Competing players in illiquid markets: predatory trading vs. liquidity provision

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Overview

Outline

Introduction

Formalization as an optimization problem

A one stage framework

A two stage framework

Three example cases

General properties of the two stage framework

Portfolio liquidation

Classical case

- Need to liquidate a large portfolio in a short time frame
- Objective: minimize market impact of quick liquidation
- Several market models available, e.g., Almgren and Chriss (2001), Obizhaeva and Wang (2006)

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The multi player setting

New issues when other market participants know that our client is selling:

- The market impact of our client creates a drift in the market price
- This drift can be exploited by the other market participants
- Since we know about this danger, we will adjust our strategy
- Brunnermeier and Pedersen (2005),
  Carlin, Lobo, and Viswanathan (2005)

Outline

The market model

- One risk free asset and one risky asset
- Trading in continuous time, interest rate $= 0$
- $n+1$ strategic players and a number of noise traders
- $X_0(t)$, $X_1(t)$, ..., $X_n(t)$ = risky asset positions of the strategic players
- Trades at time t are executed at the price

$$P(t) = \underbrace{\tilde{P}(t)}_{\text{Arithmetic Brownian motion}} + \gamma \sum_{i=0}^{n} \underbrace{(X_i(t) - X_i(0))}_{\text{Permanent Impact}} + \lambda \sum_{i=0}^{n} \underbrace{\dot{X}_i(t)}_{\text{Temporary Impact}}$$

Description as an optimization problem

Economic assumptions

- Each player i knows
  - all other players' initial asset positions $X_j(0)$
  - their binding target asset positions $X_j(T)$ for some fixed point $T > 0$ in the future
- All players are risk neutral

⇒ Players want to maximize their own *expected* return by choosing an optimal trading strategy $X_i(t)$ given their boundary constraints on $X_i(0)$ and $X_i(T)$

Description as an optimization problem

Mathematical formalization

Return of player i:

$$R_i := \max_{X_i} \mathbb{E}(\text{Return for player } i) = \max_{X_i} \mathbb{E}\left(\int_0^T (-\dot{X}_i(t))P(t)dt\right)$$

A set of strategies $(X_0, X_1, ..., X_n)$ satisfying this maximization equation for all $i = 0, 1, ..., n$ constitutes a Nash equilibrium; we call such a set of strategies *optimal*. We require strategies to be deterministic. Hence,

$$R_i = \max_{X_i} \mathbb{E}\left(\int_0^T (-\dot{X}_i(t))P(t)dt\right) = \max_{X_i} \int_0^T (-\dot{X}_i(t))\bar{P}(t)dt$$

with

$$\bar{P}(t) = P_0 + \gamma \sum_{j=0}^{n} (X_j(t) - X_j(0)) + \lambda \sum_{j=0}^{n} \dot{X}_j(t)$$

Outline

Optimal strategies in a one stage framework

Theorem ((Carlin, Lobo, and Viswanathan 2005))

$n+1$ players trade simultaneously and exclusively in a time period $t \in [0, T_1]$. Then the unique optimal strategies for these $n+1$ players are given by:

$$\dot{X}_i(t) = a e^{-\frac{n}{n+2}\frac{\gamma}{\lambda}t} + b_i e^{\frac{\gamma}{\lambda}t}$$

with

$$\begin{aligned} a &= \frac{n}{n+2}\frac{\gamma}{\lambda}\left(1 - e^{-\frac{n}{n+2}\frac{\gamma}{\lambda}T_1}\right)^{-1} \frac{\sum_{i=0}^{n}(X_i(T_1) - X_i(0))}{n+1} \\ b_i &= \frac{\gamma}{\lambda}\left(e^{\frac{\gamma}{\lambda}T_1} - 1\right)^{-1}\left(X_i(T_1) - X_i(0) - \frac{\sum_{j=0}^{n}(X_j(T_1) - X_j(0))}{n+1}\right). \end{aligned}$$

The predatory setting

We can express our initial situation in this framework:

