The Term Structure of CDO Losses with Lévy Random Fields

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This papers considers a general model for markets where CDOs are traded. Using model-free formulas the CDO prices can be traced back to prices of digital options, which therefore may be used as primary assets on the CDO market. Then a model for the forward rates based on Lévy random fields is proposed and conditions which ensure that the market is free of arbitrage are given. The model is able to capture the monotonicity of CDO tranches in a simple form.

1 Introduction

A collateralized debt obligation (CDO) is a security backed by a pool of reference entities such as bonds, loans or credit default swaps (CDSs). The reference entities form the *asset side* of the CDO. On the other side, there are issued notes on tranches of different seniorities which build the *liability side* of the CDO. In this work we consider the tranche prices as given and state a general model for the evolution of theses prices which are consistent with the market views and derive conditions under which the model is free of arbitrage.

Quite recently there emerged several attempts on this topic. Schönbucher (2005) uses a Loss process which lives on a discrete grid and arrives at a multivariate forward rate setting. Bennani (2005) and Sidenius, Piterbarg, and Andersen (2005) work with a continuous loss distribution. The paper of Sidenius, Piterbarg, and Andersen (2005) is closest to our setup. The new contribution of this work is the following: first, we provide an existence result which gives a general class of models which ensure the required monotonicity of forward loss rates. Second, we give an easy representation of tranches in terms of the forward rates.

This approach is some times called a top-down approach. The approach where first individual defaults are modelled and where then, typically with some dependence assumption (e.g. factor-approach or copula), the distribution of the loss process is computed is called bottom-up approach. It has been pointed out by Errais, Giesecke, and Goldberg (2006) that a top-down approach can be linked to individual default intensities via random thinning. The main reason for this is that under a no joint jump assumption the individual default intensities sum up to the jump intensity of the loss process. This paper generalizes Filipović, Overbeck, and Schmidt (2007) in two ways: first, we consider an infinite-dimensional random driver of the forward rate model and second we allow it to have jumps.

2 Setup

Consider a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ satisfying the usual conditions and assume there is an to $\mathbb{P}$ equivalent martingale measure Q. We consider a CDO whose overall nominal is normalized to 1. The losses in the CDO are described by the loss process $(l_t)_{t\geq 0}$, which is a càdlàg, adapted and non-decreasing process with values in $[0, 1]$. Typically, the loss

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process is assumed to be a marked point process with random, positive and bounded jump heights and at most a bounded number of jumps.

We place ourselves in the spirit of Filipović, Overbeck, and Schmidt (2007) hence do not consider CDOs directly, but rather start with some basic instruments which determine the prices of relevant instruments in an easy form. This basic instruments are so-called *digital options*, a security which pays 1 at T if less than $x \times 100\%$ of the nominal has defaulted until time T and zero otherwise. Denote its price by $C(t,T,x)$, where $0 \le t \le T$ and $x \in [0,1]$. Then $C(t,t,x) = \mathbb{1}_{\{l_t < x\}}$. Derivatives on the loss process or tranches of CDOs then can be priced on the basis of C. Our aim is to impose a general dynamics of C and show under which conditions this leads to an arbitrage-free model.

Consider a separable Hilbert space H which is a subspace of $L^2(\mathbb{R}^+ \times [0,1])$ and denote the inner product on H by $\langle \cdot,\cdot \rangle$ and its norm by $\| \cdot \|$. The elements of H will later determine $C(t,\cdot,\cdot)$, so for $h = h(T,x) \in H$ the parameter T denotes maturity while x refers to the credit riskiness in the CDO. Depending on the required smoothness one may choose a particular H, compare for example Filipović (2001). By $L(H)$ we denote the Banach space of linear operators on H. For $h \in H$ and $\Phi \in L(H)$ we write $\Phi \cdot h := \Phi(h)$. Let T^* be a finite time horizon. Throughout the paper, let $(L_s)_{s \in [0,T^*]}$ be a Lévy process with values in H which has second moments. The value of L at a certain time is an element of H and we write $L(t,T,x) := L_t(T,x)$.

We repeat some basic facts about Lévy random fields from Özkan and Schmidt (2006). First, the Lévy random field is determined by the characteristic function at $t=1$,

$$\phi_{L_1}(z) = \exp\Big(i\langle b_L, z\rangle - \frac{1}{2}\langle D_L \cdot z, z\rangle + \int_H \big(e^{i\langle z,x\rangle} - 1 - i\langle z,x\rangle 1_B(x)\big)\, F_L(dx) \Big),$$

where $d_L \in H$, D_L is an element of $L(H)$, the covariance operator of the continuous part of (L_t), and $B := \{h \in H : \| h \| < 1\}$. The Lévy measure F_L is a measure on H with $\int_H (\| x \|^2 \wedge 1)\, F_L(dx) < \infty$ and $F_L(\{0\}) = 0$. For any orthnormal basis $\{e_k : k \in \mathbb{N}\}$ of H we have the decomposition

$$L_t(u) = \sum_{k=1}^{\infty} \langle L(t), e_k\rangle e_k(u), \tag{1}$$

where the series converges in L^2. Furthermore the process $(\langle L_t, e_k\rangle)_{t\in[0,T^*]}$ is a real-valued Lévy process.

