}(t)dt\Big] < \infty,$$

Then for a.a. $t \in [0, T]$, we have

$$\text{(3.12)} \qquad \begin{aligned} & E[\nabla_\theta H(t, \hat{X}(t), \hat{\theta}(t), \hat{\pi}(t), \hat{p}(t), \hat{q}(t), \hat{r}(t, .)) \mid \mathcal{E}_t] \\ & = E[\nabla_\pi H(t, \hat{X}(t), \hat{\theta}(t), \hat{\pi}(t), \hat{p}(t), \hat{q}(t), \hat{r}(t, .)) \mid \mathcal{E}_t] = 0 \end{aligned}$$

Proof. Since h is minimum at $y = 0$ we have

$$\begin{aligned} 0 = \frac{\partial}{\partial y} h(y, 0) \mid_{y=0} &= E\Big[\int_0^T \nabla_x f(t, \hat{X}(t), \hat{\theta}(t), \hat{\pi}(t))^T \frac{d}{dy} X^{\hat{\theta}+y\beta}(t) \mid_{y=0} dt \\ & \quad + \int_0^T \nabla_\theta f(t, \hat{X}(t), \hat{\theta}(t)\hat{\pi}(t))^T \beta(t) dt + \nabla g(\hat{X}(T)^T \frac{d}{dy} X^{\hat{\theta}+y\beta}(t) \mid_{y=0} \Big] \\ & = E\Big[\int_0^T \nabla_x f(t, \hat{X}(t), \hat{\theta}(t)\hat{\pi}(t))^T Y^{\hat{\theta}}(t) dt \end{aligned}$$

$$+\int_0^T \bigtriangledown_\theta f(t,\hat{X}(t),\hat{\theta}(t)\hat{\pi}(t))^T\beta(t)dt + \bigtriangledown g(\hat{X}(T)^T Y^{\hat{\theta}}(T)\Big] \tag{3.13}$$

By Itô formula,

$$\begin{aligned}
E[\bigtriangledown g(\hat{X}(T)^T Y^{\hat{\theta}}(T)] &= E[\hat{p}^T(T)Y^{\hat{\theta}}(T)] \\
&= E\Big[\sum_{i=1}^n \int_0^T \Big\{\hat{p}_i(t)(\bigtriangledown_x b_i(t,\hat{X}(t),\hat{\theta}(t),\hat{\pi}(t))^T Y^{\hat{\theta}}(T) \\
&\qquad + \bigtriangledown_\theta b_i(t,\hat{X}(t),\hat{\theta}(t),\hat{\pi}(t))^T\beta(t)) \\
&\quad + Y_i^{\hat{\theta}}(t)(-\bigtriangledown_x H(t,\hat{X}(t),\hat{\theta}(t),\hat{\pi}(t),\hat{p}(t),\hat{q}(t),\hat{r}(t,.))_i \\
&\quad + \sum_{j=1}^n \hat{q}_{ij}(t)(\bigtriangledown_x \sigma_{ij}(t,\hat{X}(t),\hat{\theta}(t),\hat{\pi}(t))^T Y^{\hat{\theta}}(t) \\
&\qquad + \bigtriangledown_\theta \sigma_{ij}(t,\hat{X}(t),\hat{\theta}(t),\hat{\pi}(t))^T\beta(t)) \\
&\quad + \sum_{j=1}^n \int_{\mathbb{R}^n} \hat{r}_{ij}(t^-,z)(\bigtriangledown_x \gamma_{ij}(t,\hat{X}(t),\hat{\theta}(t),\hat{\pi}(t))^T Y^{\hat{\theta}}(T) \\
&\qquad + \bigtriangledown_\theta \gamma_{ij}(t,\hat{X}(t),\hat{\theta}(t),\hat{\pi}(t))^T\beta(t))\Big\}dt\Big]
\end{aligned} \tag{3.14}$$

On the other hand,

$$\begin{aligned}
\bigtriangledown_x H(t,x,\theta,\pi,p,q,r) &= \bigtriangledown_x f(t,x,\theta,\pi) + \sum_{i=1}^n \bigtriangledown_x b_i(t,x,\theta,\pi)p_i \\
&+ \sum_{j,i=1}^n \bigtriangledown_x \sigma_{ji}(t,x,\theta,\pi)q_{ji} + \sum_{j,i=1}^n \int_{\mathbb{R}} \bigtriangledown_x \gamma_{ji}(t,x,\theta,\pi)r_{ji}(t,z)\nu_j(dz)
\end{aligned} \tag{3.15}$$