The seller

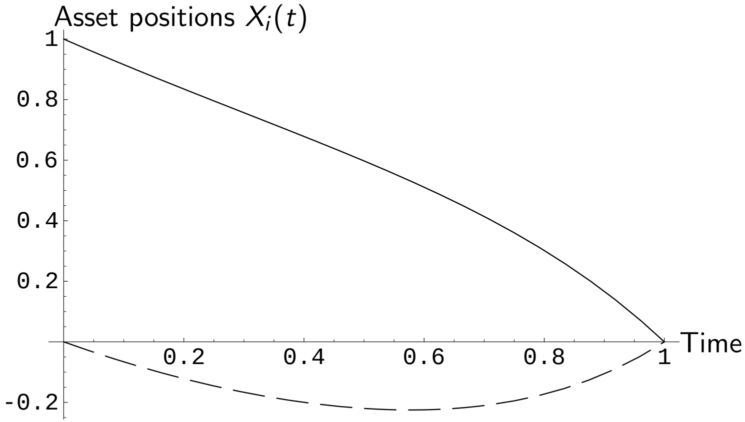
- Single seller = player 0
- $X_0(0) = X_0 > 0$, $X_0(T_1) = 0$

The predators

- n predators: players 1, 2, ..., n
- $X_i(0) = X_i(T_1) = 0$ for all $1 \leq i \leq n$

The buying situation is dual to this setup.

Selling and predation for $n = 1$



Solid line ≈ seller, dashed line ≈ predator

Key properties of predation in the one stage framework

Properties of the one stage framework

- Predation occurs irrespective of the market parameters
  - Temporary impact
  - Permanent impact
- Predation becomes fiercer when the number of predators increases
- Predators are always decreasing the liquidation return of the seller

Open questions

? Why are market participants providing liquidity *at all*?

? Why are market participants revealing their trading intentions, e.g., by sunshine trading or by indications of interest?

Outline

Introduction to the two stage framework

The one stage model considered so far...

Key assumption: The predators have to revert to their original asset position at the same time that the seller has to finish the asset liquidation

... vs. a practical example

- The seller needs to liquidate by day's close to cover a margin call
- The predators are willing to maintain asset positions until the end of the week

$\Rightarrow$ We need a two stage model!

A two stage framework

Two stages

- In stage 1 ($t \in [0, T_1]$), *all* players (the seller and the predators) are trading
  - Seller needs to liquidate in stage 1
  - No restriction on the predators' asset positions at the end of stage 1
- In stage 2 ($t \in [T_1, T_2]$), *only* the predators are trading; the seller is not active
  - Predators need to have reverted to their initial asset position at the end of the second stage
- We assume the same market model as in the one stage framework

Optimal strategies

Observation

Optimal asset positions $X_i(T_1)$ of the predators at the end of stage 1 completely determine optimal strategies.

Theorem

In the unique Nash equilibrium, all predators acquire the same asset position during stage 1:

$$X_i(T_1) = \frac{A_2 n^2 + A_1 n + A_0}{B_3 n^3 + B_2 n^2 + B_1 n + B_0} X_0.$$

The coefficients A_i and B_i are functions of n that converge in the limit $n \to \infty$.

Coefficients A_i and B_i

- Coefficients A_i and B_i in the previous theorem can be explicitly derived
- Complex formulas, e.g., A_0:

$$
\begin{aligned}
A_0 \;=\; & 2\Bigg(- e^{\frac{\gamma(-T_1+(2+n)T_2)}{(1+n)\lambda}} - e^{\frac{\gamma(n(3+2n)T_1+(2+n)T_2)}{\left(2+3n+n^2\right)\lambda}} + \\
& e^{\frac{\gamma\left(\left(2+2n+n^2\right)T_1+n(2+n)T_2\right)}{\left(2+3n+n^2\right)\lambda}} + e^{\frac{\gamma\left(\left(-2+n^2\right)T_1+(2+n)^2T_2\right)}{\left(2+3n+n^2\right)\lambda}} + \\
& e^{\frac{\gamma(-nT_1+(1+2n)T_2)}{(1+n)\lambda}} - e^{\frac{\gamma\left(-nT_1+\left(2+5n+2n^2\right)T_2\right)}{\left(2+3n+n^2\right)\lambda}} + e^{\frac{n\gamma T_1+\gamma T_2}{\lambda+n\lambda}} - \\
& e^{\frac{\gamma T_1+n\gamma T_2}{\lambda+n\lambda}} \Bigg)
\end{aligned}
$$

