Control of Some Stochastic Systems with a Fractional Brownian Motion

T. E. Duncan

Some stochastic control systems that are described by stochastic differential equations with a fractional Brownian motion are considered. The solutions of these systems are defined by weak solutions. These weak solutions are obtained by the transformation of the measure for a fractional Brownian motion by a Radon-Nikodym derivative. This weak solution approach is used to solve a control problem for a controlled stochastic differential equation with a fractional Brownian motion and to verify the existence of an optimal control. The control occurs in the drift term of the stochastic differential equation and the drift term satisfies a convexity condition.

Fractional Brownian motion denotes a family of Gaussian processes that have continuous sample paths indexed by the Hurst parameter $H \in (0, 1)$ and that have properties that empirically appear in a wide variety of physical phenomena such as finance, economic data, hydrology, telecommunications, and medicine. These processes were defined by Kolmogorov [9] and some important properties were given by Mandelbrot and Van Ness [11]. Hurst [8] initiated the statistical analysis associated with these processes. Mandelbrot [10] used these processes to model some ecomonic data.

Since fractional Brownian motions seem to be reasonable models for many physical phenomena, it is important to study stochastic systems with a fractional Brownian motion, in particular, stochastic differential equations. The solution of a stochastic differential equation with a fractional Brownian motion is not readily obtained as it is for Brownian motion. Some work (e.g. [12]) has exhibited pathwise (or nonprobabilistic) solutions. However, many probabilistic computations are not available for those solutions. Strong or mild solutions of linear, bilinear and semilinear equations have been obtained in various formulations (e.g. [2, 3, 4, 5, 13]). Weak solutions have been obtained for some families of stochastic differential equations [6]. These weak solutions are obtained by transforming the measure of a fractional Brownian motion by a suitable Radon-Nikodym derivative. This method of weak solution is applied here to verify the existence of an optimal control for a controlled stochastic system described by a stochastic differential equation with a fractional Brownian motion. The optimal control problem is solved only for $H \in (0, \frac{1}{2})$.

A standard fractional Brownian motion $(B(t), t \geq 0)$ with $H \in (0, 1)$ is a Gaussian process with continuous sample paths on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that

$$\mathbb{E}[B(t)] = 0, \tag{1}$$

$$\mathbb{E}[B(s)B(t)] = \frac{1}{2}(s^{2H} + t^{2H} - |t-s|^{2H}) \tag{2}$$

for all $s, t \in \mathbb{R}_+$. For $H = \frac{1}{2}$, the process is a standard Brownian motion. For $H \neq \frac{1}{2}$, these processes are neither Markov nor semimartingales. For $H \in (\frac{1}{2}, 1)$, these processes have a long range dependence.

Let $H \in (0,1)$ be fixed and consider the stochastic differential equation

$$\begin{aligned} dX(t) &= f(X(t))dt + dB(t), \\ X(0) &= X_0 \end{aligned} \tag{3}$$

A weak solution of the equation (3) is a triple $(X(t), B(t), (\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}), t \geq 0)$ where $(B(t), t \geq 0)$ is a (real-valued) standard fractional Brownian motion defined on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ and a (real-valued) process $(X(t), t \geq 0)$ that satisfies

$$X(t) = X_0 + \int_0^t f(X(s))ds + B(t). \tag{4}$$

A stochastic optimal control problem is formulated and solved for a stochastic system given by

$$\begin{aligned} dX(t) &= f(t, X(\cdot), u(t, X(\cdot)))dt + dB(t), \\ X(0) &= X_0 \end{aligned} \tag{5}$$

where $X_0 \in \mathbb{R}$ is fixed, $X(t) \in \mathbb{R}$ for $t \in [0,T]$. $T > 0$ is fixed and $(B(t), t \in [0,T])$ is a fractional Brownian motion with $H \in (0, \frac{1}{2})$. $u(t) \in U \subset \mathbb{R}$ compact, $f(t, z, U)$ is closed and convex for each $t \in [0,T]$ and $z \in C([0,T])$. This formulation is analogous to a similar problem with Brownian motion [1, 6].

Let $\mathcal{S}(T)$ be the Borel σ-algebra on $C[0,T] = \mathcal{C}$ with the topology of uniform convergence. The following conditions are used subsequently.

