A Maximum Principle for Stochastic Differential Games with Partial Information

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Fixed a probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$. The dynamics of a stochastic system is given of the form

$$\begin{aligned} dX^{(u)}(t) &= b(t, X(t), u_0(t))\,dt + \sigma(t, X(t), u_0(t))\,dB(t) \\ &\quad + \int_{\mathbb{R}^n} \gamma(t, X(t^-), u_1(t^-, z), z)\widetilde{N}(\,dt,\,dz), \qquad t \in [0, T] \\ X^{(u)}(0) &= x \in \mathbb{R}^n \end{aligned}$$

where

b, σ, γ are given continuous functions.

$B(t)$ is n-dimensional Brownian motion

$\widetilde{N}(.,.)$ are n independent compensated Poisson random measures.

K is a given closed subset of $\mathbb{R}^n$.

Let $u_0(t) = u_0(t, \omega) \in K_1$ and $u_1(t) = u_1(t, z, \omega) \in K_2$ be our *control processes.*

Definition

The set $\mathcal{A}_{\mathcal{E}}$ of all admissible control consists of all $u = (u_0, u_1)$ satisfying the following conditions:

i) $u_0(t), u_1(t, z)$ are càdlàg.

ii) adapted to a given filtration $\{\mathcal{E}_t\}_{t \geq 0}$, where

$$\mathcal{E}_t \subseteq \mathcal{F}_t, \qquad t \geq 0.$$

iii)

$$E^x \Big[\int_0^T \mid f(t, X(t), u_0(t)) \mid \, dt + \mid g(X(T)) \mid \Big] < \infty$$

Now suppose that the controls $u_0(t)$ and $u_1(t,z)$ have the form

$$\begin{aligned} u_0(t) &= (\theta_0(t), \pi_0(t)); \qquad t \geq 0 \\ u_1(t,z) &= (\theta_1(t,z), \pi_1(t,z)); \qquad (t,z) \in [0,\infty) \times \mathbb{R}^n \end{aligned}$$

Denote Θ and Π to be the family of admissible controls $\theta = (\theta_0, \theta_1)$ and $\pi = (\pi_0, \pi_1)$, respectively.

Partial information zero-sum stochastic differential game problems

Find $(\theta^*, \pi^*) \in \Theta \times \Pi$ such that

$$\Phi_{\mathcal{E}}(x) = J(\theta^*, \pi^*) = \sup_{\pi \in \Pi} \Big(\inf_{\theta \in \Theta} J(\theta, \pi) \Big)$$

where

$$J(u) = E^x \Big[\int_0^T f(t, X^{(u)}(t), u_0(t))\, dt + g(X^{(u)}(T)) \Big]$$

(θ^*, π^*) is called an *optimal control* (if it exists)

The same problem in the paper by Mataramvura and Øksendal (2005)

The Hamiltonian function

Let $\theta(t,z) \in K_1$ and $\pi(t,z) \in K_2$ for a.a t,z. The *Hamiltonian*

$$H : [0,T] \times \mathbb{R}^n \times K_1 \times K_2 \times \mathbb{R}^n \times \mathbb{R}^{n\times n} \times \mathcal{R} \mapsto \mathbb{R}$$

is defined by

$$\begin{aligned} H(t,x,\theta,\pi,p,q,r) =& f(t,x,\theta,\pi) + b^T(t,x,\theta,\pi)p + tr(\sigma^T((t,x,\theta,\pi)q) \\ &+ \sum_{i,j=1}^{n} \int_{\mathbb{R}_0} \gamma_{ij}(t,x,\theta,\pi,z) r_{ij}(t,z)\nu_j(dz_j) \end{aligned}$$

From now on we assume that H is continuously differentiable with respect to x.

The adjoint equation

$$\begin{cases} dp(t) & = & -\bigtriangledown_x H(t, X^{(u)}(t), \theta(t), \pi(t), p(t), q(t), r(t,.))\, dt \\ & & \quad +q(t)\, dB(t) + \int_{\mathbb{R}^n} r(t^-, z)\widetilde{N}(\, dt,\, dz), \qquad t < T \\ p(T) & = & \bigtriangledown g(X^{(u)}(T)) \end{cases}$$

where $\bigtriangledown_y \varphi(.) = \left(\frac{\partial \varphi}{\partial y_1}, ..., \frac{\partial \varphi}{\partial y_n}\right)^T$ is the gradient of $\varphi : \mathbb{R}^n \to \mathbb{R}$ with respect to $y = (y_1, ..., y_n)$.

