VALUATION AND HEDGING OF DEFAULTABLE GAME OPTIONS IN A HAZARD PROCESS MODEL

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Valuation of Defaultable Game Options

We assume that the primary market is composed of the savings account B and of d risky assets, such that, given a finite horizon date $T > 0$:

• the *discount factor* process β, that is, the inverse of the savings account B, is a $\mathbb{G}$-adapted, finite variation, continuous, positive and bounded process;

• the risky assets are $\mathbb{G}$-semimartingales with càdlàg sample paths.

• All the processes are defined on a filtered probability space $(\Omega, \mathbb{G}, \mathbb{P})$, where $\mathbb{P}$ denotes the statistical (objective) probability measure.

Given a $\mathbb{G}$-stopping time τ_d representing the *default time* of a reference entity, we introduce the concept of a defaultable, dividend paying game option (concept of a game option (GO) is originally due to Y. Kifer).

Let 0 (respectively T) stand for the *inception date* (respectively the *maturity date*) of a GO.

Let $H_t = \mathbb{1}_{\{\tau_d \leq t\}}$ denote the default indicator process of the reference entity.

We denote by $\mathcal{G}_T^t$, the set of all $\mathbb{G}$-stopping times with values in $[t, T]$.

In broad terms, a **dividend paying game option** initiated at time $t = 0$ and maturing at time T, is a contract with the following cash flows that are paid by the issuer of the contract and received by the holder of the contract:

• a **dividend stream**, whose cumulative value at time t is denoted by D_t,

• a **put payment** L_t made at time $t = \tau_p$ if $\tau_p \leq \tau_c$ and $\tau_p < T$; time τ_p is called the put time and is chosen by the holder,

• a **call payment** U_t made at time $t = \tau_c$ provided that $\tau_c < \tau_p \wedge T$; time τ_c, known as the call time, is chosen by the issuer and may be subject to the constraint that $\tau_c \geq \bar{\tau}$, where $\bar{\tau}$ is the lifting time of the call protection,

• a **payment at maturity** ξ made at time T provided that $T \leq \tau_p \wedge \tau_c$, and subject to rules specified in the contract.

A **defaultable game option** has cumulative discounted cash flows $\beta_t \pi(t; \tau_p, \tau_c)$, where, denoting by $\tau = \tau_p \wedge \tau_c$

$$\beta_t \pi(t; \tau_p, \tau_c) = \int_t^\tau \beta_u \, dD_u + \mathbb{1}_{\{\tau_d > \tau\}} \beta_\tau \Big(\mathbb{1}_{\{\tau = \tau_p < T\}} L_{\tau_p} + \mathbb{1}_{\{\tau = \tau_c < \tau_p\}} U_{\tau_c} + \mathbb{1}_{\{\tau = T\}} \xi \Big)$$

• the **dividend process** $D = (D_t)_{t \in [0,T]}$ equals

$$D_t = C_{t \wedge \tau_d} + \mathbb{1}_{\{\tau_d \le t\}} R_{\tau_d} = \int_{[0,t]} (1 - H_u) \, dC_u + \int_{[0,t]} R_u \, dH_u$$

for some finite variation **coupon process** $C = (C_t)_{t \in [0,T]}$ and some $\mathbb{G}$-predictable **recovery process** $R = (R_t)_{t \in [0,T]}$;

• the **put payment** process $L = (L_t)_{t \in [0,T]}$

• the **call payment** $U = (U_t)_{t \in [0,T]}$ such that

$$L_t \le U_t \quad \text{for} \quad t \in [0, \tau_d \wedge T) \, ;$$

• the **payment at maturity** ξ is a $\mathcal{G}_T$-measurable real random variable, such that $L_T \le \xi \le U_T$.

• All the processes C, R, U, L are $\mathbb{G}$-adapted, real-valued, càdlàg process

• It is also possible to add the call protection $\widehat{\tau}$, i.e. a stopping time such that τ_c is constrained to satisfy $\tau_c \geq \widehat{\tau}$.