Of course we will make use of stochastic calculus. For more details on stochastic calculus w.r.t. Lévy random fields we refer to Peszat and Zabczyk (2006) and van Gaans (2005). First we introduce a suitable class of integrands.

Definition 2.1. Consider a Lévy process $(L_t)_{t\in[0,T^*]}$ with values in H. We call $\mathcal{L}_{T^*}(H;\tilde H)$ the space of all predictable processes $(\beta_t)_{t\in[0,T^*]}$ taking values in the space of linear operators from H to $\tilde H$, namely $L(H;\tilde H)$, such that the process $(\beta_t)_{t\in[0,T^*]}$ is locally bounded[2]. We write $\mathcal{L}_{T^*}(H;H) = \mathcal{L}_{T^*}(H)$.

Then, for $\sigma \in \mathcal{L}_T$ the process $(\int_0^t \sigma(u)\cdot dL_u)_{t\in[0,T^*]}$ is an H-valued semi-martingale and it is a local martingale, if (L_t) is. Finally, from Kunita (1970) we obtain the following Itô-formula for Lévy random fields

Theorem 2.2. *Let $(L_t)_{t\in[0,T^*]}$ be a Lévy process with values in the Hilbert space H and $\beta \in \mathcal{L}_{T^*}(H)$. Set $X_t := \int_0^t \beta(u)\cdot dL_u$ for all $t \in [0,T^*]$ and denote by λ_k and e_k, $k \in \mathbb{N}$ the*

[2] Here we consider the space $L(H;\tilde H)$ endowed with the operator topology, which makes it a Banach space.

eigenvalues and eigenvectors, respectively, of D_L. For an open subset $A \subset H$ and a function $F(t,x) : \mathbb{R}^+ \times A \to H$ which is once continuously differentiable in t and has a uniformly continuous second derivative in x on bounded subsets of H it holds, that

$$F(t,X_t) = F(0,X_0) + \int_0^t \frac{\partial}{\partial u} F(u,X_{u-})du + \int_0^t DF(u,X_{u-}) \cdot dX_u$$
$$+ \frac{1}{2} \int_0^t \sum_{k=1}^{\infty} \lambda_k D^2 F(u,X_{u-}) \cdot (\beta(u) \cdot e_k, \beta(u) \cdot e_k)\, du$$
$$+ \sum_{u \le t} \big[\Delta F(u,X_u) - \big(DF(u,X_{u-}) \cdot \Delta X_u \big) \big]. \tag{2}$$

3 An infinite factor Lévy model for the CDO

In this section we specify the model and give conditions which guarantee that Q is a martingale measure. If it is equivalent to the objective measure, then it is also an equivalent martingale measure. The risk-free market is given in terms of forward rates. The forward rates are given by a H-valued process $(f_t)_{t\in[0,T^*]}$, where $f(t,T,x) = f(t,T,y)$ for all $x,y \in [0,1]$, hence this process is flat in the last dimension. This approach is chosen to have a convenient representation of risky and risk-free term structures. Throughout we use the notation $f(t,T,x)$ for $f_t(T,x)$ whenever convenient.

In contrast to Özkan and Schmidt (2006) we do not consider a Musiela-parametrization in T, mainly because the dependence on y does not depend on maturity and therefore it is more concise to work with the HJM-parametrization in T. Therefore, we first consider the default-free market.

3.1 Default-free market

The model for the default-free case not in Musiela parametrization goes back to Björk, Di Masi, Kabanov, and Runggaldier (1997) and for the reader's convenience we give a short proof. Denote the risk-free forward rate by $f(t,T)$. Here $0 \le t \le T \le T^*$ and t denotes the current time, while T is maturity. The relation between bonds and forward rates is

$$B(t,T) = \exp\Big(- \int_t^T f(t,u)\, du \Big).$$

The risk-free rates of course do not depend on y, such that a sensible model consists of elements $f \in H$ with $f(t,T,y) = f(t,T,x)$ for all $x,y \in [0,1]$. Denote this subspace of H by $\tilde{H}$ and write $f(t,T)$ for $f(t,T,0)$.