Substituting this into (3.14) and combining to (3.13) we get,

$$\begin{aligned}
0 &= E\Big[\int_0^T \sum_{i=1}^n \Big\{\frac{\partial f}{\partial\theta_i}(t,\hat{X}(t),\hat{\theta}(t),\hat{\pi}(t)) \\
&+ \sum_{j=1}^n \big(\hat{p}_j(t)\frac{\partial b_j}{\partial\theta_i}(t,\hat{X}(t),\hat{\theta}(t),\hat{\pi}(t)) + \sum_{k=1}^n[\hat{q}_{kj}(t)\frac{\partial\sigma_{kj}}{\partial\theta_i}(t,\hat{X}(t),\hat{\theta}(t),\hat{\pi}(t)) \\
&+ \int_{\mathbb{R}} \hat{r}_{kj}(t,z)\frac{\partial\gamma_{kj}}{\partial\theta_i}(t,\hat{X}(t),\hat{\theta}(t),\hat{\pi}(t))\nu_j(dz)]\big)\beta_i(t)\Big\}dt\Big] \\
&= E\Big[\int_0^T \bigtriangledown_\theta H(t,\hat{X}(t),\hat{\theta}(t),\hat{\pi}(t),\hat{p}(t),\hat{q}(t),\hat{r}(t,.))^T\beta(t)dt\Big]
\end{aligned} \tag{3.16}$$

Since assumption **(A1)**, the equation (3.16) leads to

$$E\Big[\int_t^{t+h} \frac{\partial}{\partial\theta_i}H(s,\hat{X}(s),\hat{\theta}(s),\hat{\pi}(s),\hat{p}(s),\hat{q}(s),\hat{r}(s,.))\alpha_i(s)ds\Big] = 0$$

Differentiating with respect to h at $h = 0$ gives

$$E\Big[\frac{\partial}{\partial \theta_i} H(s, \hat{X}(s), \hat{\theta}(s), \hat{\pi}(s), \hat{p}(s), \hat{q}(s), \hat{r}(s,.))\alpha_i(s)\Big] = 0$$

Since this holds for all bounded $\mathcal{E}_t$-measurable α_i, we conclude that

$$E\Big[\frac{\partial}{\partial \theta_i} H(s, \hat{X}(s), \hat{\theta}(s), \hat{\pi}(s), \hat{p}(s), \hat{q}(s), \hat{r}(s,.)) \mid \mathcal{E}_t\Big] = 0$$

as claimed.

Prove in the same way by differentiating the function $h(0, v)$ with respect to v we get

$$E\Big[\frac{\partial}{\partial \pi_i} H(s, \hat{X}(s), \hat{\theta}(s), \hat{\pi}(s), \hat{p}(s), \hat{q}(s), \hat{r}(s,.)) \mid \mathcal{E}_t\Big] = 0$$

This is end of proof. □

4. Applications to Finance

In this section, we will use our result to solve the problem that is given in [7].

Consider the following jump market

$$\text{(risky free asset)} \quad dS_0(t) = \rho(t)S_0(t)dt; \qquad S_0(0) = 1 \tag{4.1}$$

$$\text{(risky asset)} \quad dS_1(t) = S_1(t^-)\Big[\alpha(t)dt + \beta(t)dB(t) + \int_{\mathcal{R}} \gamma(t,z)\widetilde{N}(dt,dz)\Big]; \qquad S_1(0) > 0 \tag{4.2}$$

where $\rho(t)$ is a deterministic function, $\alpha(t), \beta(t)$ and $\gamma(t,z)$ are given functions and satisfying the following integrability condition:

$$E\Big[\int_0^T \{ \mid \rho(s) \mid + \mid \alpha(s) \mid + \frac{1}{2}\beta(s)^2 + \int_{\mathbb{R}} \mid \log(1+\gamma(s,z)) - \gamma(s,z) \mid \nu(dz)\}ds\Big] < \infty \tag{4.3}$$

where T is fixed. We assume that

$$\gamma(t,z) \geq 1 \qquad \text{for a.a. } t, z \in [0,T] \times \mathbb{R}_0 \tag{4.4}$$

where $\mathbb{R}_0 = \mathbb{R}\backslash\{0\}$.