• The r.v. $\pi(t; \tau_p, \tau_c)$ is $\mathcal{G}_{\tau_p \wedge \tau_c}$-measurable and is defined for any $t \in [0, T]$ and $(\tau_p, \tau_c) \in \mathcal{G}_T^t \times \mathcal{G}_T^t$

Example 0.1 (Zero-Coupon) Convertible Bond on the underlying asset S :

$$L_t = \bar{P} \vee \kappa S_t, \quad U_t = \bar{C} \vee \kappa S_t, \quad \xi = \bar{N} \vee \kappa S_T \ ,$$

for some constants $\bar{P} \leq \bar{N} \leq \bar{C}$.

Valuation in the General Set-Up

• The primary risky assets, with $\mathbb{R}^d$-valued price process X, pay dividends, whose cumulative value process, denoted by D^X, is assumed to be a $\mathbb{G}$-adapted, càdlàg and $\mathbb{R}^d$-valued process of finite variation.

• We define the **cumulative price** $\widehat{X}$ of the asset as

$$\widehat{X}_t = X_t + \beta_t^{-1} \int_{[0,t]} \beta_u \, dD_u^X .$$

In the financial interpretation, the last term in the formula above represents the current value at time t of all dividend payments of the asset over the period $[0, t]$, under the assumption that all dividends are immediately reinvested in the savings account.

We assume that the primary market model is free of arbitrage opportunities (though presumably incomplete) in the sense that $(X_t)_{t\in[0,T]}$ is an arbitrage price for our primary market with dividends, which means that there exists a **risk-neutral measure** $\mathbb{Q} \in \mathcal{M}$, where $\mathcal{M}$ denotes the set of probability measures $\mathbb{Q} \sim \mathbb{P}$ for which $\beta \widehat{X}$ is a $\mathbb{G}$-martingale transform (or σ-martingale) under $\mathbb{Q}$.

The following theorem characterizes the set of arbitrage prices of a game option in terms of values of related Dynkin games. The notion of arbitrage price of a game option referred to in this theorem is the dynamic notion of arbitrage price for game options, defined in Kallsen and Kühn .

If Π *is a* $\mathbb{G}$*-semimartingale and if there exists* $\mathbb{Q} \in \mathcal{M}$ *such that* Π *is the value of the Dynkin game related to a game option, specifically,*

$$\operatorname*{ess\,sup}_{\tau_{\rm p} \in \mathcal{G}^{\rm t}_{\rm T}} \operatorname*{ess\,inf}_{\tau_{\rm c} \in \bar{\mathcal{G}}^{\rm t}_{\rm T}} \mathbb{E}_{\mathbb{Q}}\big(\pi({\rm t}; \tau_{\rm p}, \tau_{\rm c}) \,\big|\, \mathcal{G}_{\rm t}\big) = \Pi_{\rm t} = \operatorname*{ess\,inf}_{\tau_{\rm c} \in \bar{\mathcal{G}}^{\rm t}_{\rm T}} \operatorname*{ess\,sup}_{\tau_{\rm p} \in \mathcal{G}^{\rm t}_{\rm T}} \mathbb{E}_{\mathbb{Q}}\big(\pi({\rm t}; \tau_{\rm p}, \tau_{\rm c}) \,\big|\, \mathcal{G}_{\rm t}\big),$$

then Π *is an arbitrage (ex-dividend) price of the game option. Moreover, the converse holds true under the following integrability assumption:*

$$\sup_{\mathbb{Q} \in \mathcal{M}} \mathbb{E}_{\mathbb{Q}}\Big(\sup_{t \in [0,T]} \widehat{\mathcal{L}}_t \,\Big|\, \mathcal{G}_0 \Big) < \infty \ \ a.s.$$

where we write

$$\widehat{\mathcal{L}}_t = \beta_t^{-1} \int_{[0,t]} \beta_u \, dD_u + \mathbb{1}_{\{\tau_d > t\}} \Big(\mathbb{1}_{\{t < T\}} L_t + \mathbb{1}_{\{t = T\}} \xi \Big).$$

Valuation in the Hazard Process Set-Up

In this section, our objective is to derive convenient pricing formulae for an arbitrage price of a game option in the hazard process set-up.