Sketch of proof of the optimal $X_i(T_1)$

The case $n = 1$

1. Expected market price $\bar{P}(t)$ is a *linear* function of
   - The seller's asset position X_0
   - The predators asset position $Z_1 = X_{1,1}(T_1)$ at the end of stage 1.
2. Return for the predator in the two stages as a *quadratic* function of X_0 and Z_1:

$$Return_{Predator} = Return_{Stage1}(X_0, Z_1) + Return_{Stage2}(X_0, Z_1)$$

3. Determine the optimal Z_1 by maximizing the quadratic function $Return_{Predator}$, giving a linear function in X_0

Sketch of proof of the optimal $X_i(T_1)$

The case $n \geq 2$

1. Assume that $n-1$ predators acquire optimal $X_i(T_1) = Y_1^{(i)}$. Now solve for optimal $X_n(T_1) =: Z_1^{(n)}$ depending on X_0 and $Y_1^{(i)}$
2. Similar to the case $n = 1$, we find that the optimal $Z_1^{(n)}$ is a *linear* function of X_0 and $Y_1^{(i)}$: $Z_1^{(n)} = f(X_0, Y_1^{(1)}, ..., Y_1^{(n-1)})$
3. Same approach gives linear equations for optimal $Z_1^{(i)}$ for all i
4. This set of linear equations has full rank and is symmetric

$\Rightarrow$ A unique Nash equilibrium exists in which all predators acquire the same asset position

Outline

Market illiquidity characteristics

Recap: The market model

$$\bar{P}(t) = P_0 + \gamma \sum_{j=0}^{n} (X_j(t) - X_j(0)) + \lambda \sum_{j=0}^{n} \dot{X}_j(t)$$

Two illiquidity components

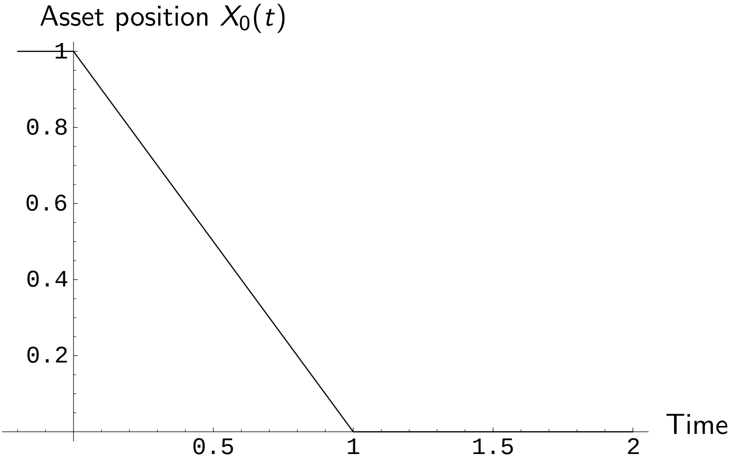
- Premium required to attract buyers *in a short timeframe*
  - Quantified by the temporary impact
  - Determined by the parameter λ
- Premium required to attract buyers *at all*
  - Modeled through the permanent price impact
  - Characterized by the parameter γ

Market illiquidity characteristics

Two polar types of illiquid markets

- **Truly illiquid markets:**
  temporary impact $\lambda >>$ permanent impact γ
- **Nervous markets:**
  permanent impact $\gamma >>$ temporary impact λ

