Conditional Small Balls and No-Arbitrage

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ABSTRACT

The talk is based on an ongoing joint work with C. Bender and E. Valkeila. In [1] we considered model classes $\mathcal{M}_\sigma$ of continuous discounted market models that have the quadratic variation $d\langle S\rangle_t = \sigma^2(S_t)dt$ and satisfy the small ball property

$$\mathbf{P}\left[\sup_{t\in[0,T]} |S_t - \eta(t)| \le \epsilon\right] > 0 \qquad \text{(-11)}$$

for all $\epsilon > 0$ and $\eta \in C_{s0,+}[0,T]$. We proved:

(i) Within allowed strategies *all* models in $\mathcal{M}_\sigma$ are free of arbitrage provided one admits an equivalent martingale measure.

(ii) If a continuous functional can be replicated with an allowed strategy in one model $\mathcal{M}_\sigma$ then it can be replicated in any model $\mathcal{M}_\sigma$ with the same initial capital and the hedge is – as a functional of the stock path – independent of the model.

(iii) The class of allowed strategies is sufficiently large. E.g. allowed hedges for European, lookback, and Asian options can be constructed via PDEs.

Our results indicated that path properties, viz. the quadratic variation, is of more importance for pricing than probabilistic properties. Indeed, hedges and prices are determined by the quadratic variation which need not be the standard deviation of the log-returns.

While the robustness of hedges result (items (ii) and (iii)) was quite satisfactory the no-arbitrage result (item (i)) was not. Indeed, allowed strategies

are continuous in the path of the stock price. But natural strategies are of type

$$\Phi_t = \sum_{k=1}^{n} \Phi^k \mathbf{1}_{(\tau_{k-1},\tau_k]}(t), \tag{-10}$$

where Φ^k is $\mathcal{F}_{\tau_{k-1}}$-measurable and τ_k's are stopping times. These kind of strategies are typically not continuous in the stock path. So, they are (or actually were) not allowed. In this talk we show how one can extend the no-arbitrage result (i) of [1] to strategies of the type (2). To do this we must strengthen the small ball condition (1) to hold for conditional laws:

$$\mathbf{P}\left[\sup_{t\in[0,T]} |S_t - \eta(t)| \le \epsilon \middle| \mathcal{F}_\tau\right] > 0 \tag{-9}$$

P-almost surely for all $\epsilon > 0$ and $\eta \in C_{S_\tau,+}[\tau, T]$. Condition (3), while much stronger than (1), does not force S to be a semimartingale.

References

[1] Bender, C., Sottinen, T., and Valkeila, E. No-arbitrage pricing beyond semimartingales. WIAS Preprint No. 1110, 2006.