Given a filtered probability space $(\Omega, \mathbb{G}, \mathbb{Q})$ with $\mathbb{G} = (\mathcal{G}_t)_{t \geq 0}$, and an $[0, +\infty]$-valued $\mathbb{G}$-stopping time τ_d, we assume that $\mathbb{G} = \mathbb{H} \vee \mathbb{F}$, where the filtration $\mathbb{H}$ is generated by the process $H_t = \mathbb{1}_{\{\tau_d \leq t\}}$, and where $\mathbb{F}$ is some reference filtration.

We also assume that

(i) the process G given by

$$G_t := \mathbb{Q}(\tau_d > t \,|\, \mathcal{F}_t), \quad t \in [0, T],$$

is strictly positive and continuous so that the $\mathbb{F}$-*hazard process* $\Gamma_t = \ln G_t, t \in [0, T]$, is well defined and it is continuous as well.

(ii) All $\mathbb{F}$-martingales are $\mathbb{G}$-martingales

(iii) $\mathbb{Q}(\tau_d = \tau) = 0$, for any $\mathbb{F}$-stopping time τ.

(iv) The discount factor process β is $\mathbb{F}$-adapted.

(v) The coupon process C, the recovery process R and payoff processes L, U are $\mathbb{F}$-predictable.

(vi) The random variable ξ is $\mathcal{F}_T$-mesurable.

For any $t \in [0, T]$, we denote by $\mathcal{F}_T^t$ the set of all $\mathbb{F}$-stopping times with values in $[t, T]$.

We shall now reduce the study of a game option to the study of Dynkin games with respect to the reference filtration $\mathbb{F}$. This essential simplification will appear crucial in the further study of valuation and hedging of defaultable game options in the intensity-based set-up.

In a first step, the computation of the lower and upper value of the Dynkin games with respect to $\mathbb{G}$-stopping times is reduced to the computation of lower and upper value with respect to $\mathbb{F}$-stopping times.

We have:

$$
\begin{aligned}
\operatorname*{ess\,sup}_{\tau_p \in \mathcal{G}_T^t} \operatorname*{ess\,inf}_{\tau_c \in \bar{\mathcal{G}}_T^t} \mathbb{E}_{\mathbb{Q}}\Big(\pi(t;\tau_p,\tau_c)\,\Big|\,\mathcal{G}_t\Big) &= \operatorname*{ess\,sup}_{\tau_p \in \mathcal{F}_T^t} \operatorname*{ess\,inf}_{\tau_c \in \bar{\mathcal{F}}_T^t} \mathbb{E}_{\mathbb{Q}}\Big(\pi(t;\tau_p,\tau_c)\Big|\,\mathcal{G}_t\Big)\ , \\
\operatorname*{ess\,inf}_{\tau_c \in \bar{\mathcal{G}}_T^t} \operatorname*{ess\,sup}_{\tau_p \in \mathcal{G}_T^t} \mathbb{E}_{\mathbb{Q}}\Big(\pi(t;\tau_p,\tau_c)\,\Big|\,\mathcal{G}_t\Big) &= \operatorname*{ess\,inf}_{\tau_c \in \bar{\mathcal{F}}_T^t} \operatorname*{ess\,sup}_{\tau_p \in \mathcal{F}_T^t} \mathbb{E}_{\mathbb{Q}}\Big(\pi(t;\tau_p,\tau_c)\Big|\,\mathcal{G}_t\Big)\ .
\end{aligned}
$$

In a second step, the computation of conditional expectations of $\mathbb{G}$-cash flows with respect to $\mathcal{G}_t$ can be reduced to the computation of conditional expectations of $\mathbb{F}$-*equivalent* cash flows with respect to $\mathcal{F}_t$: Let $\alpha_t := \beta_t \exp(-\Gamma_t)$ stand for the **credit-risk adjusted discount factor**.

