Control of Some Stochastic Systems with a Fractional Brownian Motion

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Outline

- Fractional Brownian motion (FBM)
  - A brief history
  - Definition and some properties
- Fractional calculus (Riemann-Liouville)
  - Relation to FBM
  - Some linear operators
- Optimal control
  - Controlled stochastic equations with a FBM
  - Absolute continuity of measures
  - Weak solutions: existence and uniqueness
  - Randon-Nikodym derivatives and properties
  - Existence of an optimal control

Some Development of Fractional Brownian Motion

- I. J. Schonberg 1937
- A. N. Kolmogorov 1940
- H. E. Hurst 1951
- B. Mandelbrot 1965

Some Applications

- Hydrology
- Economic data
- Telecommunications
- Device noise
- Medicine

Fractional Brownian motion (FBM)

A standard FBM $(B^H(t), t \geq 0)$ is a Gaussian process with continuous sample paths such that

$$\mathbb{E}[B^H(t)] = 0$$

$$\mathbb{E}[B^H(t)B^H(s)] = \frac{1}{2}[|t|^{2H} + |s|^{2H} - |t-s|^{2H}]$$

$H \in (0,1)$ is called the Hurst parameter

$H = \frac{1}{2} \Longrightarrow$ standard Brownian motion

Some Properties of FBM

1. Self-similarity

$$(B^H(\alpha t), t \geq 0) \overset{L}{\sim} (\alpha^H B^H(t), t \geq 0)$$

   for $\alpha > 0$

2. Long range dependence for $H \in (\frac{1}{2}, 1)$

$$r(n) = \mathbb{E}[B^H(1)(B^H(n+1) - B^H(n))]$$

$$\Sigma_{n=0}^{\infty} r(n) = \infty$$

3. A sample path property
$(B^H(t), t \geq 0)$ is of unbounded variation so the sample paths are not differentiable a.s.

$$\Sigma_i |B^H(t_{i+1}^{(n)}) - B^H(t_i^{(n)})|^p \to \begin{cases} 0 & pH > 1 \\ c(p) & pH = 1 \\ +\infty & pH < 1 \end{cases}$$

where $c(p) = \mathbb{E}|B^H(1)|^p$ and $(t_i^{(n)}, i = 0, 1, \cdots, n)$ is a sequence of nested partitions of $[0, 1]$

Remark

- $p = 2$ and $H \in (\frac{1}{2}, 1) \Longrightarrow$ zero quadratic variation
- $p = 2$ and $H \in (0, \frac{1}{2}) \Longrightarrow$ infinite quadratic variation
- FBM is not a semimartingale for $H \neq \frac{1}{2}$

Fractional Calculus

Let $(V, \|\cdot\|, \langle\cdot,\cdot\rangle)$ be a separable Hilbert space and let $\alpha \in (0,1)$. If $\varphi \in L^1([0,T], V)$ then the left-sided and the right-sided fractional (Riemann-Liouville) integrals of φ are defined (for almost all $t \in [0,T]$) by

$$\left(I_{0+}^{\alpha}\varphi\right)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\,\varphi(s)\,ds$$

and

$$\left(I_{T-}^{\alpha}\varphi\right)(t) = \frac{1}{\Gamma(\alpha)} \int_t^T (s-t)^{\alpha-1}\varphi(s)\,ds$$

respectively, where $\Gamma(\cdot)$ is the gamma function.

The inverse operators of these fractional integrals are called fractional derivatives and can be given by their respective Weyl representations

$$\left(D_{0+}^{\alpha}\psi\right)(t) = \frac{1}{\Gamma(1-\alpha)}\left(\frac{\psi(t)}{t^{\alpha}} + \alpha\int_{0}^{t}\frac{\psi(t)-\psi(s)}{(t-s)^{\alpha+1}}\,ds\right)$$

and

$$\left(D_{T-}^{\alpha}\psi\right)(t) = \frac{1}{\Gamma(1-\alpha)}\left(\frac{\psi(t)}{(T-t)^{\alpha}} + \alpha\int_{t}^{T}\frac{\psi(s)-\psi(t)}{(s-t)^{\alpha+1}}\,ds\right)$$

where $\psi \in I_{0+}^{\alpha}\left(L^{1}\left([0,T],V\right)\right)$ and $\psi \in I_{T-}^{\alpha}\left(L^{1}\left([0,T],V\right)\right)$ respectively.

