The term structure of CDO Losses

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joint work with D. Filipović and L. Overbeck

Crashcourse in CDOs

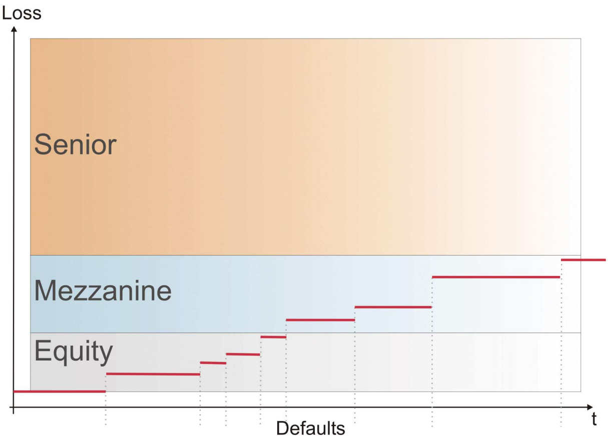
- A CDO is a pool of m defaultable objects, with notionals N_i, $\sum_{i=1}^m N_i = 1$
- Default times are denoted τ_i, loss on default of i by $l_i \in [0,1]$
- Cumulative loss:

$$L(t) = \sum_{i=1}^{m} l_i N_i 1_{\{\tau_i \le t\}}$$

- Loss is split into tranches: a tranche refers to an interval $(x_i, x_{i-1}] \subset [0,1]$,

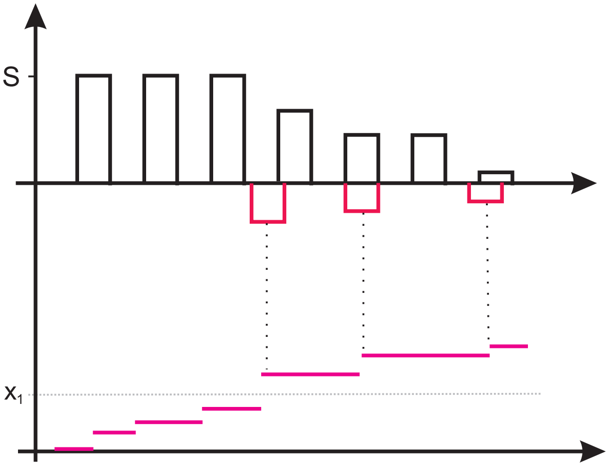
$$0 = x_0 < x_1 < \cdots < x_k = 1$$

Partition of losses into tranches



Pricing a tranche

Investing in a tranche $(x_1, x_2]$ is done via a swap. Set $\mathbf{T} = (T_1, \dots, T_n)$. The coupon is paid at beginning of a period, while the loss payments are exchanged at the end of a period:


S
x1

Pricing a tranche

Investing in a tranche $(x_1, x_2]$ is done via a swap. Set $\mathbf{T} = (T_1, \ldots, T_n)$. The coupon is paid at beginning of a period, while the loss payments are exchanged at the end of a period:

1. The investor receives at T_{i-1}, $2 \leq i \leq n$ the payment

$$S \cdot \left(1_{\{L_{T_{i-1}} \leq x_1\}} + \frac{(x_2 - L_{T_{i-1}})^+}{x_2 - x_1} 1_{\{L_{T_{i-1}} > x_1\}} \right).$$

2. In turn, the investor has to cover eventual losses, i.e. pays at T_i, $2 \leq i \leq n$

$$\left(\left(x_2 - L_{T_{i-1}}\right)^+ - \left(x_2 - L_{T_i}\right)^+ \right) 1_{\{L_{T_i} > x_1\}}$$

Define $f_{PS}(l, x_1, x_2) := (x_2 - l)^+ - (x_1 - l)^+ = \int_{x_1}^{x_2} 1_{\{l<y\}}\, dy$. Then

Receive:	$S \cdot \dfrac{f_{PS}(L_{T_{i-1}}, x_1, x_2)}{x_2 - x_1}$	at T_{i-1}
Pay:	$f_{PS}(L_{T_{i-1}}, x_1, x_2) - f_{PS}(L_{T_i}, x_1, x_2)$	at T_i

Digital options

- Filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, Q)$, Q is pricing measure
- Fundamental property: $L_s \leq L_t$ for $s < t$
- Consider digital option

$$C(t,T,y) = \mathbb{E}\left(\exp(-\int_t^T r_u du) 1_{\{L_T < y\}} \Big| \mathcal{F}_t\right)$$

1. $L_t > y \Rightarrow C(t,T,y) = 0$
2. $C(t,T,y_1) \leq C(t,T,y_2)$ for $y_1 < y_2$

Lemma

Consider a derivative with payoff $F(L_T)$ at T, and assume that F is absolutely continuous w.r.t. the Lebesgue-measure, i.e. $F(x) = F(1) - \int_0^1 f(y) 1_{\{x<y\}} dx$. Then the price of this derivative at time t is given by

$$B(t,T)F(1) - \int_0^1 f(y)\, C(t,T,y)\, dy.$$

Easy example:

$$(K-L)^+ = \int_0^K 1_{\{L<y\}}\, dy$$

Hence, we are able to price puts, calls and similar European options on L_T once $C(t,T,\cdot)$ is available.