Given stopping times $\tau_p \in \mathcal{F}_T^t$ and $\tau_c \in \mathcal{F}_T^t$, we have that

$$\mathbb{E}_{\mathbb{Q}}\big(\pi(t;\tau_p,\tau_c)\,\big|\,\mathcal{G}_t\big) = \mathbb{1}_{\{t<\tau_d\}}\mathbb{E}_{\mathbb{Q}}\big(\widetilde{\pi}(t;\tau_p,\tau_c)\,\big|\,\mathcal{F}_t\big),$$

where the $\mathcal{F}_\tau$-measurable random variable $\widetilde{\pi}(t;\tau_p,\tau_c)$ is given by

$$\alpha_t\widetilde{\pi}(t;\tau_p,\tau_c) = \int_t^\tau \alpha_u(dC_u + R_u\,d\Gamma_u) + \alpha_\tau\big(\mathbb{1}_{\{\tau=\tau_p<T\}}L_{\tau_p} + \mathbb{1}_{\{\tau<\tau_p\}}U_{\tau_c} + \mathbb{1}_{\{\tau=T\}}\xi\big).$$

Note that the original cash flows $\pi(t;\tau_p,\tau_c)$ are given as $\mathcal{G}_\tau$-measurable random variables, whereas the 'equivalent' cash flows $\widetilde{\pi}(t;\tau_p,\tau_c)$ are $\mathcal{F}_\tau$-measurable and depend of the default time τ_d only via the hazard process Γ.

Let Π *be an arbitrage* $\mathbb{Q}$*-price for a game option. Then we have, for any* $t \in [0,T]$,

$$\Pi_t = \mathbb{1}_{\{t<\tau_d\}} \widetilde{\Pi}_t,$$

where

$$\operatorname*{ess\,sup}_{\tau_p \in \mathcal{F}_T^t} \operatorname*{ess\,inf}_{\tau_c \in \mathcal{F}_T^t} \mathbb{E}_{\mathbb{Q}}\Big(\widetilde{\pi}(t;\tau_p,\tau_c)\,\Big|\,\mathcal{F}_t\Big) = \widetilde{\Pi}_t = \operatorname*{ess\,inf}_{\tau_c \in \mathcal{F}_T^t} \operatorname*{ess\,sup}_{\tau_p \in \mathcal{F}_T^t} \mathbb{E}_{\mathbb{Q}}\Big(\widetilde{\pi}(t;\tau_p,\tau_c)\Big|\,\mathcal{F}_t\Big)\ .$$

Hence the Dynkin game with cost criterion $\mathbb{E}_{\mathbb{Q}}\Big(\widetilde{\pi}(t;\tau_p,\tau_c)\,\Big|\,\mathcal{F}_t\Big)$ on $\mathcal{F}_T^t \times \bar{\mathcal{F}}_T^t$ admits a value $\widetilde{\Pi}$, which coincides with the pre-default $\mathbb{Q}$-value of the game option under the pricing measure $\mathbb{Q}$.

Given a martingale measure $\mathbb{Q} \in \mathcal{M}$, *assume that* $\widetilde{\Pi}$ *is the value of the Dynkin game with conditional cost criterion* $\mathbb{E}_{\mathbb{Q}}\big(\widetilde{\pi}(t;\tau_p,\tau_c)\,\big|\,\mathcal{F}_t\big)$ *on* $\mathcal{F}_T^t \times \mathcal{F}_T^t$ *for any* $t \in [0,T]$.

Then Π *defined via* $\Pi_t = \mathbb{1}_{\{t<\tau_d\}}\widetilde{\Pi}_t$ *is the value of the Dynkin game with conditional cost criterion* $\mathbb{E}_{\mathbb{Q}}\big(\pi(t;\tau_p,\tau_c)\,\big|\,\mathcal{G}_t\big)$ *on* $\mathcal{G}_T^t \times \mathcal{G}_T^t$, *for any* $t \in [0,T]$.