Example 1: Truly illiquid market (large temp. impact) *without predators*



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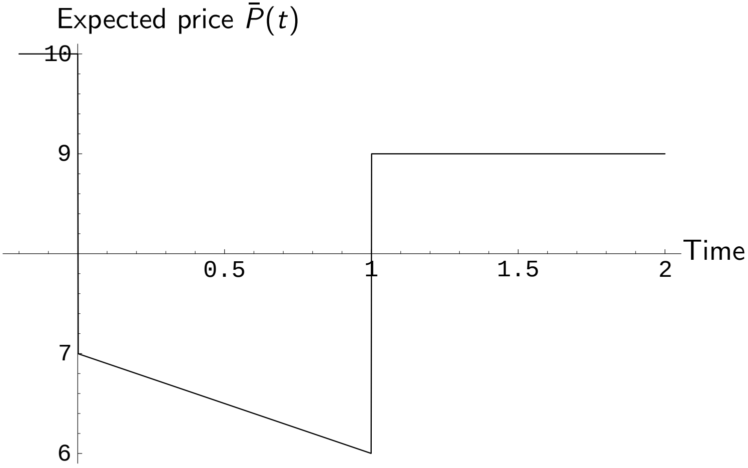
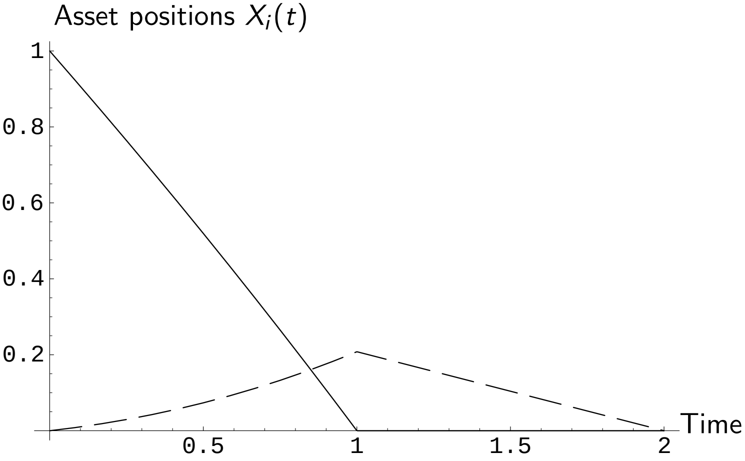
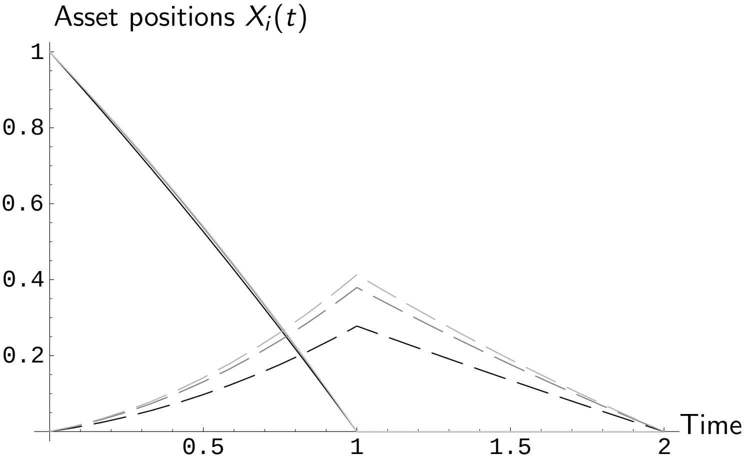


Example 1: Truly illiquid market (large temp. impact) with *one predator*



Solid line ≈ seller, dashed line ≈ predator

Example 1: A truly illiquid market (large temp. impact)



Solid lines ≈ seller, dashed lines ≈ n predators
Black ≈ $n = 2$, dark grey ≈ $n = 10$, light grey ≈ $n = 100$