Assumptions

Let $(\hat{\theta}, \hat{\pi}) \in \Theta \times \Pi$ with corresponding state process $\hat{X}(t) = X^{(\hat{\theta}, \hat{\pi})}(t)$. Put

$$X^{\pi}(t) := X^{(\hat{\theta}, \pi)}(t) \text{ and } X^{\theta}(t) := X^{(\theta, \hat{\pi})}(t)$$

(A1) $E\Big[\int_0^T (\hat{X} - X^{(\pi)})^T \{ \hat{q}\hat{q}^T + \int_{\mathbb{R}^n} \hat{r}\hat{r}^T \nu(dz) \} (\hat{X} - X^{(\pi)}) dt \Big] < \infty,$

(A2) $E\Big[\int_0^T (\hat{X} - X^{(\theta)})^T \{ \hat{q}\hat{q}^T + \int_{\mathbb{R}^n} \hat{r}\hat{r}^T \nu(dz) \} (\hat{X} - X^{(\theta)}) dt \Big] < \infty,$

(A3) $E\Big[\int_0^T \hat{p}^T . \{ \sigma^{(\theta)} \sigma^{(\theta)^T} + \int_{\mathbb{R}_0} \gamma^{(\theta)} \gamma^{(\theta)^T} \nu(dz) \} p.dt \Big] < \infty,$

(A4) $E\Big[\int_0^T \hat{p}^T \{ \sigma^{(\pi)} \sigma^{(\pi)^T} + \int_{\mathbb{R}_0} \gamma^{(\pi)} \gamma^{(\pi)^T} \nu(dz) \} p.dt \Big] < \infty.$

(A5) (partial information maximum principle)
For all $t \in [0, T]$

$$\begin{aligned}
&\inf_{\theta \in K_1} E[H(t, X(t), \theta, \hat{\pi}(t), \hat{p}(t), \hat{q}(t), \hat{r}(t,.)) \mid \mathcal{E}_t] \\
&= E[H(t, X(t), \hat{\theta}(t), \hat{\pi}(t), \hat{p}(t), \hat{q}(t), \hat{r}(t,.)) \mid \mathcal{E}_t] \\
&= \sup_{\pi \in K_2} E[H(t, X(t), \hat{\theta}(t), \pi, \hat{p}(t), \hat{q}(t), \hat{r}(t,.)) \mid \mathcal{E}_t]
\end{aligned}$$

A Sufficient Theorem

Suppose the assumptions **(A1)-(A5)** hold. We have, for all $t \in [0,T]$

i) If $g(x)$ is concave in x and $(x,\pi) \rightarrow H(t,x,\hat{\theta}(t),\pi,\hat{p}(t),\hat{q}(t),\hat{r}(t,.))$ is concave. Then

$$J(\hat{\theta},\hat{\pi}) \geq J(\hat{\theta},\pi) \text{ and } J(\hat{\theta},\hat{\pi}) = \sup_{\pi\in\Pi} J(\hat{\theta},\pi), \qquad \forall\pi\in\Pi.$$

ii) If $g(x)$ is convex in x and $(x,\theta) \rightarrow H(t,x,\theta,\hat{\pi}(t),\hat{p}(t),\hat{q}(t),\hat{r}(t,.))$ is convex. Then

$$J(\hat{\theta},\hat{\pi}) \leq J(\theta,\hat{\pi}) \qquad \text{and } J(\hat{\theta},\hat{\pi}) = \inf_{\theta\in\Theta} J(\theta,\hat{\pi}), \qquad \forall\theta\in\Theta.$$

iii) If both cases (i) and (ii) hold then $(\theta^*,\pi^*) := (\hat{\theta},\hat{\pi})$ is an optimal control and

$$\Phi_{\mathcal{E}}(x) = \sup_{\pi\in\Pi}\left(\inf_{\theta\in\Theta} J(\theta,\pi)\right) = \inf_{\theta\in\Theta}\left(\sup_{\pi\in\Pi} J(\theta,\pi)\right).$$

Proof

See Baghery and Øksendal (2006)

Complete Information Case

Corollary

Suppose $\mathcal{E}_t = \mathcal{F}_t$ for all t and that the assumptions (A1)-(A4) hold. Moreover, suppose that for all t the following maximum principle holds

$$\begin{aligned}
&\inf_{\theta \in K_1} H(t, X(t), \theta, \hat{\pi}(t), \hat{p}(t), \hat{q}(t), \hat{r}(t,.)) \\
&\quad = H(t, X(t), \hat{\theta}(t), \hat{\pi}(t), \hat{p}(t), \hat{q}(t), \hat{r}(t,.)) \\
&\quad = \sup_{\pi \in K_2} H(t, X(t), \hat{\theta}(t), \pi, \hat{p}(t), \hat{q}(t), \hat{r}(t,.))
\end{aligned}$$

Then we get all conclusions as the same in The Sufficient Theorem.