Consider a Lévy process L with values in H. We assume that L is a martingale, more precisely, L has no drift and the jump part is already compensated. We assume that $\mathbb{E} \parallel L_1 \parallel < \infty$, such that L can be stated in the following form:

$$L_t = W_t + \int_0^t \int_H x\, (\mu^L - \nu^L)(ds,dx),$$

where W is a D^L-Wiener process on H and μ^L is the random measure of jumps with Q-compensator $\nu^L(ds,dx) = ds\, F(dx)$. That is, for any Borel set $\mathcal{T}$ of $\mathbb{R}^+$ and any Borel set of H,

μ^L denotes the number of jumps in the time interval $\mathcal{T}$ which have sizes in Λ,

$$\mu^L(\mathcal{T},\Lambda) = \sum_{s\in\mathcal{T}} 1_\Lambda(\Delta L_s).$$

We will need exponential moments for L, which are guaranteed by the following condition:

$$\int \mathbb{1}_{\{\|x\|>1\}} e^{\langle c,x\rangle}\nu(dx) < \infty, \quad \forall c \in H. \tag{3}$$

It has been shown in Jakubowski and Zabczyk (2005) that under some mild conditions the HJM-drift condition already ensures the existence of exponential moments.

Assumption 3.1. *Assume that for all $0 \le t \le T^*$, $(f_t)_{t\ge0}$ is a $\tilde{H}$-valued process, and the unique solution of*

$$f_t = f_0 + \int_0^t a_t dt + \int_0^t b_t \cdot dL_t, \tag{4}$$

where a is a predictable process with values in $\tilde{H}$, $b \in \mathcal{L}_{T^*}(H;\tilde{H})$ and $f_0 \in \tilde{H}$.

Absence of arbitrage is characterized in the following theorem. Define $a^*(t,T) := \int_t^T a_t(u)du$ and $b^*(t,T) \in L(\tilde{H})$ by $b^*(t,T)\cdot v := \int_t^T (b_t \cdot v)(s)\,ds$. As previously, denote the eigenvectors of D_L by $\{e_k : k \in \mathbb{N}\}$. Furthermore, set $b_k^*(t,T) := b^*(t,T)\cdot e_k$.

Proposition 3.2. *All discounted bond prices are local martingales, iff for all (t,T), such that $0 \le t \le T \le T^*$ the following condition holds Q-a.s.:*

$$0 = -a^*(t,T) + \frac{1}{2}\sum_{k=1}^{\infty}\lambda_k(\sigma_k^*(t,T))^2 + \int_H \Big(\exp\big(b^*(t,T)\cdot x\big) - 1 - b^*(t,T)\cdot x\Big) F_L(dx). \tag{5}$$

Proof. Define $D_t := \exp(-\int_0^t f(u,u)\,du)$ and, for fixed T, the mapping $F : \mathbb{R}^+ \times H$ by $F(t,v) := \int_t^T v(s)\,ds$. As F is a linear mapping we easily obtain its derivatives, namely $DF(t,v)\cdot u = \int_t^T u(s)ds$ and $D^2F = 0$. Moreover the partial derivative w.r.t. t equals $\partial/\partial t F(t,v) = -v(t)$, if $t \le T$.

Then the Itô -formula (2) yields

$$\begin{aligned} F(t,f_t) - F(0,f_0) &= -\int_0^t f(s,s)ds + \int_0^t\int_t^T a(s,u)\,du\,ds + \int_0^t\int_t^T b(s,u)\cdot dL_s^c\,du + \sum_{s\le t}\Delta F(s,f_s) \\ &= -\int_0^t f(s,s)ds + \int_0^t a^*(s,T)\,ds + \int_0^t b^*(s,T)\cdot dL_s \end{aligned}$$

as we have that $\Delta F(s,f_s) = \int_s^T \Delta f_s(u)du$. Secondly, we have that $D_tB(t,T) = D_t\exp(-F(t,f_t))$. Note that D_t is of finite variation. Write short $r_t = f(t,t)$. Then, using the standard Itô -formula we get

$$
\begin{aligned}
D_tB(t,T) - B(0,T) &= -\int_0^t r_s D_s B(s,T)\,ds - \int_0^t D_s B(s,T) d(F(s,f_s)^c) \\
&+ \frac{1}{2}\int_0^t D_s B(s-,T) d\langle F(\cdot, f.)^c\rangle_s + \sum_{s\le t} D_s \Delta B(s,T) \\
&= \int_0^t (f(s,s) - r_s) D_s B(s,T)\,ds \\
&- \int_0^t D_s B(s,T) a^*(s,T) ds - \int_0^t D_s B(s,T) b^*(s,T) dL_s^c \\
&+ \frac{1}{2}\sum_{k=1}^{\infty} \lambda_k \int_0^t D_s B(s,T)(b_k^*(s,T))^2 ds + \sum_{s\le t} D_s \Delta B(s,T),
\end{aligned}
$$

where we used $b_k^*(s,T) = b^*(s,T)\cdot e_k$. Later it will be important to distinguish between r_s and $f(s,s)$ such that we keep the first term, even if it is zero here. Next,

$$
\begin{aligned}
\Delta B(s,T) &= B(s-,T)\left[\frac{B(s,T)}{B(s-,T)} - 1\right] \\
&= B(s-,T)\Big[\exp\Big(\int_s^T \Delta f_s(u)\,du\Big) - 1\Big] \\
&= B(s-,T)\Big[\exp\Big(\int_s^T (b_s\cdot \Delta L_s)(u)\,du\Big) - 1\Big] \\
&= B(s-,T)\Big[\exp\Big(b^*(s,T)\cdot \Delta L_s\Big) - 1\Big].
\end{aligned}
$$