Let $\mathcal{E}_t \subseteq \mathcal{F}_t$ be a given sub-filtration. Let $\pi(t)$ be a portfolio, which is $\mathcal{E}_t$-measurable random variable and represented by the *fraction* of the wealth invested in the risky asset at time t. The the dynamics of the corresponding wealth process $X^{(\pi)}(t)$ is

$$\begin{aligned} dX^{(\pi)}(t) &= X^{(\pi)}(t^-)\Big[\{\rho(t) + (\alpha(t) - \rho(t))\pi(t)\}dt \\ &\quad + \pi(t)\beta(t)dB(t) + \pi(t^-)\int_{\mathbb{R}} \gamma(t,z)\widetilde{N}(dt,dz)\Big] \qquad (4.5) \\ X^{(\pi)}(0) &= x > 0. \end{aligned}$$

(4.6)

A portfolios π is called *admissible* if π is measurable càdlàg stochastic process adapted to filtration $\mathcal{E}_t$ and satisfies

$$\pi(t^-)\gamma(t,z) > -1 \qquad \text{a.s.}$$

and

$$\int_0^T \{ \mid \rho(t) + (\alpha(t) - \rho(t))\pi(t) \mid + \pi^2(t)\beta^2(t) + \pi^2(t) \int_{\mathbb{R}} \gamma^2(t,z)\nu(dz)\} dt < \infty \qquad \text{a.s.} \tag{4.7}$$

Then the final wealth of the admissible portfolio π is the solution of (4.5):

$$\begin{aligned} X^{(\pi)}(t) =& x \exp\Big[\int_0^t \{\rho(t) + (\alpha(t) - \rho(t))\pi(t) - \frac{1}{2}\pi^2(t)\beta^2(t) \\ &+ \int_{\mathbb{R}} (\ln(1 + \pi(s)\gamma(s,z)) - \pi(z)\gamma(s,z))\nu(dz)\} ds \\ &+ \int_0^t \pi(s)\beta(s)dB(s) + \int_0^t \int_{\mathbb{R}} \ln(1 + \pi(s)\gamma(s,z))\widetilde{N}(ds,dz)\Big] \end{aligned} \tag{4.8}$$

The family of admissible portfolios is denoted by Π.

Now we introduce a family $\mathcal{Q}$ of measures Q_θ parameterized by processes $\theta = (\theta_0(t), \theta_1(t,z))$ such that

$$dQ_\theta(\omega) = Z_\theta(T)dP(\omega) \qquad \text{on } \mathcal{F}_T, \tag{4.9}$$

where

$$\begin{cases} dZ_\theta(t) &= Z_\theta(t^-)[-\theta_0(t)dB(t) - \int_{\mathbb{R}} \theta_1(t,z)\widetilde{N}(dt,dz)] \\ Z_\theta(0) &= 1 \end{cases} \tag{4.10}$$

We assume that $\theta_1(t,z) \leq 1$ for a.a. t,z and

$$\int_0^T \{\theta_0^2(s) + \int_{\mathbb{R}} \theta_1^2(s,z)\} ds < \infty \qquad a.s. \tag{4.11}$$

Then by Itô formula the solution of (4.10) is given by

$$\begin{aligned} Z_\theta(t) = \exp\Big[&- \int_0^t \theta_0(s)dB(s) - \frac{1}{2}\int_0^t \theta_0^2(s)ds \\ &+ \int_0^t \int_{\mathcal{R}} \log(1 - \theta_1(s,z))\widetilde{N}(dt,dz) \\ &+ \int_0^t \int_{\mathcal{R}} \{\log(1 - \theta_1(s,z)) + \theta_1(s,z)\}\nu(dz)ds\Big] \end{aligned} \tag{4.12}$$

If $\theta = (\theta_0(t), \theta_1(t,z))$ satisfy

$$E[Z_\theta(T)] = 1 \tag{4.13}$$

then Q_θ is a probability measure.

In addition, if

$$\beta(t)\theta_0(t) + \int_{\mathbb{R}} \gamma(t,z)\theta_1(t,z)\nu(dz) = \alpha(t) - r(t); \qquad t \in [0,T] \tag{4.14}$$

then $dQ_\theta(\omega) = Z_\theta(T)dP(\omega)$ is an *equivalent local martingale measure*. See e.g. [6], Ch.1. But here we will not assume (4.14) holds.

For all $\theta = (\theta_0, \theta_1)$ adapted to sub-filtration $\mathcal{E}_t$ and satisfies (4.11)-(4.13) is called *admissible controls of the market*. The families of admissible controls θ is denoted by Θ.

The problem is to find $(\theta, \pi) \in \Theta \times \Pi$ such that

$$\inf_{\theta\in\Theta} \big(\sup_{\pi\in\Pi} E_{Q_\theta}[U(X^\pi(T))] \big) = E_{Q_{\theta^*}}[U(X^{\pi^*}(T))] \tag{4.15}$$

where $U : [0,\infty) \to [-\infty, \infty)$ is utility function, which is increasing, concave and twice continuously differentiable on $(0,\infty)$.