Assumptions

(B1) For all t, h such that $0 \leq t < t + h \leq T$, all bounded $\mathcal{E}_t$-measurable α, ρ, and for $s \in [0, T]$ the controls $\beta(s) := (0, ..., \beta_i(s), ..., 0)$ and $\eta(s) := (0, ..., \eta_i(s), ..., 0), i = 1, ..., n$ with

$$\beta_i(s) := \alpha_i \chi_{[t,t+h]}(s), \qquad \text{and} \qquad \eta_i(s) := \rho_i \chi_{[t,t+h]}(s)$$

belong to Θ and Π, respectively.

(B2) For given $(\theta, \pi), (\beta, \eta) \in \Theta \times \Pi$ and (β, η) are bounded, there exists $\delta > 0$ such that

$$\theta + y\beta \in \Theta, \qquad \text{and} \qquad \pi + v\eta \in \Pi$$

where $y, v \in (-\delta, \delta)$.

Denote $X^{\theta+y\beta}(t) = X^{(\theta+y\beta,\pi)}(t)$ and $X^{\pi+v\eta}(t) = X^{(\theta,\pi+v\eta)}(t)$. Define

$$Y^{\theta}(t) = \frac{d}{dy} X^{\theta+y\beta}(t) \mid_{y=0} = (Y_1^{\theta}(t), ..., Y_n^{\theta}(t))^T$$

We have that

$$dY_i^{\theta}(t) = \lambda_i^{\theta}(t)dt + \sum_{j=1}^{n} \xi_{ij}^{\theta}(t) dB_j(t) + \sum_{j=1}^{n} \int_{\mathbb{R}} \zeta_{ij}^{\theta}(t,z) \widetilde{N}_j(dt,dz),$$

where $i = 1, ..., n$ and

$$\begin{cases} \lambda_i^{\theta}(t) & = \bigtriangledown_x b_i(t, X(t), \theta(t), \pi(t))^T Y^{\theta}(t) \\ & \qquad + \bigtriangledown_{\theta}\, b_i(t, X(t), \theta(t), \pi(t))^T \beta(t) \\ \xi_{ij}^{\theta}(t) & = \bigtriangledown_x \sigma_{ij}(t, X(t), \theta(t), \pi(t))^T Y^{\theta}(t) \\ & \qquad + \bigtriangledown_{\theta}\, \sigma ij(t, X(t), \theta(t), \pi(t))^T \beta(t) \\ \zeta_{ij}^{\theta}(t) & = \bigtriangledown_x \gamma_{ij}(t, X(t), \theta(t), \pi(t))^T Y^{\theta}(t) \\ & \qquad + \bigtriangledown_{\theta}\, \gamma_{ij}(t, X(t), \theta(t), \pi(t))^T \beta(t) \end{cases}$$

Similarly define

$$Y^{\pi}(t) = \frac{d}{dv} X^{\pi+v\eta}(t) \mid_{v=0} = (Y_1^{\pi}(t), ..., Y_n^{\pi}(t))^T$$

We have

$$dY_i^{\pi}(t) = \lambda_i^{\pi}(t)dt + \sum_{j=1}^{n} \xi_{ij}^{\pi}(t)dB_j(t) + \sum_{j=1}^{n} \int_{\mathbb{R}} \zeta_{ij}^{\pi}(t,z)\widetilde{N}_j(dt,dz),$$

where $i = 1, ..., n$ and

$$\begin{cases} \lambda_i^{\pi}(t) & = \bigtriangledown_x b_i(t, X(t), \theta(t), \pi(t))^T Y^{\pi}(t) \\ & \qquad + \bigtriangledown_{\pi} b_i(t, X(t), \theta(t), \pi(t))^T \eta(t) \\ \xi_{ij}^{\pi}(t) & = \bigtriangledown_x \sigma_{ij}(t, X(t), \theta(t), \pi(t))^T Y^{\pi}(t) \\ & \qquad + \bigtriangledown_{\pi} \sigma ij(t, X(t), \theta(t), \pi(t))^T \eta(t) \\ \zeta_{ij}^{\pi}(t) & = \bigtriangledown_x \gamma_{ij}(t, X(t), \theta(t), \pi(t))^T Y^{\pi}(t) \\ & \qquad + \bigtriangledown_{\pi} \gamma_{ij}(t, X(t), \theta(t), \pi(t))^T \eta(t) \end{cases}$$

Local minimum and maximum

Definition

$\hat{\theta} \in \Theta$ and $\hat{\pi} \in \Pi$ are respectively local minimum and maximum for $J(\theta, \pi)$ if for all bounded $(\beta, \eta) \in \Theta \times \Pi$ there exist $\delta > 0$ such that $(\hat{\theta} + y\beta, \hat{\pi} + v\eta) \in \Theta \times \Pi$ and

$$h(y, v) := J(\hat{\theta} + y\beta, \hat{\pi} + v\eta) \qquad \forall y, v \in (-\delta, \delta)$$

has a minimum at $y = 0$ and a maximum at $v = 0$.