We have

Proposition

Assume that $r = 0$ and $T_1 = t$. Then the value of the CDS with spread S investing in the tranche $(x_1, x_2]$ equals

$$V(t, \mathbf{T}, S, x_1, x_2) = \sum_{i=1}^{n-1} \frac{S}{x_2 - x_1} \int_{x_1}^{x_2} C(t, T_i, y)\, dy - \int_{x_1}^{x_2} \Big(B(t, T_2) 1_{\{L_t < y\}} - C(t, T_n, y) \Big)\, dy$$

Solving $V = 0$ for S gives the fair spread.

Drift condition: general setup

Assumption 1 Assume

$$C(t,T,1-x) = 1_{\{L_t<1-x\}} \exp\Big(- \int_t^T \Big(f(t,u) + \phi(t,u,x) \Big)\, du \Big). \qquad (1)$$

Consider a separable Hilbert space of real valued functions on $[0,T^*] \times [0,1]$. The dynamics of f and ϕ are given by

$$f_t = f_0 + \int_0^t a_u\, du + \int_0^t b_u \cdot dL_u, \qquad (2)$$

$$\phi_t = \phi_0 + \int_0^t \alpha_u\, du + \int_0^t \beta_u \cdot dL_u \qquad (3)$$

where a, b and α, β are predictable processes with values in H and $L(H)$, L be a H-valued Lévy process (L has stationary and independent increments), and the risk-free market is free of arbitrage.

Note that $C(t,T,1-x)$ now should be non-increasing in x. Furthermore,

$$1_{\{L_t<1-x\}} + \int_0^t \lambda^x(u) 1_{\{L_{u-}<1-x\}}\, du$$

is a martingale.

The Lévy process L is determined by the characteristic function at $t = 1$,

$$\phi_{L_1}(z) = \exp\Big(i\langle b_L, z\rangle - \frac{1}{2}\langle D_L \cdot z, z\rangle + \int_H \big(e^{i\langle z,x\rangle} - 1 - i\langle z, x\rangle 1_B(x)\big)\, F_L(dx)\Big),$$

where $b_L \in H$, D_L is an element of $L(H)$, the covariance operator of the continuous part of (L_t), and $B := \{h \in H : \|h\| < 1\}$. F_L is the Lévy measure. In the following we denote by $\{e_k : k \in \mathbb{N}\}$ the eigenvectors of D_L.

Under technical assumptions (exponential moments, ...) we have:

Theorem

All discounted digital options are martingales under Q, iff:
(i) For any $x \in [0,1]$ and $t \geq 0$, such that $L_t < 1 - x$,

$$\phi(t,t,x) = \lambda^x(t). \tag{4}$$

(ii) For any $x \in [0,1]$ and $0 \leq t \leq T$, such that $L_t < 1 - x$,

$$\alpha^*(s,T,x) = \frac{1}{2}\sum_{k=1}^{\infty}\lambda_k \int_0^t \Big(2b_k^*(s,T)\beta_k^*(s,T,x) + (\beta_k^*(s,T,x))^2\Big)$$
$$+ \int_H \Big[\exp\big(b^*(s,T)\cdot z\big)\Big(\exp\big(\beta^*(s,T,x)\cdot z\big) - 1\Big) - \beta^*(s,T,x)\cdot z\Big] F_L(dz).$$

where $\alpha^*(t,T,x) := \int_t^T \alpha(t,u,x)du$, also for b, β and moreover $\beta_k(t,T) := \int_t^T (\beta_t \cdot e_k)(u,x)du$.

Special case

In the case where L is a n-dimensional Brownian motion the drift condition reads

$$\alpha(t,T,x) = b(t,T)^\top \int_t^T \beta(t,u,x)\,du + \beta(t,T,x)^\top \int_t^T \Big(b(t,u) + \beta(t,u,x) \Big)\,du.$$

One possibility to guarantee monotonicity of $C(t, T, \cdot)$ is to assume that

$$\phi(t, T, x) = \alpha(t, T, x) + L(t, T, x)$$

where L is a subordinator in its coordinate x.