We can consider this problem as our stochastic differential game between the *agent* and the *market*. The agent wants to maximize her expected discounted utility over all portfolios π and the market wants to minimize the maximal expected utility of the representative agent over all probability measures Q_θ.

Putting

$$\begin{aligned} dY(t) &= \begin{bmatrix} dY_1(t) \\ dY_2(t) \end{bmatrix} = \begin{bmatrix} dZ_\theta(t) \\ dX(t) \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ X^{(\pi)}(t^-)\{\rho(t) + (\alpha(t) - \rho(t))\pi\} \end{bmatrix} dt \\ &+ \begin{bmatrix} -Z_\theta(t^-)\theta_0(t) \\ X^{(\pi)}(t^-)\beta(t)\pi(t) \end{bmatrix} dB(t) + \begin{bmatrix} -Z_\theta(t^-)\int_{\mathbb{R}}\theta_1(t,z) \\ X^{(\pi)}(t^-)\pi(t)\int_{\mathbb{R}}\gamma(t,z) \end{bmatrix} \widetilde{N}(dt,dz) \end{aligned} \tag{4.16}$$

To solve this problem we first write down the Hamiltonian function

$$\begin{aligned} H(t,y_1,y_2,\theta,\pi,p,q,r) &= y_2\{\rho(t) + (\alpha(t)-\rho(t))\pi(t)\}p_2 - y_1\theta_0 q_1 \\ &+ y_2\beta(t)\pi(t)q_2 + \int_{\mathbb{R}} \{-y_1\theta_1(t,z)r_1(t,z) + y_2\pi(t)\gamma(t,z)r_2(t,z)\}\nu(dz) \end{aligned} \tag{4.17}$$

and adjoint equations is

$$\begin{cases} dp_1(t) = -\big(\theta_0(t)q_1(t) + \int_{\mathbb{R}}\theta_1(t,z)r_1(t,z)\nu(dz)\big)dt + q_1(t)dB(t) \\ \qquad\qquad + \int_{\mathbb{R}} r_1(t,z)\widetilde{N}(dt,dz) \\ p_1(T) = \nabla_{y_1}U(Y_2(T)) \end{cases} \tag{4.18}$$

and

(4.19)

$$\begin{cases} dp_2(t) &= -\Big[\{\rho(t)+(\alpha(t)-\rho(t))\pi(t)\}p_2(t)+\beta(t)\pi(t)q_2(t) \\ & \quad +\int_{\mathbb{R}}\pi(t)\gamma(t,z)r_2(t,z)\nu(dz)\Big]dt+q_2(t)dB(t)+\int_{\mathbb{R}}r_2(t,z)\widetilde{N}(dt,dz) \\ p_2(T) &= \nabla_{y_2}U(Y_2(T)) \end{cases}$$

Let $(\hat{\theta},\hat{\pi})$ be candidate for an optimal control and let $\hat{Y}(t)=(\hat{Y}_1(t),\hat{Y}_2(t))$ be the corresponding optimal processes with corresponding solution $\hat{p}(t)=(\hat{p}_1(t),\hat{p}_2(t)),\hat{q}(t)=(\hat{q}_1(t),\hat{q}_2(t)),\hat{r}(t,.)=(\hat{r}_1(t,.),\hat{r}_2(t,.))$ of the adjoint equations.

We first maximize the Hamiltonian $E[H(t,y_1,y_2,\theta,\pi,p,q,r)\mid\mathcal{E}_t]$ over all π gives the condition for maximum point $\hat{\pi}$

$$E[(\alpha(t)-\rho(t))\hat{p}_2(t)\mid\mathcal{E}_t]+E[\beta(t)\hat{q}_2(t)\mid\mathcal{E}_t] \\ +\int_{\mathbb{R}}\gamma(t,z)E[\gamma(t,z)\hat{r}_2(t,z)\mid\mathcal{E}_t]\nu(dz)=0 \tag{4.20}$$

And then minimize $E[H(t,y_1,y_2,\theta,\pi,p,q,r)\mid\mathcal{E}_t]$ over all $\theta\in\Theta$ and get the conditions for minimum point $\hat{\theta}=(\theta_0,\theta_1)$

$$E[-\hat{Y}_1(t)\hat{q}_1(t)\mid\mathcal{E}_t]=0 \tag{4.21}$$

and

$$\int_{\mathbb{R}}E[-\hat{Y}_1(t)\hat{r}_1(t,z)\mid\mathcal{E}_t]\nu(dz)=0 \tag{4.22}$$