Assumptions

(B3)

$$E\Big[\int_0^T Y^{\hat{\theta}^T}(t)\Big\{\hat{q}\hat{q}^T + \int_{\mathbb{R}} \hat{r}\hat{r}^T(t,z)\nu(dz)\Big\}Y^{\hat{\theta}}(t)dt\Big] < \infty,$$

(B4)

$$E\Big[Y^{\hat{\pi}^T}(t)\Big\{\hat{q}\hat{q}^T + \int_{\mathbb{R}} \hat{r}\hat{r}^T(t,z)\nu(dz)\Big\}Y^{\hat{\pi}}(t)dt\Big] < \infty,$$

(B5)

$$E\Big[\int_0^T \hat{p}^T(t)\Big\{\xi^{\hat{\theta}}\xi^{\hat{\theta}^T} + \int_{\mathbb{R}} \gamma\gamma^T\nu(dz)\Big\}\hat{p}(t)dt\Big] < \infty,$$

(B6)

$$E\Big[\int_0^T \hat{p}^T(t)\Big\{\xi^{\hat{\pi}}\xi^{\hat{\pi}^T} + \int_{\mathbb{R}} \gamma\gamma^T\nu(dz)\Big\}\hat{p}(t)dt\Big] < \infty,$$

A Necessary Theorem

Suppose **(B1)-(B6)** hold and $\hat{\theta} \in \Theta$ and $\hat{\pi} \in \Pi$ are respectively local minimum and maximum for $J(\theta, \pi)$. Then for a.a. $t \in [0, T]$, we have

$$E[\,\bigtriangledown_\theta\, H(t, \hat{X}(t), \hat{\theta}(t), \hat{\pi}(t), \hat{p}(t), \hat{q}(t), \hat{r}(t,.)) \mid \mathcal{E}_t]$$
$$= E[\bigtriangledown_\pi H(t, \hat{X}(t), \hat{\theta}(t), \hat{\pi}(t), \hat{p}(t), \hat{q}(t), \hat{r}(t,.)) \mid \mathcal{E}_t] = 0$$

Proof of Necessary Theorem

We have

$$\nabla_x H = \nabla_x f + \sum_{i=1}^{n} \nabla_x b_i.p_i + \sum_{j,i=1}^{n} \nabla_x \sigma_{ji}.q_{ji} + \sum_{j,i=1}^{n} \int_{\mathbb{R}} \nabla_x \gamma_{ji}.r_{ji}(t,z)\nu_j(dz)$$

and

$$\nabla_\theta H = \nabla_\theta f + \sum_{i=1}^{n} \nabla_\theta b_i.p_i + \sum_{j,i=1}^{n} \nabla_\theta \sigma_{ji}.q_{ji} + \sum_{j,i=1}^{n} \int_{\mathbb{R}} \nabla_\theta \gamma_{ji}.r_{ji}(t,z)\nu_j(dz)$$

Proof of Necessary Theorem

By the It formula,

$$E[\bigtriangledown g(\hat{X}(T)^T.Y^{\hat{\theta}}(T)] = E[\hat{p}^T(T).Y^{\hat{\theta}}(T)]$$

$$= E\Big[\sum_{i=1}^{n}\int_0^T \Big\{\hat{p}_i(t).(\bigtriangledown_x b_i^T.Y^{\hat{\theta}}(t) + \bigtriangledown_\theta b_i^T.\beta(t))$$

$$+ Y_i^{\hat{\theta}}(t).(-\bigtriangledown_x H)_i + \sum_{j=1}^{n}\hat{q}_{ij}(t)(\bigtriangledown_x \sigma_{ij}^T.Y^{\hat{\theta}}(t) + \bigtriangledown_\theta \sigma_{ij}^T.\beta(t))$$

$$+ \sum_{j=1}^{n}\int_{\mathbb{R}^n}\hat{r}_{ij}(t^-,z)(\bigtriangledown_x \gamma_{ij}^T.Y^{\hat{\theta}}(t) + \bigtriangledown_\theta \gamma_{ij}^T.\beta(t))\Big\}dt\Big]$$

$$= E\Big[\int_0^T \Big\{-\bigtriangledown_x f.Y^{\hat{\theta}}(t) + \bigtriangledown_\theta H.\beta(t)\Big\}dt\Big]$$