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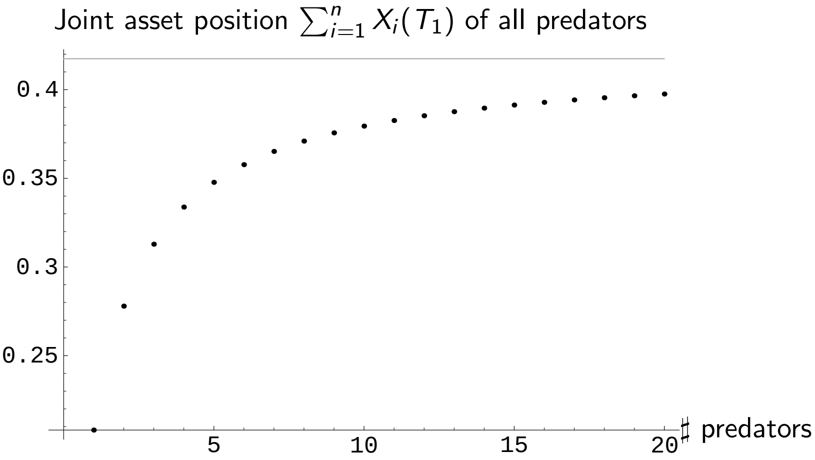

The grey line represents the limit $\lim_{n\to\infty} \sum_{i=1}^{n} X_i(T_1)$

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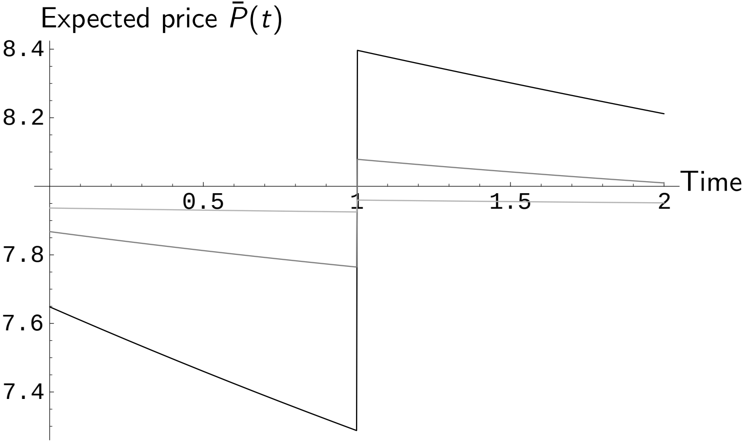

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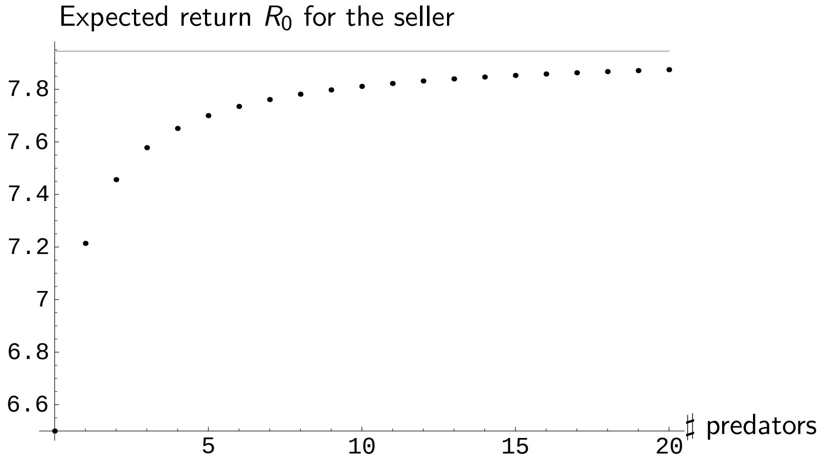
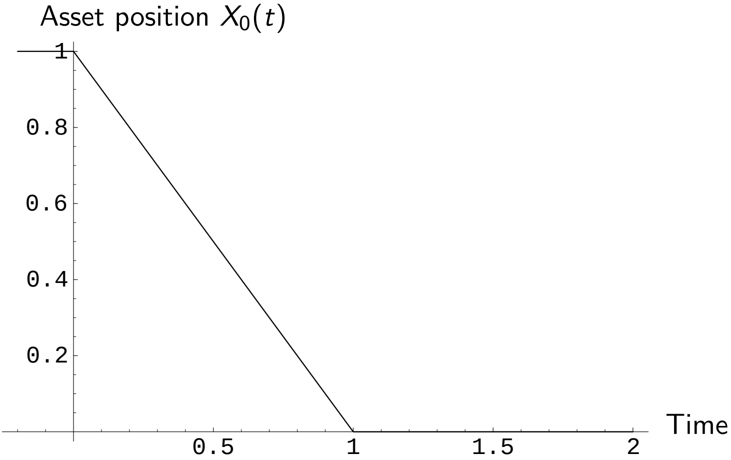


The grey line represents the limit $n \to \infty$. The return for the seller without predators is at the intersection of x- and y-axis.