We try a process $\hat{p}_1(t)$ of the form

$$\hat{p}_1(t)=U(f(t)\hat{Y}_2(t)) \tag{4.23}$$

with f is deterministic differentiable function. Differentiating (4.23) and use (4.5) we get

$$\begin{aligned} d\hat{p}_1(t) &= f'(t)\hat{Y}_2(t)U'(f(t)\hat{Y}_2(t))dt \\ &\quad +\hat{Y}_2(t)[(\rho(t)-(\alpha(t)-\rho(t))\hat{\pi}(t))dt+\beta\hat{\pi}dB(t)]f(t)U'(f(t)\hat{Y}_2(t)) \\ &\quad +\frac{1}{2}f^2(t)\hat{Y}_2^2(t)\beta^2(t)\pi^2(t)U''(f(t)\hat{Y}_2(t))dt \\ &\quad +\int_{\mathbb{R}}\{U(\hat{Y}_2(t)(f(t)+\hat{\pi}(t)\gamma)-U(f(t)\hat{Y}_2(t))) \\ &\qquad -\hat{Y}_2(t)\hat{\pi}(t)\gamma f(t)U'(f(t)\hat{Y}_2(t))\}\nu(dz)dt \\ &\quad +\int_{\mathbb{R}}\{U(\hat{Y}_2(t)(f(t)+\hat{\pi}(t)\gamma)-U(f(t)\hat{Y}_2(t)))\}\widetilde{N}(dt,dz) \\ &= \Big\{f'(t)\hat{Y}_2(t)U'(f(t)\hat{Y}_2(t))+\frac{1}{2}f^2(t)\hat{Y}_2^2(t)\beta^2(t)\pi^2(t)U''(f(t)\hat{Y}_2(t)) \\ &\qquad +\hat{Y}_2(t)(\rho(t)-(\alpha(t)-\rho(t))\hat{\pi}(t))f(t)U'(f(t)\hat{Y}_2(t)) \end{aligned}$$

$$
\begin{aligned}
&\quad + \int_{\mathbb{R}} \{U(\hat{Y}_2(t)(f(t) + \hat{\pi}(t)\gamma) - U(f(t)\hat{Y}_2(t)))\\
&\quad - \hat{Y}_2(t)\hat{\pi}(t)\gamma f(t)U^{'}(f(t)\hat{Y}_2(t))\}\nu(dz)\Big\}dt\\
&+ \hat{Y}_2(t)\beta(t)\hat{\pi}(t)f(t)U^{'}(f(t)\hat{Y}_2(t))dB(t)\\
&+ \int_{\mathbb{R}} \{U(\hat{Y}_2(t)(f(t) + \hat{\pi}(t)\gamma) - U(f(t)\hat{Y}_2(t)))\}\widetilde{N}(dt, dz)
\end{aligned}
$$

Comparing this with (4.18) by equating the $dt, dB(t)$ and $\widetilde{N}(dt, dz)$ coefficients respectively, we get

$$\hat{q}_1(t) = \hat{Y}_2(t)\beta(t)\hat{\pi}(t)U^{'}(f(t)\hat{Y}_2(t)) \tag{4.24}$$

$$\hat{r}_1(t, z) = U(\hat{Y}_2(t)(f(t) + \hat{\pi}(t)\gamma(t, z))) - U(f(t)\hat{Y}_2(t)) \tag{4.25}$$

and

$$
\begin{aligned}
&f^{'}(t)\hat{Y}_2(t)U^{'}(f(t)\hat{Y}_2(t)) + \frac{1}{2}\hat{Y}_2^2(t)\beta^2(t)\pi^2(t)U^{''}(f(t)\hat{Y}_2(t))\\
&\quad + \hat{Y}_2(t)(\rho(t) - (\alpha(t) - \rho(t))\hat{\pi}(t))f(t)U^{'}(f(t)\hat{Y}_2(t))\\
&\quad + \int_{\mathbb{R}} \{U(\hat{Y}_2(t)(f(t) + \hat{\pi}(t)\gamma) - U(f(t)\hat{Y}_2(t)))\\
&\qquad - \hat{Y}_2(t)\hat{\pi}(t)\gamma f(t)U^{'}(f(t)\hat{Y}_2(t))\}\nu(dz)\\
&\quad = \hat{\theta}_0(t)\hat{q}_1(t) + \int_{\mathbb{R}} \hat{\theta}_1(t, z)\hat{r}_1(t, z)\nu(dz)
\end{aligned} \tag{4.26}
$$