Proof of Necessary Theorem

Since h has a minimum at $y = 0$ we have

$$0 = \frac{\partial}{\partial y} h(y,0) \mid_{y=0} = E\Big[\int_0^T \bigtriangledown_x f . \frac{d}{dy} X^{\hat{\theta}+y\beta}(t) \mid_{y=0} dt$$

$$+ \int_0^T \bigtriangledown_\theta f . \beta(t) dt + \bigtriangledown g(\hat{X}(T)^T . \frac{d}{dy} X^{\hat{\theta}+y\beta}(T) \mid_{y=0} \Big]$$

$$= E\Big[\int_0^T \bigtriangledown_x f . Y^{\hat{\theta}}(t) dt + \int_0^T \bigtriangledown_\theta f . \beta(t) dt + \bigtriangledown g(\hat{X}(T)^T . Y^{\hat{\theta}}(T)\Big]$$

$$= E\Big[\int_0^T \bigtriangledown_\theta H . \beta(t) dt\Big]$$

Proof of Necessary Theorem

By assumption **(B1)**, this leads to

$$E\Big[\int_t^{t+h}\frac{\partial}{\partial\theta_i}H.\alpha_i(s)ds\Big]=0$$

Differentiating with respect to h at $h=0$ gives

$$E\Big[\frac{\partial}{\partial\theta_i}H.\alpha_i(s)\Big]=0$$

Since this holds for all bounded $\mathcal{E}_t$-measurable α_i, we conclude that

$$E\Big[\frac{\partial}{\partial\theta_i}H\Big|\mathcal{E}_t\Big]=0,$$

Proceeding in the same way by differentiating the function $h(0,v)$ with respect to v we get

$$E\Big[\frac{\partial}{\partial\pi_i}H\Big|\mathcal{E}_t\Big]=0$$

This section solves a partial information version of the problem studied in Øksendal and Sulem (2006)

Consider the following jump diffusion market

$$\begin{aligned} &\text{(risky free asset)} \quad dS_0(t) = \rho(t)S_0(t)dt; \qquad S_0(0) = 1 \\ &\text{(risky asset)} \quad dS_1(t) = S_1(t^-)\Big[\alpha(t)dt + \beta(t)dB(t) \\ &\qquad + \int_{\mathcal{R}} \gamma(t,z)\widetilde{N}(dt,dz)\Big]; \qquad S_1(0) > 0 \end{aligned}$$

where $\rho(t)$ is a deterministic function, $\alpha(t), \beta(t)$ and $\gamma(t,z) \geq -1$ are given $\mathcal{F}_t$-predictable functions satisfying

$$E\Big[\int_0^T \{\mid \rho(s) \mid + \mid \alpha(s) \mid + \frac{1}{2}\beta(s)^2 + \int_{\mathbb{R}} \mid \log(1+\gamma(s,z)) - \gamma(s,z) \mid \nu(dz)\}ds\Big] < \infty$$

Wealth process

Portfolio $\pi(t)$ is *fraction* of the total wealth invested in the risky asset at time t.

Wealth process

$$\begin{aligned} dV^{(\pi)}(t) &= V^{(\pi)}(t^-)\Big[\{\rho(t)+(\alpha(t)-\rho(t))\pi(t)\}dt \\ &\quad +\pi(t)\beta(t)dB(t)+\pi(t^-)\int_{\mathbb{R}}\gamma(t,z)\widetilde{N}(dt,dz)\Big] \\ V^{(\pi)}(0) &= v>0. \end{aligned}$$

Definition

A portfolio π is called *admissible* if

- $\pi(t)$ is $\mathcal{E}_t$-adapted
- $\pi(t^-)\gamma(t,z) > -1 \qquad$ a.s.
- $$\int_0^T \{ \mid \rho(t) + (\alpha(t) - \rho(t))\pi(t) \mid + \pi^2(t)\beta^2(t) + \pi^2(t) \int_{\mathbb{R}} \gamma^2(t,z)\nu(dz)\} dt < \infty \qquad \text{a.s.}$$

The family of admissible portfolios is denoted by Π.

Set of measures

Introduce a family $\mathcal{Q}$ of measures Q_θ parameterized by processes $\theta = (\theta_0(t), \theta_1(t,z))$ such that

$$dQ_\theta(\omega) = Z_\theta(T)dP(\omega) \qquad \text{on } \mathcal{F}_T,$$

where

$$\begin{cases} dZ_\theta(t) & = & Z_\theta(t^-)[-\theta_0(t)dB(t) - \int_{\mathbb{R}} \theta_1(t,z)\widetilde{N}(dt,dz)] \\ Z_\theta(0) & = & 1 \end{cases}$$

Definition

$\theta = (\theta_0, \theta_1)$ is called *admissible control of the market* if the following satisfies:

- θ is $\mathcal{E}_t$-adapted
- $\theta_1(t, z) \leq 1$ for a.a. t,z
- $\int_0^T \{\theta_0^2(s) + \int_{\mathbb{R}} \theta_1^2(s, z)\} ds < \infty \qquad a.s.$

The families of admissible controls θ is denoted by Θ.