Example 2: Nervous market (large perm. impact) *without predators*



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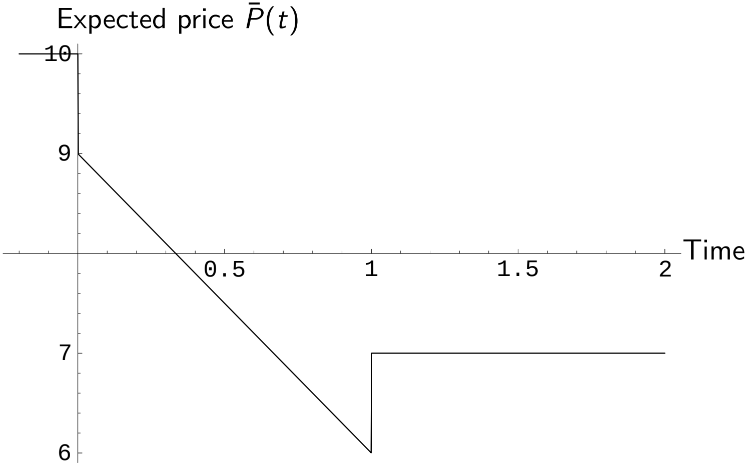
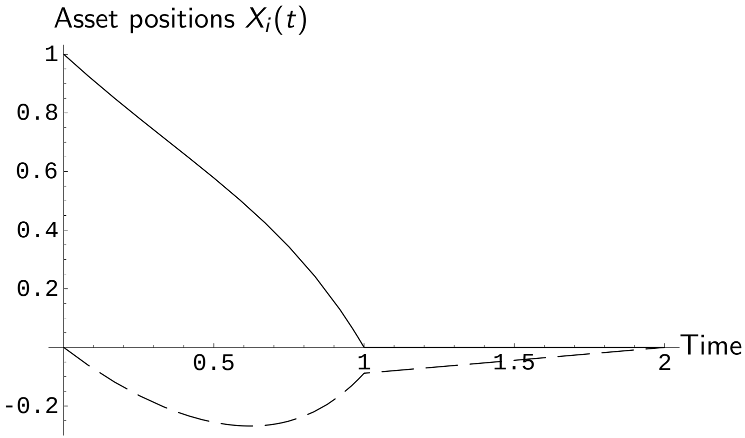


Example 2: Nervous market (large perm. impact) with *one predator*



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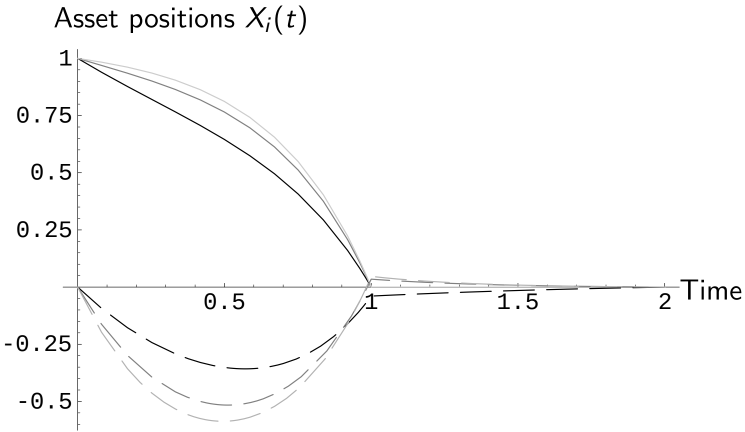
Example 2: Nervous market (large perm. impact)