If, in addition, $\widetilde{\Pi}$ *is a* $\mathbb{G}$*-semimartingale, then* Π *is a* $\mathbb{Q}$*-price for the game option.*

Valuation via BSDEs

Given a risk-neutral measure $\mathbb{Q} \in \mathcal{M}$, we shall now characterize an arbitrage $\mathbb{Q}$-price of a game option as a solution to a suitably chosen doubly reflected BSDE. Existence and uniqueness of the solutions ares studied in companion papers (Crepey, Jeanblanc and Matoussi).

Definition of a 2RBSDE: A **doubly reflected BSDE with data** F, ξ, L, U is given as:

$$\alpha_t \Theta_t = \alpha_T \xi + \alpha_T F_T - \alpha_t F_t + K_T - K_t - (Z_T - Z_t), \quad t \in [0,T],$$
$$L_t \leq \Theta_t \leq U_t, \quad t \in [0,T],$$
$$\int_0^T (\Theta_{u-} - L_{u-})\, dK_u^+ = \int_0^T (U_{u-} - \Theta_{u-})\, dK_u^- = 0,$$

where F is a given $\mathbb{F}$-adapted process with finite variation

By a *($\mathbb{Q}$-)solution* to the doubly reflected BSDE, we mean a triplet $(\Theta, Z, \mathbf{K})$ such that:

- the *state process* Θ is a real valued, $\mathbb{F}$-adapted, càdlàg process,
- Z is an $(\mathbb{F}, \mathbb{Q})$-martingale vanishing at time 0,
- $\mathbf{K} = (K^+, K^-)$ is a pair of $\mathbb{F}$-adapted, non-decreasing processes,
- the above equations are satisfied, with $K = K^+ - K^-$ in the first line.

Note that the formula which defines $\widetilde{\pi}$ can be rewritten:

$$\alpha_t \widetilde{\pi}(t;\tau_p,\tau_c) = \alpha_\tau F^0_\tau - \alpha_t F^0_t + \alpha_\tau \Big(\mathbb{1}_{\{\tau=\tau_p<T\}} L_{\tau_p} + \mathbb{1}_{\{\tau<\tau_p\}} U_{\tau_c} + \mathbb{1}_{\{\tau=T\}} \xi \Big) \ ,$$

with

$$F^0_t := \alpha_t^{-1} \int_{[0,t]} \alpha_u \left(dC_u + R_u d\Gamma_u \right).$$

Let

$$F_t^0 := \alpha_t^{-1} \int_{[0,t]} \alpha_u \, (dC_u + R_u d\Gamma_u).$$

We denote by E_0 the doubly reflected BSDE

$$\left.\begin{array}{c} \alpha_t \Theta_t^0 = \alpha_T \xi^0 + K_T - K_t - (Z_T - Z_t), \quad t \in [0,T], \\ L_t^0 \le \Theta_t^0 \le U_t^0, \quad t \in [0,T], \\ \int_0^T (\Theta_{u-}^0 - L_{u-}^0) \, dK_u^+ = \int_0^T (U_{u-}^0 - \Theta_{u-}^0) \, dK_u^- = 0, \end{array}\right\} (E_0)$$

with

$$\xi^0 = \xi + F_T^0, \quad L_t^0 = L_t + F_t^0, \quad U_t^0 = U_t + F_t^0.$$

We assume that the doubly reflected BSDE E_0 admits a $\mathbb{Q}$-solution $(\Theta^0, Z, \mathbf{K})$.

We define, for $t \in [0, T]$,

$$\Pi_t = \mathbb{1}_{\{t < \tau_d\}} (\Theta_t^0 - F_t^0)$$

Verification Principle for a Defaultable Game Option *The process* Π *is an arbitrage ex-dividend* $\mathbb{Q}$*-price for the game option. Moreover, for any* $t \in [0, T]$ *and* $\varepsilon > 0$*, the pair of* ε*-optimal stopping times* $(\tau_p^\varepsilon, \tau_c^\varepsilon) \in \mathcal{F}_T^t \times \mathcal{F}_T^t$ *for the related Dynkin game on* $\mathcal{G}_T^t \times \mathcal{G}_T^t$ *is given by*

$$\tau_p^\varepsilon = \inf \Big\{ u \in [t, T]\,;\, \widetilde{\Pi}_u \leq L_u + \varepsilon \Big\} \wedge T, \ \tau_c^\varepsilon = \inf \Big\{ u \in [t, T]\,;\, \widetilde{\Pi}_u \geq U_u - \varepsilon \Big\} \wedge T.$$