The Problem

Find $(\theta, \pi) \in \Theta \times \Pi$ such that

$$\inf_{\theta \in \Theta} \Big(\sup_{\pi \in \Pi} E_{Q_\theta}[U(V^{(\pi(T))}]\Big) = E_{Q_{\theta^*}}[U(V^{(\pi^*)}(T))]$$

where $U : [0, \infty) \to [-\infty, \infty)$ is a given utility function, which is increasing, concave and twice continuously differentiable on $(0, \infty)$.

Put

$$dX(t) = \begin{bmatrix} dX_1(t) \\ dX_2(t) \end{bmatrix} = \begin{bmatrix} dZ_\theta(t) \\ dV^{(\pi)}(t) \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ V^{(\pi)}(t^-)\{\rho(t) + (\alpha(t) - \rho(t))\pi\} \end{bmatrix} dt$$

$$+ \begin{bmatrix} -Z_\theta(t^-)\theta_0(t) \\ V^{(\pi)}(t^-)\beta(t)\pi(t) \end{bmatrix} dB(t)$$

$$+ \int_{\mathbb{R}} \begin{bmatrix} -Z_\theta(t^-)\theta_1(t,z) \\ V^{(\pi)}(t^-)\pi(t)\gamma(t,z) \end{bmatrix} \widetilde{N}(dt,dz)$$

The Hamiltonian function

$$H(t, x_1, x_2, \theta, \pi, p, q, r) = x_2\{\rho(t) + (\alpha(t) - \rho(t))\pi(t)\}p_2 - x_1\theta_0 q_1$$
$$+ x_2\beta(t)\pi(t)q_2 + \int_{\mathbb{R}}\{-x_1\theta_1(t,z)r_1(t,z) + x_2\pi(t)\gamma(t,z)r_2(t,z)\}\nu(dz)$$

The adjoint equations

$$\begin{cases} dp_1(t) &= \Big(\theta_0(t)q_1(t) + \int_{\mathbb{R}} \theta_1(t,z)r_1(t,z)\nu(dz)\Big)dt + q_1(t)dB(t) \\ & \qquad + \int_{\mathbb{R}} r_1(t,z)\widetilde{N}(dt,dz) \\ p_1(T) &= \nabla_{x_1} U(X_2(T)) \end{cases}$$

and

$$\begin{cases} dp_2(t) &= -\Big[\{\rho(t) + (\alpha(t)-\rho(t))\pi(t)\}p_2(t) + \beta(t)\pi(t)q_2(t) \\ &+ \int_{\mathbb{R}} \pi(t)\gamma r_2(t,z)\nu(dz)\Big]dt + q_2(t)dB(t) + \int_{\mathbb{R}} r_2(t,z)\widetilde{N}(dt,dz) \\ p_2(T) &= \nabla_{x_2} U(X_2(T)) \end{cases}$$

The condition for maximum point

$\hat{\pi}$ is given by maximizing the Hamiltonian function over all $\pi \in K_2$:

$$E[(\alpha(t) - \rho(t))\hat{p}_2(t) \mid \mathcal{E}_t] + E[\beta(t)\hat{q}_2(t) \mid \mathcal{E}_t] + \int_{\mathbb{R}} \gamma(t,z) E[\gamma(t,z)\hat{r}_2(t,z) \mid \mathcal{E}_t]\nu(dz) = 0$$

The condition for minimum point

$\hat{\theta} = (\theta_0, \theta_1)$ by minimizing the Hamiltonian function over all $\theta \in K_1$:

$$E[-\hat{X}_1(t)\hat{q}_1(t) \mid \mathcal{E}_t] = 0$$

and

$$\int_{\mathbb{R}} E[-\hat{X}_1(t)\hat{r}_1(t,z) \mid \mathcal{E}_t]\nu(dz) = 0$$

Try a process $\hat{p}_1(t)$ of the form

$$\hat{p}_1(t) = U(f(t)\hat{X}_2(t))$$

with f is a deterministic differentiable function.

Differentiating $\hat{p}_1(t)$, we get

$$
\begin{aligned}
d\hat{p}_1(t) = \Big\{ & f^{'}(t)\hat{X}_2(t)U^{'}(f(t)\hat{X}_2(t)) + \frac{1}{2}f^2(t)\hat{X}_2^2(t)\beta^2(t)\pi^2(t)U^{''}(f(t)\hat{X}_2(t)) \\
& + \hat{X}_2(t)(\rho(t) - (\alpha(t) - \rho(t))\hat{\pi}(t))f(t)U^{'}(f(t)\hat{X}_2(t)) \\
& + \int_{\mathbb{R}} \{U(\hat{X}_2(t)(f(t) + \hat{\pi}(t)\gamma(t,z)) - U(f(t)\hat{X}_2(t))) \\
& - \hat{X}_2(t)\hat{\pi}(t)\gamma(t,z)f(t)U^{'}(f(t)\hat{X}_2(t))\}\nu(dz)\Big\}dt \\
& + \hat{X}_2(t)\beta(t)\hat{\pi}(t)f(t)U^{'}(f(t)\hat{X}_2(t))dB(t) \\
& + \int_{\mathbb{R}} \{U(\hat{X}_2(t)(f(t) + \hat{\pi}(t)\gamma(t,z)) - U(f(t)\hat{X}_2(t)))\}\widetilde{N}(dt,dz)
\end{aligned}
$$