Hence the dynamics of the discounted bond price computes to

$$
\begin{aligned}
D_tB(t,T) - B(0,T) &= \int_0^t (f(s,s) - r_s) D_s B(s,T)\,ds \\
&- \int_0^t D_s B(s,T) a^*(s,T) ds + \frac{1}{2}\sum_{k=1}^{\infty}\lambda_k \int_0^t D_s B(s,T)(b_k^*(s,T))^2 ds \\
&+ \int_0^t \int_H D_s B(s-,T)\Big[\exp\big(b^*(s,T)\cdot x\big) - 1 - b^*(s,T)\cdot x\Big] F_L(dx) ds \\
&- \int_0^t D_s B(s-,T) b^*(s,T) dL_s \\
&+ \int_0^t \int_H D_s B(s-,T)\Big[\exp\big(b^*(s,T)\cdot x\big) - 1 - b^*(s,T)\cdot x\Big]\big(\mu^L(ds,dx) - \nu^L(ds,dx)\big), \quad (6)
\end{aligned}
$$

where the terms in the last two lines are local martingales. For the discounted bond price to be a martingale it is necessary, that the first two lines are equal to zero and deriving w.r.t. t we arrive at the drift condition (5). On the other side, if (5) holds, than clearly $(D_t B(t,T))_{0\le t\le T}$ are local martingales. ∎

3.2 The defaultable market

Now we are in the position to state the model for the defaultable market and derive the drift conditions. Recall that we consider a market consisting of all digital options, hence options with payoff $\mathbb{1}_{\{L_T<y\}}$, whose prices were denoted by $C(t,T,y)$.

Assumption 3.3. *Assume that for $x\in[0,1)$ and all let $\le T\le T^*$, digital options may be represented as*

$$C(t,T,1-x)=\mathbb{1}_{\{L_t<1-x\}}\exp\Big[-\int_t^T\Big(f(t,u)+\phi(t,u,x)\Big)\,du\Big], \tag{7}$$

where ϕ is a H-valued process which is the unique solution of

$$\phi_t=\phi_0+\int_0^t\alpha(u)\,du+\int_0^t\beta(u)\cdot dL_u; \tag{8}$$

here α is a predictable H-valued process and $\beta\in\mathcal{L}_{T^*}(H)$. Finally, we assume that for any $x\in[0,1]$ there exists a nonnegative, predictable process $(\lambda^x(t))_{t\ge0}$, s.t. the following process is a martingale

$$\mathbb{1}_{\{L_t<1-x\}}+\int_0^t\lambda^x(u)\mathbb{1}_{\{L_{u-}<1-x\}}\,du. \tag{9}$$

The main result of this section gives conditions under which the measure Q is a martingale measure.

Theorem 3.4. *Assume that the risk-free market is free of arbitrage and Assumptions 3.3 and 3.6 hold. Then, all discounted digital options are martingales under Q, iff the following two conditions hold:*
(i) For any $x\in[0,1)$ and $t\ge0$, such that $L_t<1-x$,

$$\phi(t,t,x)=\lambda^x(t). \tag{10}$$

(ii) For any $x\in[0,1)$ and $0\le t\le T$, such that $L_t<1-x$,

$$\begin{aligned}0=&-\alpha^*(s,T,x)+\frac{1}{2}\sum_{k=1}^{\infty}\lambda_k\int_0^t\Big(2b_k^*(s,T)\beta_k^*(s,T,x)+(\beta_k^*(s,T,x))^2\Big)\\&+\int_{\tilde H}\Big[\exp\big(b^*(s,T)\cdot z\big)\Big(\exp\big(\beta^*(s,T,x)\cdot z\big)-1\Big)-\beta^*(s,T,x)\cdot z\Big]\,F_L(dz).\end{aligned} \tag{11}$$

Proof. We need some additional notation. Set

$$P(t,T,x):=\exp\Big(-\int_t^T\big(f(t,u)+\phi(t,u,x)\big)\,du\Big),$$

such that $C(t,T,x) = \mathbb{1}_{\{L_t<1-x\}}P(t,T,x)$. In the following we compute the dynamics of $D_tC(t,T,x)$. First, consider $0 \le t \le T \le T^*$ and $x \in [0,1)$. The dynamics of the discounted digital options is

$$\begin{aligned} d\big(D_tC(t,T,x)\big) &= d\big(\mathbb{1}_{\{L_t<1-x\}}D_tP(t,T,x)\big) \\ &= D_tP(t-,T,x)d\mathbb{1}_{\{L_t<1-x\}} + \mathbb{1}_{\{L_{t-}<1-x\}}d(D_tP(t,T,x)) \\ &= D_tP(t-,T,x)dM_t - D_tP(t,T,x)\lambda^x(t)\mathbb{1}_{\{L_t<1-x\}}\,dt \\ &+ \mathbb{1}_{\{L_{t-}<1-x\}}d(D_tP(t,T,x)), \end{aligned} \tag{12}$$

because $\mathbb{1}_{\{L_t<1-x\}}$ is of bounded variation and $(M_t)_{t\ge0}$ is the martingale given in (9).