If $\mathbf{K}$ *is continuous then the pair of stopping times* $(\tau_p^*, \tau_c^*) \in \mathcal{F}_T^t \times \mathcal{F}_T^t$*, obtained by setting* $\varepsilon = 0$*, is a saddle-point of the defaultable game option.*

Hedging of Defaultable Game Options

We now examine the implications of existence of a solution to the doubly reflected BSDE on hedging strategies of a game option, under the assumption that the primary market is complete.

The Model

We place ourselves within the hazard process set-up of the first Section for some fixed $\mathbb{Q} \in \mathcal{M}$. We define the compensated jump martingale M^d of the non-decreasing default indicator process $H_t = \mathbb{1}_{\{\tau_d \le t\}}$ by setting

$$M_t^d = H_t - \Gamma_{t \wedge \tau_d}, \quad t \in [0, T].$$

We recall that the **cumulative price** $\widehat{X}$ of the asset is

$$\widehat{X}_t = X_t + \beta_t^{-1} \int_{[0,t]} \beta_u \, dD_u^X.$$

We now assume that the dynamics of the prices are driven by the pair of martingales M, M^d, and, that there exists a $\mathbb{G}$-predictable, matrix-valued process Υ_t, such that

$$d \begin{bmatrix} M_t \\ M_t^d \end{bmatrix} = \beta_t^{-1} \Upsilon_t d(\beta_t \widehat{X}_t) \ ,$$

We now recall the notion of a portfolio with financing cost.

A **portfolio with financing cost Q** is a pair of a constant V_0 and a $\mathbb{R}^d$-valued, $\mathbb{G}$-predictable process ζ. The wealth associated to that triple (V_0, Q, ζ) is the process V given by

$$d(\beta_t V_t) = \zeta_t^{\mathsf{T}} d(\beta_t \widehat{X}_t) - \beta_t \, dQ_t, \quad t \in [0, T].$$

In the sequel, the process Q will be either equal to the dividend process D of a game option, or to $-D$.

By an **issuer ε-hedge** *for the game option we mean a triple* (V_0, ζ, τ_c) *such that:*

(i) τ_c *belongs to* $\mathcal{G}_T^0$*,*

(ii) denoting by V_t *the time-*t *value of the portfolio* (V_0, D, ζ)*,*

$$V_{t\wedge\tau_c} - \mathbb{1}_{\{t\wedge\tau_c<\tau_d\}}\Big(\mathbb{1}_{\{t<\tau_c\}}L_t + \mathbb{1}_{\{\tau_c\le t<T\}}U_{\tau_c} + \mathbb{1}_{\{t=\tau_c=T\}}\xi\Big) \ge -\varepsilon.$$

By a holder ε-hedge *for the game option we mean a triple* (V_0, ζ, τ_p) *such that:*

(i) τ_p *belongs to* $\mathcal{G}_T^0$*,*

(ii) denoting by V_t *the time-*t *value of the portfolio* $(V_0, -D, \zeta)$

$$V_{t\wedge\tau_p} + \mathbb{1}_{\{t\wedge\tau_p<\tau_d\}}\Big(\mathbb{1}_{\{\tau_p\le t<T\}}L_{\tau_p} + \mathbb{1}_{\{t<\tau_p\}}U_t + \mathbb{1}_{\{\tau_p=t=T\}}\xi\Big) \ge -\varepsilon.$$

In the case of $\varepsilon = 0$, we shall say that we deal with an *issuer hedge* and a *holder hedge* for the game option.