Comparing this with the first adjoint equation by equating the $dt, dB(t)$ and $\widetilde{N}(dt, dz)$ coefficients respectively, we get

$$\begin{aligned}\hat{q}_1(t) &= \hat{X}_2(t)\beta(t)\hat{\pi}(t)U'(f(t)\hat{X}_2(t)) \\ \hat{r}_1(t,z) &= U(\hat{X}_2(t)(f(t)+\hat{\pi}(t)\gamma(t,z))) - U(f(t)\hat{X}_2(t))\end{aligned}$$

and

$$\begin{aligned} & f'(t)\hat{X}_2(t)U'(f(t)\hat{X}_2(t)) + \frac{1}{2}\hat{X}_2^2(t)\beta^2(t)\pi^2(t)U''(f(t)\hat{X}_2(t)) \\ &\quad + \hat{X}_2(t)(\rho(t) - (\alpha(t) - \rho(t))\hat{\pi}(t))f(t)U'(f(t)\hat{X}_2(t)) \\ &\quad + \int_{\mathbb{R}} \{U(\hat{X}_2(t)(f(t)+\hat{\pi}(t)\gamma(t,z)) - U(f(t)\hat{X}_2(t))) \\ &\quad - \hat{X}_2(t)\hat{\pi}(t)\gamma(t,z)f(t)U'(f(t)\hat{X}_2(t))\}\nu(dz) \\ &\quad = \hat{\theta}_0(t)\hat{q}_1(t) + \int_{\mathbb{R}} \hat{\theta}_1(t,z)\hat{r}_1(t,z)\nu(dz)\end{aligned}$$

Substituting $\hat{q}_1(t)$ into the equation of minimum point we get **optimal portfolio**:

$$-\hat{\pi}(t)E[\hat{X}_1(t)\hat{X}_2(t)\beta(t)U^{''}(f(t)\hat{X}_2(t)) \mid \mathcal{E}_t] = 0$$

or

$$\hat{\pi}(t) = 0$$

Try the process $\hat{p}_2(t)$ of the form

$$\hat{p}_2(t) = \hat{X}_1(t)U^{'}(f(t)\hat{X}_2(t))f(t)$$

Differentiating $\hat{p}_2(t)$ and using $\hat{\pi}(t) = 0$ we get

$$\begin{aligned} d\hat{p}_2(t) = \hat{X}_1(t)\Big(& f^{'}(t)U^{'}(f(t)\hat{X}_2(t)) + f(t)f^{'}(t)\hat{X}_2(t)U^{''}(f(t)\hat{X}_2(t)) \\ &+ f^2(t)\hat{X}_2(t)\rho(t)U^{''}(f(t)\hat{X}_2(t))\Big)dt \\ &- f(t)\hat{X}_1(t)\theta_0(t)U^{'}(f(t)\hat{X}_2(t))dB(t) \\ &- \int_{\mathbb{R}} f(t)\hat{X}_1(t)\theta_1(t,z)U^{'}(f(t)\hat{X}_2(t))\widetilde{N}(dt,dz) \end{aligned}$$

Compare this with the second adjoint equation we get

$$\begin{aligned}\hat{q}_2(t) &= -f(t)\hat{X}_1(t)\theta_0(t)U^{'}(f(t)\hat{X}_2(t))\\ \hat{r}_2(t,z) &= -f(t)\hat{X}_1(t)\theta_1(t,z)U^{'}(f(t)\hat{X}_2(t))\end{aligned}$$

and

$$f^{'}(t)U^{'}(f(t)\hat{X}_2(t)) + f(t)\hat{X}_2(t)U^{''}(f(t)\hat{X}_2(t))(f^{'}(t)+f(t)\rho(t)) \\ = -\rho(t)f(t)U^{'}(f(t)\hat{X}_2(t))$$

$$\Updownarrow$$

$$\Big(U^{'}(f(t)\hat{X}_2(t)) + \hat{X}_2(t)f(t)U^{''}(f(t)\hat{X}_2(t))\Big)(f^{'}(t)+r(t)f(t)) = 0$$

$$\Updownarrow$$

$$f(t) = \exp\Big(\int_t^T r(s)ds\Big)$$

Substituting $\hat{q}_2(t), \hat{r}_2(t,z)$ into the equation of maximum point we get

$$\begin{aligned} &E[(\alpha(t)-\rho(t))f(t)\hat{X}_1(t)U^{'}(f(t)\hat{X}_2(t)) \mid \mathcal{E}_t] \\ &\quad -\theta_0(t)E[\beta(t)f(t)\hat{X}_1(t)U^{'}(f(t)\hat{X}_2(t)) \mid \mathcal{E}_t] \\ &\quad -\int_{\mathbb{R}}\theta_1(t,z)E[\gamma(t,z)f(t)\hat{X}_1(t)U^{'}(f(t)\hat{X}_2(t)) \mid \mathcal{E}_t] = 0 \end{aligned}$$