Substituting (4.24) into (4.21) we get

$$-\hat{\pi}(t)E[\hat{Y}_1\hat{Y}_2(t)\beta(t)U^{''}(f(t)\hat{Y}_2(t)) \mid \mathcal{E}_t] = 0 \tag{4.27}$$

or

$$\hat{\pi}(t) = 0 \tag{4.28}$$

Now we try the process $\hat{p}_2(t)$ of the form

$$\hat{p}_2(t) = \hat{Y}_1(t)U^{'}(f(t)\hat{Y}_2(t))f(t) \tag{4.29}$$

Differentiating (4.29) and using (4.28) we get

$$
\begin{aligned}
d\hat{p}_2(t) &= f^{'}(t)\hat{Y}_1(t)U^{'}(f(t)\hat{Y}_2(t))dt + f(t)U^{'}(f(t)\hat{Y}_2(t))d\hat{Y}_1(t)\\
&\quad + f(t)\hat{Y}_1(t)dU^{'}(f(t)\hat{Y}_2(t))\\
&= \hat{Y}_1(t)\Big(f^{'}(t)U^{'}(f(t)\hat{Y}_2(t)) + f(t)f^{'}(t)\hat{Y}_2(t)U^{''}(f(t)\hat{Y}_2(t))\\
&\quad + f^2(t)\hat{Y}_2(t)\rho(t)U^{''}(f(t)\hat{Y}_2(t))\Big)dt\\
&\quad - f(t)\hat{Y}_1(t)\theta_0(t)U^{'}(f(t)\hat{Y}_2(t))dB(t)\\
&\quad - \int_{\mathbb{R}} f(t)\hat{Y}_1(t)\theta_1(t, z)U^{'}(f(t)\hat{Y}_2(t))\widetilde{N}(dt, dz)
\end{aligned} \tag{4.30}
$$

Compare this with (4.19) we get

$$\hat{q}_2(t) = -f(t)\hat{Y}_1(t)\theta_0(t)U^{'}(f(t)\hat{Y}_2(t)) \tag{4.31}$$

$$\hat{r}_2(t,z) = -f(t)\hat{Y}_1(t)\theta_1(t,z)U^{'}(f(t)\hat{Y}_2(t)) \tag{4.32}$$

and

$$\begin{aligned} f^{'}(t)U^{'}(f(t)\hat{Y}_2(t)) + f(t)\hat{Y}_2(t)U^{''}(f(t)\hat{Y}_2(t))(f^{'}(t) + f(t)\rho(t)) \\ = -\rho(t)f(t)U^{'}(f(t)\hat{Y}_2(t)) \end{aligned} \tag{4.33}$$

Substituting (4.31), (4.32) into (4.20) we get

$$\begin{aligned} &E[(\alpha(t)-\rho(t))f(t)\hat{Y}_1(t)U^{'}(f(t)\hat{Y}_2(t)) \mid \mathcal{E}_t] \\ &\quad - \theta_0(t)E[\beta(t)f(t)\hat{Y}_1(t)U^{'}(f(t)\hat{Y}_2(t)) \mid \mathcal{E}_t] \\ &\quad - \int_{\mathbb{R}} \theta_1(t,z)E[\gamma(t,z)f(t)\hat{Y}_1(t)U^{'}(f(t)\hat{Y}_2(t)) \mid \mathcal{E}_t] = 0 \end{aligned} \tag{4.34}$$

This can be written as

(4.35)
$$\hat{\theta}_0(t)E[\beta(t) \mid \mathcal{E}_t] - \int_{\mathbb{R}} \hat{\theta}_1(t,z)E[\gamma(t,z) \mid \mathcal{E}_t]\nu(dz) = E[(\alpha(t) \mid \mathcal{E}_t] - \rho(t)$$

From (4.33) we get

$$\big(U^{'}(f(t)\hat{Y}_2(t)) + \hat{Y}_2(t)f(t)U^{''}(f(t)\hat{Y}_2(t))\big)(f^{'}(t) + r(t)f(t)) = 0 \tag{4.36}$$

or

$$f^{'}(t) + r(t)f(t) = 0 \tag{4.37}$$

i.e.

$$f(t) = \exp\Big(\int_t^T r(s)ds\Big) \tag{4.38}$$

We have proved:

Theorem 4. *The optimal portfolio $\pi \in \Pi$ for the agent is*

$$\pi(t) = \hat{\pi}(t) = 0 \tag{4.39}$$

and the optimal measure $Q_{\hat{\theta}}$ for the market is to choose $\hat{\theta} = (\hat{\theta}_0, \hat{\theta}_1)$ such that

(4.40)
$$\hat{\theta}_0(t)E[\beta(t) \mid \mathcal{E}_t] - \int_{\mathbb{R}} \hat{\theta}_1(t,z)E[\gamma(t,z) \mid \mathcal{E}_t]\nu(dz) = E[(\alpha(t) \mid \mathcal{E}_t] - \rho(t)$$

5. The sufficient maximum principle for nonzero-sum game

Let $X(t)$ be a dynamic of the system. We now consider the case when two controllers I and II intervene on the dynamic of the system and their advantages are not necessarily antagonistic but each one acts such as to save its own interest. This situation is a nonzero-sum game.