An Elementary Itô Formula for $H \in (\frac{1}{2}, 1)$

If $f \in C^2$, then

$$f(B^H(T)) - f(B^H(0)) = \int_0^T f'(B^H(s))dB^H(s) + H\int_0^T s^{2H-1} f''(B^H(s))ds$$

Remark

- Formally letting $H = \frac{1}{2}$ recovers the usual Itô formula.
- Recall that an FBM with $H \in (\frac{1}{2}, 1)$ has zero quadratic variation.

Let $K_H(t,s)$ for $0 \le s \le t \le T$ be the real-valued kernel function

$$K_H(t,s) = c_H(t-s)^{H-\frac{1}{2}} + c_H\left(\frac{1}{2}-H\right)\int_s^t (u-s)^{H-\frac{3}{2}}\left(1-\left(\frac{s}{u}\right)^{\frac{1}{2}-H}\right)du$$

where

$$c_H = \left[\frac{2H\Gamma\left(H+\frac{1}{2}\right)\Gamma\left(\frac{3}{2}-H\right)}{\Gamma\left(2-2H\right)}\right]^{\frac{1}{2}}$$

and $H \in (0,1)$. If $H \in (1/2,1)$, then K_H has a simpler form as

$$K_H(t,s) = c_H\left(H-\frac{1}{2}\right)s^{\frac{1}{2}-H}\int_s^t (u-s)^{H-\frac{3}{2}}u^{H-\frac{1}{2}}\,du\,.$$

Define the integral operator $\mathbb{K}_H$ induced from the kernel K_H by

$$\mathbb{K}_H \varphi(t) = \int_0^t K_H(t,s) h(s)\, ds$$

for $h \in L^2([0,T], V)$. It is well-known that

$$\mathbb{K}_H \colon L^2([0,T], V) \to I_{0+}^{H+\frac{1}{2}} \left(L^2([0,T], V) \right)$$

is a bijection.

$\mathbb{K}_H$ can be described as

$$\mathbb{K}_H h(s) = \bar{c}_H I_{0+}^{2H} \left(u_{\frac{1}{2}-H} I_{0+}^{\frac{1}{2}-H} \left(u_{H-\frac{1}{2}} h \right) \right)(s)$$

for $H \in (0, 1/2]$ and

$$\mathbb{K}_H h(s) = c_H I_{0+}^{1} \left(u_{H-\frac{1}{2}} I_{0+}^{H-\frac{1}{2}} \left(u_{\frac{1}{2}-H} h \right) \right)(s)$$

for $H \in [1/2, 1)$ where $u_a(s) = s^a$ for $s \geq 0$ and $a \in \mathbb{R}$.

The inverse operator, $\mathbb{K}_H^{-1}$ for the two cases is given by

$$\mathbb{K}_H^{-1}\varphi(s) = \bar{c}_H^{-1} s^{\frac{1}{2}-H} D_{0+}^{\frac{1}{2}-H}\left(u_{H-\frac{1}{2}} D_{0+}^{2H}\varphi\right)(s)$$

for $H \in (0, 1/2]$ and

$$\mathbb{K}_H^{-1}\varphi(s) = c_H^{-1} s^{H-\frac{1}{2}} D_{0+}^{H-\frac{1}{2}}\left(u_{\frac{1}{2}-H} D_{0+}^{2H}\varphi\right)(s)$$

for $H \in [1/2, 1)$ and $\varphi \in I_{0+}^{H+\frac{1}{2}}\left(L^2([0,T], V)\right)$.
If $\varphi(s) \in H^1([0,T], V)$ and $H \in (0, 1/2)$ then

$$\mathbb{K}_H^{-1}\varphi(s) = \bar{c}_H^{-1} s^{H-\frac{1}{2}} I_{0+}^{\frac{1}{2}-H}\left(u_{H-\frac{1}{2}}\varphi'\right)(s)$$