Let $(\Theta^0, \vartheta, \mathbf{K})$ be a solution of the BSDE

$$\begin{aligned} \alpha_t \Theta_t^0 &= \alpha_T \xi^0 + K_T - K_t - \int_t^T \vartheta_s dM_s, \quad t \in [0,T], \\ L_t^0 \le \Theta_t^0 &\le U_t^0, \quad t \in [0,T], \\ \int_0^T (\Theta_{u-}^0 - L_{u-}^0)\, dK_u^+ &= \int_0^T (U_{u-}^0 - \Theta_{u-}^0)\, dK_u^- = 0, \end{aligned}$$

and

$$\widetilde{\Pi}_t = \Theta_t^0 - F_t^0, .$$

So the process $\Pi_t = \mathbb{1}_{\{t<\tau_d\}} \widetilde{\Pi}_t$ is the arbitrage $\mathbb{Q}$-price for the game option. We further define the $\mathbb{F}$-adapted process Π^* by the formula $\alpha_t \Pi_t^* = \alpha_t \widetilde{\Pi}_t + K_t$ for every $t \in [0,T]$.

The process Y given by

$$Y_t := \mathbb{1}_{\{t<\tau_d\}}\beta_t\Pi^*_t + \mathbb{1}_{\{\tau_d\le t\}}\beta_{\tau_d}R_{\tau_d} + \int_0^{t\wedge\tau_d}\beta_u\,dC_u$$

is a $\mathbb{G}$-martingale, such that

$$dY_t = \mathbb{1}_{\{t\le\tau_d\}}e^{\Gamma_t}\vartheta_t^{\mathsf{T}}\,dM_t + \mathbb{1}_{\{t\le\tau_d\}}\beta_t\big(R_t - \Pi^*_{t-}\big)\,dM^d_t\ ,\ t\in[0,T]\ .$$

Proof. From definition

$$\alpha_t \widetilde{\Pi}_t = \alpha_T \xi + \int_t^T \alpha_u \, dC_u + \int_t^T \alpha_u R_u \, d\Gamma_u + K_T - K_t - \int_t^T \vartheta_u^{\mathsf{T}} \, dM_u.$$

Note that

$$\alpha_t \Pi_t^* := \alpha_t \widetilde{\Pi}_t + K_t = \widetilde{\Pi}_0 - \int_0^t \alpha_u \, dC_u - \int_0^t \alpha_u R_u \, d\Gamma_u + \int_0^t \vartheta_u^{\mathsf{T}} dM_u.$$

Therefore, the process Y^* given by

$$Y_t^* := \alpha_t \Pi_t^* + \int_0^t \alpha_u \, dC_u + \int_0^t \alpha_u R_u \, d\Gamma_u = \widetilde{\Pi}_0 + \int_0^t \vartheta_u^{\mathsf{T}} \, dM_u$$

is an $\mathbb{F}$-martingale. It is easy to check that we have, for any $0 \le t \le u \le T$,

$$\mathbb{E}_{\mathbb{Q}}(Y_u - Y_t \,|\, \mathcal{G}_t) = \mathbb{1}_{\{t<\tau_d\}} e^{\Gamma_t} \, \mathbb{E}_{\mathbb{Q}}(Y_u^* - Y_t^* \,|\, \mathcal{F}_t).$$

Since Y^* is an $\mathbb{F}$-martingale, the process Y is a $\mathbb{G}$-martingale.

Moreover, by definition, this process is stopped at τ_d. Note that

$$Y_t = (1 - H_t)\beta_t \Pi_t^* + \int_0^t \beta_u R_u \, dH_u + \int_0^t \beta_u (1 - H_u) \, dC_u$$

so that the result follows by direct computations. □

We are now in the position to state the hedging result for a defaultable game option.