Let $\mathcal{E}_t^1, \mathcal{E}_t^2$ be filtration satisfying

$$\mathcal{E}_t^i \subseteq \mathcal{F}_t, \qquad t \geq 0, i = 1, 2$$

Let $u = (\theta, \pi)$, where $\theta = (\theta_0, \theta_1)$ and $\pi = (\pi_0, \pi_1)$ are controls for player I and II, respectively. We assume that $\theta = (\theta_0, \theta_1)$ are adapted to $\mathcal{E}_t^1$ and $\pi = (\pi_0, \pi_1)$ are adapted to $\mathcal{E}_t^2$. Denote by Θ and Π the sets of admissible controls θ and π, respectively. Suppose the players acts on the system with strategy $(\theta, \pi) \in \Theta \times \Pi$, then the costs associated with I and II are, respectively, $J_1^{(\theta,\pi)}(x)$ and $J_2^{(\theta,\pi)}(x)$ of the form

$$J_i^{(\theta,\pi)}(x) = E^x\Big[\int_0^T f_i(t, X(t), u(t))\, dt + g_i(X(T))\Big], \qquad i = 1, 2. \tag{5.1}$$

The problem is to find a control $(\theta^*, \pi^*) \in \Theta \times \Pi$ such that

$$J_1^{(\theta,\pi^*)}(x) \leq J_1^{(\theta^*,\pi^*)}(x) \qquad \text{for all} \qquad \theta \in \Theta \tag{5.2}$$

$$J_2^{(\theta^*,\pi)}(x) \leq J_2^{(\theta^*,\pi^*)}(x) \qquad \text{for all} \qquad \pi \in \Pi \tag{5.3}$$

The pair of control (θ^*, π^*) is called a *Nash equilibrium* for the game because when player I (resp. II) acts with the strategy θ^* (resp. π^*), the best that has to do II (resp. I) is to act with π^* (resp. θ^*).

Let us introduce the Hamiltonian functions associated with this game, namely H_1 and H_2, from $[0, T] \times \mathbb{R}^n \times K_1 \times K_2 \times \mathbb{R}^n \times \mathbb{R}^{n\times m} \times \mathcal{R}$ to $\mathbb{R}$ which are defined by

$$\begin{aligned} H_i(t, x, \theta, \pi, p_i, q_i, r_1) &= f_i(t, x, \theta, \pi) + b^{\intercal}(t, x, \theta, \pi)p_i + tr(\sigma^{\intercal}((t, x, \theta, \pi)q_i) \\ &+ \sum_{i,j=1}^n \int_{\mathbb{R}_0} \gamma_{ij}(t, x, \theta, \pi, z) r_i^{ij}(t, z)\nu_j(dz_j), \qquad i = 1, 2. \end{aligned} \tag{5.4}$$

And we also have the adjoint equations for the game as following

$$\begin{cases} dp_i(t) &= -\bigtriangledown_x H_i(t, X(t), \theta(t), \pi(t), p_i(t), q_i(t), r_i(t, .))dt \\ & \qquad +q_i(t)dB(t) + \int_{\mathbb{R}^n} r_i(t^-, z)\widetilde{N}(dt, dz), \qquad t < T \\ p_i(T) &= \bigtriangledown g_i(X(T)), \qquad i = 1, 2. \end{cases} \tag{5.5}$$

The following result is a generalization of Theorem 1

Theorem 5. *Let $(\hat{\theta}, \hat{\pi}) \in \Theta \times \Pi$ with corresponding state process $\hat{X}(t) = X^{(\hat{\theta},\hat{\pi})}(t)$. Suppose there exists a solution $(\hat{p}_i(t), \hat{q}_i(t), \hat{r}_i(t, z))$, $i = 1, 2$ of the* *corresponding adjoint equation* (5.5) *such that for all* $\theta \in \Theta$ *and* $\pi \in \Pi$, *we have*

$$\begin{aligned} &E[H_1(t,\hat{X}(t),\hat{\theta}(t),\hat{\pi}(t),\hat{p}_1(t),\hat{q}_1(t),\hat{r}_1(t,.)) \mid \mathcal{E}_t^1] \\ &\quad \geq E[H_1(t,\hat{X}(t),\theta(t),\hat{\pi}(t),\hat{p}_1(t),\hat{q}_1(t),\hat{r}_1(t,.)) \mid \mathcal{E}_t^1] \end{aligned} \tag{5.6}$$