Finally, we need to compute the dynamics of DP itself. Basically, we will be able to use the results of the default-free case for this. If we set $\tilde f(t,T,x) := f(t,T) + \phi(t,T,x)$, then

$$\tilde f_t - \tilde f_0 = \int_0^t \big(a_s+\alpha_s\big)\,ds + \int_0^t \big(b_t+\beta_t\big)\cdot dL_s.$$

Denote $A_t := a_t+\alpha_t$ and $B_t := b_t+\beta_t$. Recall that we use the convention $f(t,T) = f(t,T,0)$ and so by $a_s(T,x) = a_s(T,y) = a_s(T,0)$, a_s can be considered as an H-valued process, and similarly b_s may be viewed as a process with values in $\mathcal{L}_{T^*}(H)$. Next, we will use (6) to compute the dynamics of DP. We obtain

$$\begin{aligned} D_tP(t,T,x) - P(0,T,x) &= \int_0^t (\tilde f(s,s) - r_s)D_sP(s,T,x)\,ds \\ &- \int_0^t D_sP(s,T,x)A^*(s,T,x)ds + \frac{1}{2}\sum_{k=1}^\infty \lambda_k \int_0^t D_sP(s,T,x)(B_k^*(s,T,x))^2ds \\ &+ \int_0^t\int_H D_sP(s-,T,x)\Big[\exp\big(B^*(s,T,x)\cdot x\big) - 1 - B^*(s,T,x)\cdot z\Big]\,F_L(dz)ds \\ &+ d\tilde M_t \end{aligned} \tag{13}$$

where $\tilde M$ is a local martingale. As DP is a local martingale, if and only if its drift term vanishes, from (12) we obtain the following drift condition (using $D_tP(s,T,x) > 0$):

$$\begin{aligned} 0 = \tilde f(s,s) - r_s - \lambda^x(t) - A^*(s,T,x) + \frac{1}{2}\sum_{k=1}^\infty \lambda_k \int_0^t (B_k^*(s,T,x))^2 \\ + \int_H \Big[\exp\big(B^*(s,T,x)\cdot z\big) - 1 - B^*(s,T,x)\cdot z\Big]\,F_L(dz). \end{aligned}$$

First, letting $T = t$ and using that $\tilde f(t,t) = r_t + \phi(t,t,x)$ we obtain condition (i). As a second step we insert the risk-free drift condition (5) and deduce

$$\begin{aligned} 0 = -\alpha^*(s,T,x) + \frac{1}{2}\sum_{k=1}^\infty \lambda_k \int_0^t \Big(2b_k^*(s,T)\beta_k^*(s,T,x) + (\beta_k^*(s,T,x))^2\Big) \\ + \int_{\tilde H} \Big[\exp\big(b^*(s,T)\cdot z\big)\Big(\exp\big(\beta^*(s,T,x)\cdot z\big) - 1\Big) - \beta^*(s,T,x)\cdot z\Big]\,F_L(dz). \end{aligned}$$

For the converse, we need to show that (10) and (11) imply that all discounted digital options are local martingales. For fixed x, these conditions imply that on $\{L_t < 1-x\}$ discounted prices are local martingales. On the other side, on $\{L_t \geq 1-x\}$ the prices are zero by definition, and hence martingales. The conclusion follows. ■

Special cases In Filipović, Overbeck and Schmidt the case where L is continuous and finite dimensional considered: assume W is an n-dimensional Wiener process, and that the dynamics of f and ϕ are given by

$$df(t,T) = a(t,T)\,dt + b(t,T)^\top\,dW_t \tag{14}$$

$$d\phi(t,T,x) = \alpha(t,T,x)\,dt + \beta(t,T,x)^\top\,dW_t. \tag{15}$$

where a, b and α, β are predictable processes with values in $\mathbb{R}$ and $\mathbb{R}^n$, respectively, jointly continuous in (t,T). Furthermore, for all $0 \leq t \leq T$ and $x \geq 0$ it holds $\mathbb{P}$-a.s. that

$$\int_0^t \Big[\int_t^T \parallel b(u,v) + \beta(u,v,x) \parallel^2 \; dv\,du < \infty. \tag{16}$$

The drift condition (11) then simplifies to

$$\alpha(t,T,x) = b(t,T)^\top \int_t^T \beta(t,u,x)\,du + \beta(t,T,x)^\top \int_t^T \Big(b(t,u) + \beta(t,u,x)\Big)\,du. \tag{17}$$

Independence of risk-free and defaultable rate In the case of independence of f and ϕ, and under the assumption that the risk-free market is free of arbitrage, concluding that the CDO-market is free of arbitrage is equivalent to the condition that the digital options are local martingales. As the assumption of independence is typically used in CDO modelling, we examine this approach more closely.