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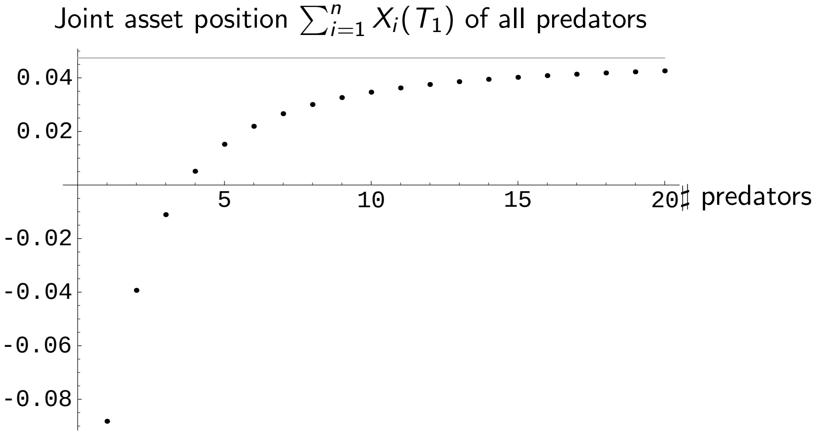

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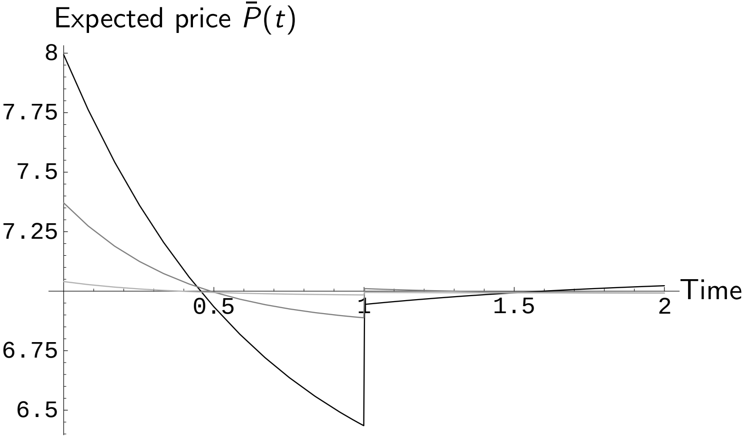

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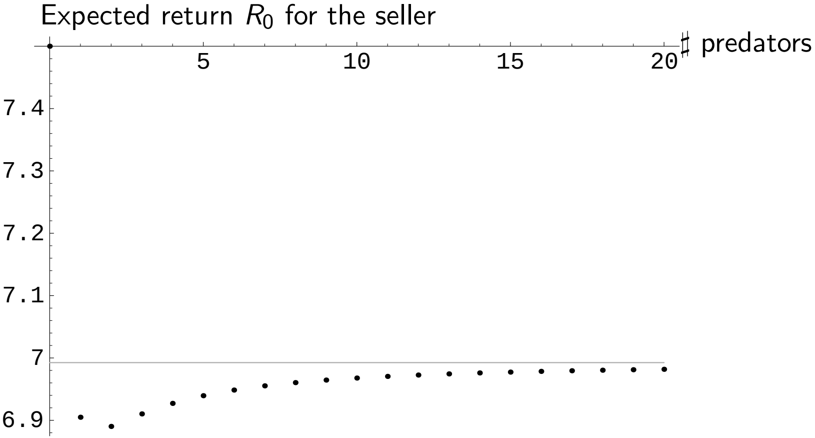