K_H and the Covariance of a FBM

$$\int_0^{s \wedge t} K_H(t,r) K_H(s,r) dr = \frac{1}{2}(s^{2H} + t^{2H} - |t-s|^{2H})$$

$$B(t) = \int_0^t K_H(t,s) dW(s)$$

W is a Wiener process

Theorem 1

Let $H \in (0,1)$ and $T > 0$ be fixed and let $(u(t), t \in [0, T])$ be a V-valued, $(\mathcal{F}_t)$-adapted process such that

$$\int_0^T \|u(t)\| \, dt < \infty \quad \text{a.s. } \mathbb{P}$$

and

$$U(t) := \int_0^t u(s) \, ds \in I_{0+}^{H+\frac{1}{2}} \left(L^2 \left([0, T], V \right) \right) \quad \text{a.s. } \mathbb{P} \, .$$

Furthermore, it is assumed that

$$\mathbb{E}\xi(T) = 1$$

where

$$\xi(T) = \exp\left[\int_0^T \left\langle \mathbb{K}_H^{-1}(U)(t), dW(t)\right\rangle - \frac{1}{2}\int_0^T \left\|\mathbb{K}_H^{-1}(U)(t)\right\|^2 dt\right]$$

where $(W(t), t \in [0, T])$ is a standard cylindrical Wiener process in V.

Then the process $\left(\tilde{B}(t), t \in [0, T]\right)$ given by

$$\tilde{B}(t) := B(t) - U(t)$$

is a standard cylindrical fractional Brownian motion in V with the Hurst parameter H on the probability space $\left(\Omega, \mathcal{F}, \tilde{\mathbb{P}}\right)$ where

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = \xi(T) \quad \text{a.s.}$$

Definition

A weak solution is a triple $(X(t), B(t), (\Omega, \mathcal{F}, \mathbb{P}), t \geq 0)$ where $(B(t), t \geq 0)$ is a standard cylindrical fractional Brownian motion in V that is defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $(X(t), t \geq 0)$ satisfies the stochastic differential equation.

Definition

The equation has a unique weak solution if for any two weak solutions, $(X(t), B(t), (\Omega, \mathcal{F}, \mathbb{P}), t \geq 0)$, and $(\tilde{X}(t), \tilde{B}(t), (\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}), t \geq 0)$, the processes $(X(t), t \geq 0)$ and $(\tilde{X}(t), t \geq 0)$ have the same probability law.

Example

$$dX(t) = f(X(t))dt + \Phi dB^H(t)$$
$$X(0) = x$$

where $f : \mathbb{R}^n \to \mathbb{R}^n, \Phi \in L(\mathbb{R}^n), \Phi\Phi^* > 0$.
$(B^H(t), t \geq 0)$ is a $\mathbb{R}^n$-valued standard FBM. $E = V = \mathbb{R}^n$.
$S(t) = I$ for $t \in \mathbb{R}_+$.
$(\int_0^t \Phi dB^H, t \in [0, T])$ has $C^\beta([0, T], \mathbb{R}^n)$ sample paths for $\beta \in (0, H)$.

- Let $H \in (0, \frac{1}{2})$.
  If $f : \mathbb{R}^n \to \mathbb{R}^n$ is Borel measurable and

$$||f(x)|| \leq k_1(1 + ||x||)$$

  for all $x \in \mathbb{R}^n$, then there is one and only one weak solution.
- Let $H \in (\frac{1}{2}, 1)$.
  Assume additionally that

$$||f(x) - f(y)|| \leq k||x - y||^{\gamma}$$

  for all $x, y \in \mathbb{R}^n$ and some $\gamma > 1 - \frac{1}{2H}$, then there is one and only one weak solution.

$$
\begin{aligned}
&|\mathbb{K}_H^{-1}(\int_0^{\cdot} G(Z))|^2_{L^2([0,T])} \\
&\qquad \leq c_T(1 + \|X\|^2 + \|\tilde{Z}\|^2_{C([0,T])} + |\tilde{Z}|^2_{C^\beta([0,T])})
\end{aligned}
$$

$$
\tilde{Z}(t) = \int_0^t f(s, X(s))ds
$$

Controlled Stochastic Equation

$$\begin{aligned} dX(t) &= f(t, X(\cdot), u(t, X(\cdot)))dt + dB^H(t) \\ X(0) &= X_0 \end{aligned} \tag{1}$$

where $X_0 \in \mathbb{R}^n$ is a constant, $X(t) \in \mathbb{R}^n$, $t \in [0, T]$, $T > 0$ is fixed, $(B^H(t), t \in [0, T])$ is an $\mathbb{R}^n$-valued FBM with $H \in (0, \frac{1}{2})$, $u(t) \in U \subset \mathbb{R}^m$ (compact), $f(t, z, U)$ is closed and convex for each $t \in [0, T]$ and $z \in C([0, T])$.