On the one hand it is immediate, that digital options being local martingales is equivalent to (10) and (11), where in the second equation we have $b^* \equiv 0$. In the following we examine conditions on b and β which suffice for independence and study their impact on the drift condition. The main tool for this is the series representation of L given in (1). It is well-known, that if L is a continuous process, the coefficients $\langle L(t), e_k\rangle$ are independent of each other (compare **?**), Proposition 4.1). In the Lévy -case this property does not hold anymore. However, a sufficiently rich class of Lévy -processes is obtained by imposing this property.

Definition 3.5. A H-valued Lévy process L has the *independent decomposition property*, if there exists an orthonormal basis $\{e_k : k \in \mathbb{N}\}$ of H, such that the processes $(\langle L(t), e_k\rangle)_{t\geq 0}$ are independent.

Now, if L has the independent decomposition property w.r.t. $\{e_k : k \in \mathbb{N}\}$ the following is sufficient to guarantee independence of f and ϕ: For all $t \geq 0$ and $k \in \mathbb{N}$ either $b_t \cdot e_k = 0$ or $\beta_t \cdot e_k = 0$. Inserting this in (11), we obtain $b_k^(t,T)\beta_k^*(t,T) = 0$. Second for $x \in H$, we write $x = \sum a_k e_k$ and hence $b^*(t,T) = \sum a_k b_k^*(t,T)$ as well as $\beta^*(t,T) = \sum a_k \beta_k^*(t,T)$. From this, we also obtain that

$$\exp\big(b^*(t,T)\cdot x\big)\big(\exp\big(\beta^*(t,T)\cdot x\big) - 1\big) = \big(\exp\big(\beta^*(t,T)\cdot x\big) - 1\big) \tag{18}$$

and hence the drift condition coincides with the drift condition under $b = 0$. On the other other side, (18) implies that for each x either $\exp(b^*(t,T)\cdot x) = 1$ or $\exp(\beta^*(t,T)\cdot x) = 1$.

Then this also holds for any e_k. In turn, $b^*(t,T) \cdot e_k = \int_t^T (b_t \cdot e_k)(u)du = 0$ has to hold for all $0 \le t \le T \le T^*$ to guarantee that the model is free of arbitrage in general and we obtain for all k, t that either $(b_t \cdot e_k)(u) = 0$ or $(\beta_t \cdot e_k)(u) = 0$ Lebesgue-almost surely for any $u \in [t, T^*]$. Thus, if we assume that $b_t \cdot x$ as well as $\beta_t \dot{x}$ vanish on $[0,t)$ we obtain also necessity.

Continuously increasing loss process If the loss process heights are random, it makes sense to assume that ϕ is increasing and absolutely continuous in x,

$$\phi(t,T,x) = \phi(t,T,0) + \int_0^x \phi_x(t,T,y)\,dy. \tag{19}$$

If ϕ is moreover differentiable, that ϕ_x is the partial derivative w.r.t. x. Monotonicity of ϕ in x is simply assured by assuming that $\phi_x \ge 0$.

In the special case of continuity we have the following:

Assumption 3.6. *Assume that*

$$d\phi_x(t,T,x) = \alpha_x(t,T,x)dt + \beta_x(t,T,x)^\top dW_t,$$

where α_x and β_x are the partial derivatives of α and β, respectively. Furthermore we assume that $\phi_x \ge 0$.

Lemma 3.7. *Under Assumption 3.6 we have that, for any $x \in (0,1]$ and $0 \le t \le T < \infty$,*

$$\phi(t,T,x) = \phi(t,T,0) + \int_0^t \alpha(s,T,x)\,ds + \int_0^t \beta(s,T,x)^\top dW_s.$$

Proof. The result follows from Equation (19) using the stochastic Fubini theorem, see Protter (2004, Theorem 64). Observe that continuity of α_x and β_x implies boundedness. ∎

Specifying the dynamics for ϕ_x easily yields the dynamics of ϕ itself. From the above result we immediately obtain a condition for the positive process $\phi_x(t,T,x)$.