Another possibility is to assume that

$$\phi(t, T, x) = \int_0^x \phi_x(t, T, z) dz$$

with $\phi_x \geq 0$. In the continuous case, this implies a drift condition on ϕ_x:

$$\begin{aligned} \alpha_x(t, T, x) = &\Big(b(t, T) + \beta(t, T, x)\Big)^\top \int_t^T \beta_x(t, u, x)\, du \\ &+ \beta_x(t, T, x)^\top \int_t^T \Big(b(t, u) + \beta(t, u, x)\Big)\, du. \end{aligned}$$

Affine Specification

It is also possible, to impose an affine structure:

$$C(t,T,x) = \mathbb{E}^Q\Big(\exp(-\int_t^T r_s ds) 1_{\{L_T < x\}}\Big).$$

Chosse (r, L) to have an affine structure, i.e.

$$\mathbb{E}^Q\Big(\exp\big(-\int_t^T r_s ds + v \cdot L_T\big)\Big) = \exp\Big(\phi(t,T,v) + \psi(t,T,v) \cdot L_0\Big)$$

with generalized Riccati equations for ϕ, ψ. Inversion of the Laplace transform leads to an expression for C.

Discrete case

- L_t has only values in $\{q_1, \ldots, q_N\}$ where $Q(L_T = q_i) > 0$ for all $i = 1, \ldots, N$.
- Then, for all $0 \le t \le T$ and $x, y \in (q_i, q_{i+1}]$,

$$C(t, T, x) = C(t, T, y). \tag{5}$$

Possible specification:

$$\phi(t, T, x) = \sum_{q_i \le x} \phi_i(t, T)$$

with positive $\phi_i(t, T)$, i.e.

$$\phi_i(t, T) = a_i(T - t) \cdot \theta_i(t) + \int_0^t b_i(u, T)\, du$$

and $d\theta_i(t) = (\delta_i + \kappa_i \theta_i(t))dt + \sqrt{\theta_i(t)}dW_i(t)$.

Other options

- A forward starting CDO on $(x_1, x_2]$ with starting point $T \le T_1$ and payment dates $\mathbf{T} = (T_1, \dots, T_N)$:
- The spread is agreed today
- attachment points are fixed at T to $(x_1 + L_T, x_2 + L_T]$
- Set $\Lambda_t = \int_0^t \lambda(s)ds$ and $\sigma(T_1, T_2) = \Lambda^{-1}(\Lambda_{T_2} - \Lambda_{T_1})$.

Lemma

If L is a (inhomogeneous) compound Poisson process with intensity $\lambda(\cdot)$, then the spread of the forward starting CDO is

$$\frac{\int_{x_1}^{x_2} \Big(1_{\{L_t < y\}} - C\big(t, \sigma(T, T_n), y\big) \Big) \, dy}{\sum_{i=1}^n \frac{1}{x_2 - x_1} \int_{x_1}^{x_2} C\big(t, \sigma(T, T_i), y\big) \, dy}.$$

Proof: First,

$$\mathbb{E}\left(1_{\{L_{T_2} < y + L_{T_1}\}}\right) = \mathbb{E}\left(1_{\{L_{T_2} - L_{T_1} < y\}}\right)$$

thus, we need to be able to value differences of L over a certain time period. To this, stationary assumptions or similar tools are useful. In our case, the U_i are i.i.d. and hence $L_{T_2} - L_{T_1}$ and $\sum_{i=1}^{N_{T_2} - N_{T_1}} U_i$ have the same distribution. Moreover, $N_{T_2} - N_{T_1} \sim \text{Poiss}(\Lambda_{T_2} - \Lambda_{T_1})$. Letting $\sigma = \sigma(T_1, T_2) = \Lambda^{-1}(\Lambda_{T_2} - \Lambda_{T_1})$ we obtain that

$$\mathbb{E}\left(1_{\{L_{T_2} < y + L_{T_1}\}}\right) = C\left(0, \sigma(T_1, T_2), y\right).$$

If $\lambda(\cdot) = \lambda$, then $\sigma(T_1, T_2) = T_2 - T_1$.

Relation to models for individual defaults

- L as compound Poisson process:
  $L_t = \sum_{i=1}^{N_t} U_i$, where $U_i > 0$ are i.i.d.$\sim F_U$ and N_t is a Poisson (λ) process.
  Observe that

$$\begin{cases} \lambda^{1-x}(t) = 0 & x \leq L_{t-}, \\ \lambda^{1-x}(t) = \lambda \cdot P(L_{t-} + U_i \geq x) = \lambda(1 - F_U(x - L_{t-})) & \text{otherwise.} \end{cases}$$

  Hence $\lambda = \lim_{x \downarrow L_{t-}} \lambda^{1-x}(t)$ and the distribution of the next jump, bounded by $1 - L_{t-}$, is given by

$$F_U(y) = 1 - \frac{\lambda^{1-L_{t-}-y}(t)}{\lambda}, \qquad y \in [0, 1 - L_{t-}].$$

  Easily generalizes to marked point processes.

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