Let $\mathcal{S}(T)$ be the Borel σ-algebra on $C[0,T] = \mathcal{C}$ with the topology of uniform convergence. The following conditions are used subsequently.

C1. The function $f : [0,T] \times \mathcal{C} \times U \to \mathbb{R}^n$ is jointly measurable;

C2. For each $t \in [0,T]$, the function $f(t,\cdot,\cdot) : C([0,t]) \times U \to \mathbb{R}^m$ is $\mathcal{S}(t) \otimes \mathcal{B}(U)$-measurable;

C3. The function $f(t,z,\cdot) : U \to \mathbb{R}$ is continuous for each $t \in [0,T]$ and $z \in C([0,t])$;

C4. The function $g : [0, T] \times C([0, T])$ given by $g(t, z) = f(t, z, u(t, z))$ satisfies

$$|g(t, z)| \leq k(1 + ||z||)$$

for $k > 0$ and each $z \in C([0, T])$ where $|| \cdot ||$ is the uniform norm. An admissible control is an $\mathcal{S}(T)$-measurable map such that for each $t \in [0, T]$, $u(t, \cdot)$ is $\mathcal{S}(t)$-measurable. The family of admissible controls is denoted by $\mathcal{U}$.

Proposition 2

Let $H \in (0, \frac{1}{2})$ be fixed. Let $g_u : [0, T] \times \mathbb{C}$ be the function given by

$$g_u(t, z) = f(t, z, u(t, z)).$$

for $u \in \mathcal{U}$. If C1, C2 and C4 are satisfied then

$$\mathbb{E}[\rho(g_u)] = 1$$

where

$$\rho(g_u) = \exp\Big(\int_0^T \mathbb{K}_H^{-1}\Big(\int_0^{\cdot} g_u\Big)(t)\Big)dW(t) - \frac{1}{2}\int_0^T \Big|\mathbb{K}_H^{-1}\Big(\int_0^{\cdot} g_u\Big)(t)\Big|^2 dt\Big).$$

Proposition 3

Let $u \in \mathcal{U}$ and let g_u be the associated drift term given above and $H \in (0, \frac{1}{2})$.
Then

$$\begin{aligned} dX(t) &= g_u(t, X(\cdot))dt + dB^H(t) \\ &= f(t, X(\cdot), u(t, X(\cdot)))dt + dB^H(t) \qquad (2) \\ X(0) &= X_0 \end{aligned}$$

has one and only one weak solution. The solution can be obtained from $(B^H(t), t \in [0, T])$ by a transformation of its measure by the Radon-Nikodym derivative $\rho(g_u)$ given above.

Proposition 4

Let C1-C4 be satisfied, $G = \{g_u(\cdot,\cdot) | g_u(t,z) = f(t,z,u(t,z))$ for f in (2) and $u \in \mathcal{U}\}$, and $\mathcal{D}(G) = \{\rho(g_u) | g_u \in G\}$. Then $\mathcal{D}(G)$ is a closed, convex, uniformly integrable subset of $L^1(\mathbb{P})$.

Let $L : \mathcal{C} \to \mathbb{R}$ be a bounded, continuous function. For $u \in \mathcal{U}$, let $J(u)$ be given by

$$J(u) = \int_{\mathcal{C}} L(z)\rho(g_u)d\mu_0(z) \tag{3}$$

where μ_0 is the measure on $(\mathcal{C}, \mathcal{S}(T))$ for the standard fractional Brownian motion $(B^H(t), t \in [0, T])$.

Theorem 5

Consider the stochastic control problem given by (1) and (3). If J^* is given by

$$J^* = \inf_{u \in \mathcal{U}} J(u),$$

then there is a control $u^* \in \mathcal{U}$ such that

$$J^* = J(u^*).$$