Corollary 3.8. *From the second drift condition, Equation* (11)*, it follows that*

$$\begin{aligned}\alpha_x(t,T,x) &= \Big(b(t,T) + \beta(t,T,x)\Big)^\top \int_t^T \beta_x(t,u,x)\,du \\ &\quad + \beta_x(t,T,x)^\top \int_t^T \Big(b(t,u) + \beta(t,u,x)\Big)\,du. \end{aligned} \tag{20}$$

3.1.1 Further no-arbitrage considerations and special cases

It is interesting that in special cases no-arbitrage restrictions lead to further consistency conditions. First, consider the case where $L(t) \equiv 0$, such that the underlying CDO is risk-free. Then all digital options $C(t,T,x)$ offer the same payoff at time T, namely 1 and hence must have the same price at time t. Besides this, if the loss process has a discrete distribution, similar implications can be drawn. For example, assume that there is at most one default and the loss at default is equal to the constant $q \in (0,1)$. In this case, the tranches above q are risk-free and the above considerations apply. On the other side, two different tranches below q offer the same payoff and hence have the same price at any time $0 \le t \le T$. In the following we formalize these ideas.

We first consider the case when there is no risk for the CDO. We assume that Assumption 3.3 and 3.6 hold and assume that Q is a martingale measure, i.e. Equations (10) and (11) hold.

Proposition 3.9. *Consider a fixed maturity time T and assume that $\mathbb{P}(L_t = 0) = 1$. Then, at any $t \le T$ the prices of digital tranche options coincide for all $x \in [0, 1)$ which is equivalent to*

$$\phi(t, T, x) = 0, \qquad \text{for } x \in [0, 1).$$

Proof. If no loss occurs, i.e. $L_T \equiv 0$, the digital tranche options pay $\mathbb{1}_{\{L_T > 1-x\}} = 1$ at maturity T. Moreover, discounted prices of these options are martingale by assumption, hence their present value equals the expectation of their discounted payoffs. Consider $x \ne y \in [0, 1)$. Then

$$C(t, T, x) = \mathbb{E}^Q\Big(e^{-\int_t^T r_u\,du} C(T, T, x)\Big|\,\mathcal{F}_t\Big) = \mathbb{E}^Q\Big(e^{-\int_t^T r_u\,du}\Big|\,\mathcal{F}_t\Big) = C(t, T, y)$$

and it follows that all tranche options have the same price. Of course, the price coincides with the price of the risk-free bond, $\exp\Big(-\int_t^T f(t, u)\,du\Big)$ and equation (7) reveals the equivalence to $\phi \equiv 0$. ■

Proposition 3.10. *Assume that L_t has only values in $\{q_1, \ldots, q_N\}$ with $0 = q_1 < \cdots < q_N \le 1$, where $Q(L_T = q_i) > 0$ for all $i = 1, \ldots, N$. Then, for all $0 \le t \le T$ and $x \ne y \in (q_i, q_{i+1}]$,*

$$C(t, T, x) = C(t, T, y). \tag{21}$$

Proof. The conclusion follows as in the previous proposition from

$$\{L_T > 1 - x\} = \{L_T > 1 - q_{i+1}\} = \{L_T > 1 - y\}.$$ ■

Note that Equation (21) implies that $\phi(t, T, \cdot)$ is piecewise constant.

As a special case we obtain a connection to zero-recovery bonds when we choose $N = 1$. The tranche above q_1 is again risk-free, while the digital tranche option below q act as zero-recovery bonds. We fix $x > 1 - q$, such that we can drop dependence on x from α and β and we obtain from (11) that

$$\begin{aligned} a(t,T) &+ \alpha(t,T,x) \\ &= b(t,T)^\top \int_t^T b(t,u)\,du + b(t,T)^\top \int_t^T \beta(t,u)\,du + \beta(t,T)^\top \int_t^T \Big(b(t,u) + \beta(t,u)\Big)\,du \\ &= \Big(b(t,T) + \beta(t,T)\Big)^\top \int_t^T \Big(b(t,u) + \beta(t,u)\Big)\,du, \end{aligned}$$

which is exactly the drift condition, given e.g. by Duffie and Singleton (1999, eq. (40)).

The model proposed by Schönbucher (2005) also falls in the category given by Proposition 3.10 and a similar drift condition was obtained, compare Equation (3.11) therein.

3.1.2 Implications of the no-arbitrage conditions

It is interesting to examine, how the specification of λ^x relates to the loss process. To this, we examine some examples for the dynamics of the loss processes.

1. *L is a compound Poisson process.* Assume, that $L_t = \sum_{i=1}^{N_t} U_i$, where $U_i > 0$ are i.i.d. with d.f. F_U and N_t is a Poisson process, independent of all U_i with intensity λ. In this case L_t is unbounded. However, relevant for the loss of the CDO is only $L \wedge 1$ and so the λ_t^x give only information about the jump size up to $1 - L_{t-}$. Observe that $\lambda^{1-x}(t) = 0$ for $x \le L_{t-}$ and $\lambda^{1-x}(t) = \lambda \cdot P(L_{t-} + U_i \ge x) = \lambda(1 - F_U(x - L_{t-}))$ otherwise. Hence $\lambda = \lim_{x \downarrow L_{t-}} \lambda^{1-x}(t)$ and the distribution of the next jump, bounded by $1 - L_{t-}$ is given by
$$F_U(y) = 1 - \frac{\lambda^{1-L_{t-}-y}(t)}{\lambda}, \qquad y \in [0, 1 - L_{t-}].$$
Note that the left hand side does not depend on t, so $\lambda^x(t)$ has quite restricted dynamics.