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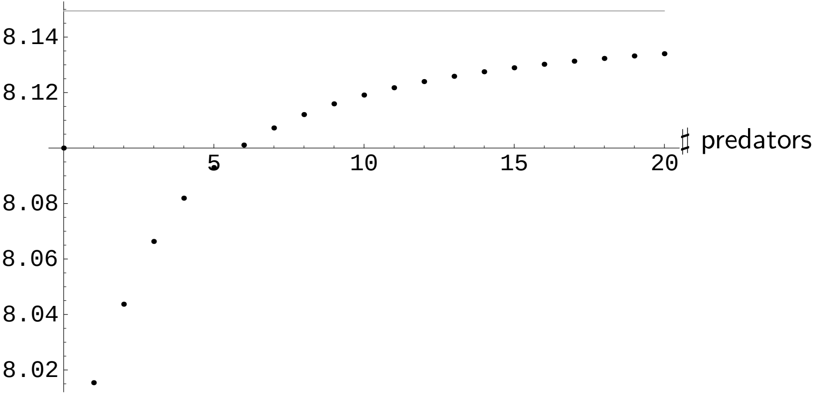
Example 3: Moderate market

Expected return R_0 for the seller



The grey line represents the limit $n \longrightarrow \infty$. The return for the seller without predators is at the intersection of x- and y-axis.

Outline

Asset position of the predators at T_1

Proposition

As the number of predators n tends to infinity, the combined asset position of all predators at the end of stage 1 converges to

$$\lim_{n\to\infty} \sum_{i=1}^{n} X_i(T_1) = \lim_{n\to\infty} nX_1(T_1) = \frac{e^{\frac{\gamma(T_2-T_1)}{\lambda}} - 1}{e^{\frac{\gamma(T_2)}{\lambda}} - 1} X_0.$$

- For large n, intra-stage cooperation always occurs
  - In stage 1, predators buy a portion of the seller's position
  - In stage 2, predators sell this portion

$\Rightarrow$ Reduces market impact in stage 1

Market efficiency

Proposition

The drift $|\dot{\bar{P}}(t)|$ is a decreasing function of n. In the limit, the expected market price instantaneously jumps to

$$P_0 - \frac{\gamma}{1 - e^{-\frac{\gamma(T_2)}{\lambda}}} X_0$$

and is constant from there on throughout stage 1 and stage 2.

- As the number n of predators increases, the price evolution exhibits less drift, i.e., the intentions of the seller are incorporated into market prices more quickly
- The market is weakly efficient but not strongly efficient in the sense of Fama (1970)

Seller's return

Theorem

By selling his asset position X_0 in stage 1, the seller receives an average total cash position of

$$R_0 = X_0 \left(P_0 - \gamma X_0 \frac{\sum_{i=0}^{7} C_i n^i}{\sum_{i=0}^{7} D_i n^i} \right)$$

The coefficients C_i and D_i are functions of n that converge in the limit $n \to \infty$. For large n, the seller's return is increasing in n and converges to:

$$\lim_{n \to \infty} R_0 = X_0 \left(P_0 - \gamma X_0 \frac{1}{1 - e^{-\frac{\gamma}{\lambda} T_2}} \right)$$

Sunshine trading vs. stealth execution

Stealth algorithm

- No predators
- Expected return:

$$X_0\,(P_0 - \gamma X_0/2 - \lambda X_0/T_1)$$

Sunshine trading

- Large number of predators
- Expected return:

$$X_0\left(P_0 - \gamma X_0 \frac{1}{1 - e^{-\frac{\gamma}{\lambda}T_2}}\right)$$

$\Downarrow$ $\Downarrow$

- Simple method to determine benefits of stealth execution vs. sunshine trading
- If the predators do not face any material time constraint ($T_2 \to \infty$), then a stealth algorithm is beneficial if $\frac{T_1}{2} > \frac{\lambda}{\gamma}$

A note on sunshine trading

Benefit of stealth algorithm is model dependent:

Drivers of benefit of a stealth algorithm

- In our model:
  - Ratio γ/λ
  - Length of the two stages T_1 and $T_2 - T_1$
- Admati and Pfleiderer (1991): stealth algorithm is *never* beneficial
- Brunnermeier and Pedersen (2005): Size of the order is the dominant driver

Summary and Outlook

Summary of results

- Liquidity provision profitable if number of predators is large
- Model market is weakly efficient
- Benefit of stealth algorithms and sunshine trading depends on market

Outlook

Many possible extensions

- Portfolio liquidation
- Risk aversion
- Front running
- ...

Thank you for your attention!

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