This can be written as

$$\hat{\theta}_0(t)E[\beta(t) \mid \mathcal{E}_t] - \int_{\mathbb{R}}\hat{\theta}_1(t,z)E[\gamma(t,z) \mid \mathcal{E}_t]\nu(dz) = E[(\alpha(t) \mid \mathcal{E}_t] - \rho(t)$$

Theorem

The optimal portfolio $\pi \in \Pi$ *for the agent is*

$$\pi(t) = \hat{\pi}(t) = 0$$

and the optimal measure $Q_{\hat{\theta}}$ *for the market is to choose* $\hat{\theta} = (\hat{\theta}_0, \hat{\theta}_1)$ *such that*

$$\hat{\theta}_0(t)E[\beta(t) \mid \mathcal{E}_t] - \int_{\mathbb{R}} \hat{\theta}_1(t,z)E[\gamma(t,z) \mid \mathcal{E}_t]\nu(dz) = E[(\alpha(t) \mid \mathcal{E}_t] - \rho(t)$$

Let $\mathcal{E}_t^1, \mathcal{E}_t^2$ be filtrations satisfying

$$\mathcal{E}_t^i \subseteq \mathcal{F}_t, \qquad t \geq 0, i = 1, 2$$

Let $u = (\theta, \pi)$, with $\theta = (\theta_0, \theta_1) \in \Theta$ are adapted to $\mathcal{E}_t^1$ and $\pi = (\pi_0, \pi_1) \in \Pi$ are adapted to $\mathcal{E}_t^2$, be admissible controls for player 1 and 2, respectively.

The costs function $J_i^{(\theta,\pi)}(x)$ associated with i is given by

$$J_i(\theta, \pi) = E^x\Big[\int_0^T f_i(t, X(t), u(t))\, dt + g_i(X(T))\Big], \qquad i = 1, 2.$$

Definition

A pair (θ^*, π^*) is called a Nash equilibrium for the stochastic game if the following holds:

$$\begin{aligned} J_1(\theta, \pi^*) &\leq J_1(\theta^*, \pi^*) \qquad \text{for all} \qquad \theta \in \Theta \\ J_2(\theta^*, \pi) &\leq J_2(\theta^*, \pi^*) \qquad \text{for all} \qquad \pi \in \Pi \end{aligned}$$

Assumption

(C1)

$$E[H_1(t, \hat{X}(t), \hat{\theta}(t), \hat{\pi}(t), \hat{p}_1(t), \hat{q}_1(t), \hat{r}^1(t,.)) \mid \mathcal{E}_t^1]$$
$$\geq E[H_1(t, \hat{X}(t), \theta(t), \hat{\pi}(t), \hat{p}_1(t), \hat{q}_1(t), \hat{r}^1(t,.)) \mid \mathcal{E}_t^1]$$

(C2)

$$E[H_2(t, \hat{X}(t), \hat{\theta}(t), \hat{\pi}(t), \hat{p}_2(t), \hat{q}_2(t), \hat{r}^2(t,.)) \mid \mathcal{E}_t^2]$$
$$\geq E[H_2(t, \hat{X}(t), \hat{\theta}(t), \pi(t), \hat{p}_2(t), \hat{q}_2(t), \hat{r}^2(t,.)) \mid \mathcal{E}_t^2]$$

Theorem

Suppose that $(\hat{p}_i(t), \hat{q}_i(t), \hat{r}^i(t,z))$, $i = 1, 2$ *is a solution of the corresponding adjoint equation and that satisfy* ***(C1)(C2)***. *Moreover, suppose that for all* $t \in [0,T]$, $H_i(t, x, \theta, \pi, \hat{p}_i(t), \hat{q}_i(t), \hat{r}^i(t,.))$, $i = 1, 2$, *is concave in* x, θ, π *and* $g_i(x)$, $i = 1, 2$, *is concave in* x. *Then* $(\hat{\theta}(t), \hat{\pi}(t))$ *is a Nash equilibrium point for the game and*

$$
\begin{aligned}
J_1(\hat{\theta}, \hat{\pi}) &= \sup\nolimits_{\theta \in \Theta} J_1(\theta, \hat{\pi}) \\
J_2(\hat{\theta}, \hat{\pi}) &= \sup\nolimits_{\pi \in \Pi} J_2(\hat{\theta}, \pi)
\end{aligned}
$$

For additional reading. . .

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