2. *L is a marked point process.* Assume that L is a marked point process with positive jumps and $L_t \in [0,1]$ for all t. We denote the local characteristics of L by $(\lambda_t, m_t(du))$. The different indexation of $\lambda^x(t)$ and λ_t should suffice to avoid confusion. Analogously to the previous case we have that $\lambda^{1-x}(t) = 0$ for $x \le L_{t-}$ and
$$\lambda^{1-x}(t) = \lambda_t \int_{x-L_{t-}}^{1-L_{t-}} m_t(du) = \lambda_t \int_{x-L_{t-}}^{1} m_t(du). \tag{22}$$
The last equation follows because $L_t \in [0,1]$ and therefore $m_t((1 - L_{t-}, \infty)) = 0$. As m_t is a probability measure with support $[0,1]$, we obtain that $\lambda_t = \lim_{x \downarrow L_{t-}} \lambda^{1-x}(t)$ and m_t is given by
$$m_t((0,y]) = 1 - \frac{\lambda^{1-L_{t-}-y}(t)}{\lambda_t}.$$

3.2 Options on CDOs

Forward - starting CDOs In this section we examine the question of pricing forward starting CDOs on the basis of $C(0,T,x)$. Fix $t = 0$ and $r = 0$.

1. In a forward starting CDO on $(x_1, x_2]$ with starting point $T \le T_1$ and payment dates $\mathbf{T} = (T_1, \ldots, T_N)$ the spread is agreed today, while the lower and attachment points will be fixed just at T to $(x_1 + L_{T_1}, x_2 + L_{T_1}]$, such that the tranche shifts upward by the losses occurred until time T_1. Compare e.g. Ehlers and Schönbucher (2006).

   Lemma 3.11. *If L is a (inhomogeneous) compound Poisson process with intensity $\lambda(\cdot)$, then the spread of the forward starting CDO is*
$$\frac{\int_{x_1}^{x_2} \Big(\mathbb{1}_{\{L_t < y\}} - C\big(t, \sigma(T,T_n), y\big) \Big)\, dy}{\sum_{i=1}^n \frac{1}{x_2 - x_1} \int_{x_1}^{x_2} C\big(t, \sigma(T,T_i), y\big)\, dy}.$$

   Proof. First,
$$\mathbb{E}\big(\mathbb{1}_{\{L_{T_2} < y + L_{T_1}\}}\big) = \mathbb{E}\big(\mathbb{1}_{\{L_{T_2} - L_{T_1} < y\}}\big)$$

thus, we need to be able to value differences of L over a certain time period. To this, stationary assumptions or similar tools are useful. In our case, the U_i are i.i.d. and hence $L_{T_2} - L_{T_1}$ and $\sum_{i=1}^{N_{T_2}-N_{T_1}} U_i$ have the same distribution. Moreover, $N_{T_2} - N_{T_1} \sim \text{Poisson}(\Lambda_{T_2} - \Lambda_{T_1})$. Letting $\sigma = \sigma(T_1, T_2) = \Lambda^{-1}(\Lambda_{T_2} - \Lambda_{T_1})$ we obtain that

$$\mathbb{E}\big(\mathbb{1}_{\{L_{T_2} < y + L_{T_1}\}}\big) = C\big(0, \sigma(T_1, T_2), y\big).$$

E.g., if $\lambda(s) = \lambda$, then $\sigma = T_2 - T_1$ and we conclude. ■

In which cases can we extract $\lambda(\cdot)$ from the observed digital option prices? On easy example is the following: assume that $L_t = \sum_{i=1}^{N_t} U_i$ with N being a inhomogeneous Poisson process with intensity $\lambda(t)$, $t \geq 0$. First, we compute $C(0, T, x)$. To this we have to compute $Q(\sum_{i=1}^{N_t} U_i)$ which involves the convolution of the U_i. In a first step we assume $U_i \sim \mathcal{N}(\mu, \sigma^2)$. Then $L_t \sim \mathcal{N}(\Lambda_T \mu, \lambda_T \sigma^2)$ where $\Lambda_T = \int_0^T \lambda(s)ds$. Overall we obtain

$$C(t, T, y) = \Phi\left(\frac{y - \mu \int_t^T \lambda(s)ds}{\sigma\sqrt{\int_t^T \lambda(s)ds}}\right)$$

and solving a quadratic equation we may compute $\int_t^T \lambda(s)ds$ from $C(t, T, y)$. This suffices to price the forward starting CDO as we shown above.

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