(i) Π_0 *is the least value* V_0 *such that for any* $\varepsilon > 0$, *there exists an issuer* ε*-hedge with initial value* V_0. *Morever, starting from* $V_0 = \Pi_0$, *such an hedge is furnished by* τ_c^ε *and*

$$\zeta_t^c = \mathbb{1}_{\{t \le \tau_d\}} \Upsilon_t^\mathsf{T} \begin{bmatrix} \alpha_t^{-1} \vartheta_t \\ R_t - \Pi_{t-}^* \end{bmatrix}, \quad t \in [0, T]. \tag{1}$$

(ii) *For any* $\varepsilon > 0$, *an holder* ε*-hedge with initial value* $-\Pi_0$ *is furnished by* τ_p^ε *and* $\zeta^p = -\zeta^c$. *Moreover, in the case of a game option with bounded cash flows* π, $-\Pi_0$ *is the least value* V_0 *such that for any* $\varepsilon > 0$, *there exists an issuer* ε*-hedge with initial value* V_0.

(iii) *If the process* $\mathbf{K}$ *is continuous, we may set* ε *equal to 0 in (i)-(ii). Hence, there exist an issuer hedge (respectively the holder hedge) of the game option with the initial value* Π_0 *(respectively* $-\Pi_0$*).*

References

[1] ANDERSEN, L. AND BUFFUM, L.: Calibration and implementation of convertible bond models. Working paper, 2003.

[2] AYACHE, E., FORSYTH, P. AND VETZAL, K.: Valuation of convertible bonds with credit risk. *The Journal of Derivatives*, Fall 2003.

[3] BIELECKI, T.R., CRÉPEY, S., JEANBLANC, M. AND RUTKOWSKI, M.: Arbitrage pricing of defaultable game options and applications. Working paper, 2005.

[4] BIELECKI, T.R., CRÉPEY, S., JEANBLANC, M. AND RUTKOWSKI, M.: Valuation and hedging of convertible bonds in a jump-diffusion model of credit risk. Work in preparation, 2006.

[5] CRÉPEY, S., JEANBLANC, M., AND MATOUSSI, A.: On a class of Doubly Reflected BSDEs. Work in preparation.

[6] Cvitanić, J. and Karatzas, I.: Backward stochastic differential equations with reflection and Dynkin games. *Annals of Probability* 24 (1996), 2024–2056.

[7] Delbaen, F. and Schachermayer, W.: The fundamental theorem of asset pricing for unbounded stochastic processes. *Mathematische Annalen* 312 (1997), 215–250.

[8] Dellacherie, C., Maisonneuve, B. and Meyer, P.-A.: *Probabilités et Potentiel, Chapitres XVII-XXIV.* Hermann, Paris, 1992.

[9] Dynkin, E.B.: Game variant of a problem on optimal stopping. *Soviet Math. Dokl.* 10 (1969), 270–274.

[10] El Karoui, N., Peng, S., and Quenez, M.-C.: Backward stochastic differential equations in finance. *Mathematical Finance* 7 (1997), 1–71.

[11] El Karoui, N. and Quenez, M.-C.: Nonlinear pricing theory

and backward stochastic differential equations. In: *Financial Mathematics, Bressanone, 1996,* W.Runggaldier, ed. *Lecture Notes in Math.* 1656, Springer, Berlin Heidelberg New York, 1997, pp. 191–246.

[12] Hamadène, S.: Mixed zero-sum differential game and American game options. Preprint, Université du Maine, 2004.

[13] Kallsen, J. and Kühn, C.: Convertible bonds: financial derivatives of game type. In: *Exotic Option Pricing and Advanced Lévy Models*, Edited by Kyprianou, A., Schoutens, W. and Wilmott, P., Wiley, 2005, pp. 277–288.

[14] Karatzas, I.: On the pricing of American options. *Appl. Math. Optim.* 17 (1988), 37–60.

[15] Kifer, Y.: Game options. *Finance and Stochastics* 4 (2000), 443–463

[16] Kifer, Y.: Optimal stopping in games with continuous time.

Theory Probab. Appl. 16 (1971), 545–550.

[17] Lepeltier, J.-P. and Maingueneau, M.: Le jeu de Dynkin en théorie générale sans l'hypothèse de Mokobodski. *Stochastics* 13 (1984), 25–44.