and

$$\begin{aligned} &E[H_2(t,\hat{X}(t),\hat{\theta}(t),\hat{\pi}(t),\hat{p}_2(t),\hat{q}_2(t),\hat{r}_2(t,.)) \mid \mathcal{E}_t^2] \\ &\quad \geq E[H_2(t,\hat{X}(t),\hat{\theta}(t),\pi(t),\hat{p}_2(t),\hat{q}_2(t),\hat{r}_2(t,.)) \mid \mathcal{E}_t^2] \end{aligned} \tag{5.7}$$

Moreover, suppose that for all $t \in [0,T]$, $H_i(t,x,\theta,\pi,\hat{p}(t),\hat{q}(t),\hat{r}(t,.))$, $i = 1,2$ *is concave in* x,θ,π *and* $g_i(x)$, $i=1,2$ *is concave in* x. *Then* $(\hat{\theta}(t),\hat{\pi}(t))$ *is an equilibrium point for the game and*

$$J_1^{(\hat{\theta},\hat{\pi})}(x) = \sup_{\theta\in\Theta} J_1^{(\theta,\hat{\pi})}(x) \tag{5.8}$$

$$J_2^{(\hat{\theta},\hat{\pi})}(x) = \sup_{\pi\in\Pi} J_2^{(\hat{\theta},\pi)}(x) \tag{5.9}$$

Proof. As in the proof of Theorem 1 we have

$$\begin{aligned} &J_1{}^{(\hat{\theta},\hat{\pi})}(x) - J_1^{(\theta,\hat{\pi})}(x) \\ &= E\Big[\int_0^T \big\{f_1(t,\hat{X}(t),\hat{\theta}(t),\hat{\pi}(t)) - f_1(t,X^{(\pi)}(t),\hat{\theta}(t),\pi(t))\big\}dt \\ &\qquad + g_1(\hat{X}(T)) - g_2(X^{(\pi)}(T))\Big] \\ &= E\Big[\int_0^T \{H_1(t,\hat{X}(t),\hat{\theta}(t),\hat{\pi}(t),\hat{p}_1(t),\hat{q}_1(t),\hat{r}_1(t,.)) \\ &\qquad - H_1(t,X^{(\pi)}(t),\hat{\theta}(t),\pi(t),\hat{p}_1(t),\hat{q}_1(t),\hat{r}_1(t,.))\}dt\Big] \\ &- E\Big[\int_0^T (\bigtriangledown_x H_1(t,\hat{X}(t),\hat{\theta}(t),\hat{\pi}(t),\hat{p}_1(t),\hat{q}_1(t),\hat{r}_1(t,.)))(\hat{X}(t) - X^{(\pi)}(t))^T dt\Big] \end{aligned} \tag{5.10}$$

From (5.6) and concavity of H_1 in x and π we have

$$\begin{aligned} &E\Big[\int_0^T \{H_1(t,\hat{X}(t),\hat{\theta}(t),\hat{\pi}(t),\hat{p}_1(t),\hat{q}_1(t),\hat{r}_1(t,.)) \\ &\qquad - H_1(t,X^{(\pi)}(t),\hat{\theta}(t),\pi(t),\hat{p}_1(t),\hat{q}_1(t),\hat{r}_1(t,.))\}dt\Big] \\ &\geq E\Big[\int_0^T (\bigtriangledown_x H_1(t,\hat{X}(t),\hat{\theta}(t),\hat{\pi}(t),\hat{p}_1(t),\hat{q}_1(t),\hat{r}_1(t,.)))(\hat{X}(t) - X^{(\pi)}(t))^T dt\Big] \end{aligned} \tag{5.11}$$

Hence

$$J_1^{(\hat{\theta},\hat{\pi})}(x) - J_1^{(\theta,\hat{\pi})}(x) \geq 0 \tag{5.12}$$

Since this holds for all $\theta \in \Theta$ we have

$$J_1^{(\hat{\theta},\hat{\pi})}(x) = \sup_{\theta\in\Theta} J_1^{(\theta,\hat{\pi})}(x) \tag{5.13}$$

In the same way we show that $J_2^{(\theta^*,\pi)}(x) \leq J_2^{(\theta^*,\pi^*)}(x)$ and $J_2^{(\hat{\theta},\hat{\pi})}(x) = \sup_{\pi\in\pi} J_2^{(\hat{\theta},\pi)}(x)$, whence the desired result. □

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