C1. The function $f : [0,T] \times \mathcal{C} \times U \to \mathbb{R}$ is jointly measurable;

C2. For each $t \in [0,T]$, the function $f(t, \cdot, \cdot) : C([0,t]) \times U \to \mathbb{R}$ is $\mathcal{S}(t) \otimes \mathcal{B}(U)$-measurable;

C3. The function $f(t, z, \cdot) : U \to \mathbb{R}$ is continuous for each $t \in [0,T]$ and $z \in C([0,t])$;

C4. The function $g : [0,T] \times C([0,T])$ given by $g(t,z) = f(t, z, u(t,z))$ satisfies

$$|g(t,z)| \leq k(1 + ||z||)$$

for $k > 0$ and each $z \in C([0,T])$ where $|| \cdot ||$ is the uniform norm.

An admissible control is an $\mathcal{S}(T)$-measurable map such that for each $t \in [0,T], u(t, \cdot)$ is $\mathcal{S}(t)$-measurable. The family of admissible controls is denoted by $\mathcal{U}$.

A weak solution exists and is unique for the controlled stochastic system (5) by a result in [6].

Proposition 2 Let $H \in (0, \frac{1}{2})$ be fixed. Let $g_u : [0,T] \times \mathbb{C}$ be the function given by

$$g_u(t,z) = f(t,z,u(t,z)). \tag{6}$$

for $u \in \mathcal{U}$. If C1, C2 and C4 are satisfied then

$$\mathbb{E}[\rho(g_u)] = 1 \tag{7}$$

where

$$\begin{aligned}\rho(g_u) = \exp(&\int_0^T \mathbb{K}_H^{-1}(\int_0^{\cdot} g_u)(t))dW(t) \\ &- \frac{1}{2}\int_0^T |\mathbb{K}_H^{-1}(\int_0^{\cdot} g_u)(t))|^2 dt)\end{aligned} \tag{8}$$

and

$$\mathbb{K}_H^{-1}\phi(s) = c_H^{-1} s^{\frac{1}{2}-H} D_{0+}^{\frac{1}{2}-H}(u_{H-\frac{1}{2}} D_{0+}^{2H}\phi)(s).$$

c_H is a constant that only depends on H and D_{0+}^a is the (Riemann-Liouville) fractional derivative for $a \in (0,1)$. [14]

Proposition 3 Let $u \in \mathcal{U}$ and let g_u be the associated drift term given by (6) and $H \in (0, \frac{1}{2})$. Then

$$\begin{aligned}dX(t) &= g_u(t, X(\cdot))dt + dB(t) \\ &= f(t, X(\cdot), u(t, X(\cdot)))dt + dB(t) \\ X(0) &= X_0\end{aligned} \tag{9}$$

has one and only one weak solution. The solution can be obtained from $(B(t), t \in [0,T])$ by a transformation of its measure by the Radon-Nikodym derivative $\rho(g_u)$ given by (8).

Some important properties of the family of Radon-Nikodym derivatives in the solution of the control problem are given in the following result.

Proposition 4 Let C1-C4 be satisfied, $G = \{g_u(\cdot,\cdot) | g_u(t,z) = f(t,z,u(t,z))$ for f in (9) and $u \in \mathcal{U}\}$, and $\mathcal{D}(G) = \{\rho(g_u) | g_u \in G\}$. Then $\mathcal{D}(G)$ is a closed, convex, uniformly integrable subset of $L^1(\mathbb{P})$.

Let $L : \mathcal{C} \to \mathbb{R}$ be a bounded, continuous function. For $u \in \mathcal{U}$, let $J(u)$ be given by

$$J(u) = \int_{\mathcal{C}} L(z)\rho(g_u) d\mu_0(z) \tag{10}$$

where μ_0 is the measure on $(\mathcal{C}, \mathcal{S}(T))$ for the standard fractional Brownian motion $(B(t), t \in [0,T])$.

The existence of an optimal control is given in the following theorem.

Theorem 5 Consider the stochastic control problem given by (5) and (10). If J^* is given by

$$J^* = \inf_{u \in \mathcal{U}} J(u), \tag{11}$$

then there is a control $u^* \in \mathcal{U}$ such that

$$J^* = J(u^*). \tag{12}$$

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