Corporate finance model : a mixed singular/switching control problem

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Motivations : a corporate finance problem

Cash-flow utilization policy

1. **Dividend policy** : payment to shareholders
   - Stock purchase (dividend or investment ?)
   - Dividend payment

   $\longrightarrow$ **Singular problem** : Jeanblanc and Shiryaev (95), Choulli, Taksar and Zhou (03)

2. **Investment policy** : capital expenditure/investment for future growth....
   - Organic growth : internal development of new products, technologies, factories...
   - Merger/Acquisition : acquire a competitor for its product portfolio, geographical reach ...

   $\longrightarrow$ **Optimal switching problem** : Brekke and Oksendal (94), Duckworth and Zervos (01), Hamadène and Jeanblanc (05), Ly Vath and Pham (05).

Motivations : a corporate finance problem

Objective : Identify the best “dividend and investment" policy maximizing shareholders’ interest.

Typical corporate dilemma :

- Total, ENI, BP : how to use the huge Cash-Flow generated ? which investment project ?

Three policies :

1. Trader Classified Media : return over 90% of stock value in cash.
2. Bouygues : return 25% of stock value in cash.
3. Microsoft : almost no dividend payment for years !

Study a mixed singular/switching control problem : an extension of Décamps-Villeneuve (05)

Main result

The characterization of our natural intuition that the manager should always delay dividend payment if growth opportunity of an investment is deemed satisfying.

A coupled singular/switching problem

- A policy strategy is a singular/switching control $\alpha = (Z, (\tau_n))_{n\geq 1} \in \mathcal{A}$.

  Z : is a $\mathbb{F}$-adapted càdlàg non-decreasing process

  (τ_n) : an increasing sequence of stopping times representing investment decision times.

- We consider process X representing the dynamics of the cash reserve of a firm :

$$\begin{aligned} dX_t &= \sum \left(\mu_i 1_{\tau_{2n}\leq t<\tau_{2n+1}} + \mu_{1-i} 1_{\tau_{2n+1}\leq t<\tau_{2n+2}}\right) dt + \sigma dW_t - dZ_t - dK_t, \\ X_{0-} &= x \end{aligned}$$

where μ_i : flow of generated cash depending on regime, $i = 0, 1$, with $\mu_0 < \mu_1$.

K : cost related to investments $g > 0$ and disinvestments $(1-\lambda)g$

Investment Problem

- Investment Problem : maximizing the value received by shareholders (Discounted Dividend Flow method)

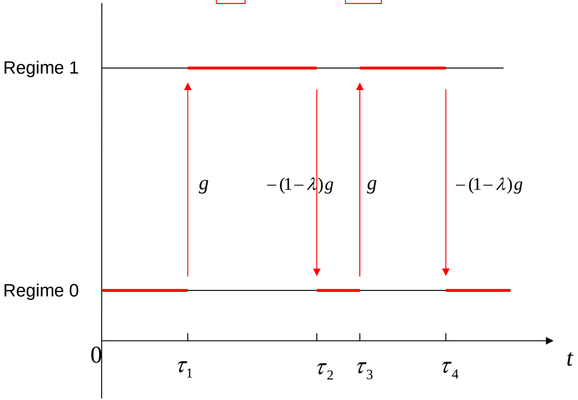
$$v_i(x) \quad = \quad \sup_{\alpha \in \mathcal{A}} \mathbb{E}\left[\int_0^{T^-} e^{-\rho t} dZ_t\right], \quad x \in \mathbb{R},\ i = 0, 1,$$

where T is the time of strict bankruptcy

$$T \quad = \quad T^{x,i,\alpha} = \inf\left\{t \geq 0 \ : X_t^{x,i} \ < \ 0\right\}$$

The model : more details

$$dX_t = \sum\left(\mu_i 1_{\tau_{2n} \le t < \tau_{2n+1}} + \mu_{1-i} 1_{\tau_{2n+1} \le t < \tau_{2n+2}}\right)dt + \sigma dW_t - dZ_t - dK_t$$



$$v_i(x) = \sup_{(Z,(\tau_n)_n)} E\left[\int_0^{T^-} e^{-\rho t} dZ_t\right]$$

Dynamic programming system and properties of v_i

- Model and problem formulation
- $\Rightarrow$ Dynamic programming system and properties of v_i
- Qualitative results on the switching regions

DPP and associated VI system

Variational inequalities system (V.I.S) associated to DPP is :

Varational Inequalities System (V.I.S)

$$\begin{aligned} \min\left[\rho v_0(x) - \mathcal{L}_0 v_0(x)\ ,\ v_0'(x) - 1\ ,\ v_0(x) - v_1(x-g)\right] &= 0, \\ \min\left[\rho v_1(x) - \mathcal{L}_1 v_1(x)\ ,\ v_1'(x) - 1\ ,\ v_1(x) - v_0(x + (1-\lambda)g)\right] &= 0,\ x > 0. \end{aligned}$$

where $\mathcal{L}_i$ is the infinitesimal generator associated to the firm cash reserve process, in the absence of dividend payement and in the regime i :

$$dR_t^i = \mu_i dt + \sigma dW_t, \quad R_0^i = x.$$

We define the following regions : $i = 0, 1$:

$$\begin{aligned} \mathcal{S}_i &= \{x \geq 0\ : v_i(x) = v_{1-i}(x - g_{i,1-i})\} :\ \text{switching region} \\ \mathcal{D}_i &= \{x > 0\ : v_i'(x) = 1\} :\ \text{dividend region} \\ \mathcal{C}_i &= (0, \infty) \setminus (\mathcal{S}_i \cup \mathcal{D}_i) :\ \text{continuation region.} \end{aligned}$$

Viscosity property and regularity

Analytical characterization by viscosity : theoretical results

Theorem

The value functions v_i, $i = 0, 1$, are continuous on $(0, \infty)$, and constitute the unique solution, satisfying linear growth and boundary conditions $v_0(0) = 0$, $v_1(0) = v_0((1-\lambda)g)$, of our (V.I.S) :

$$\min \left[\rho v_i(x) - \mathcal{L}_i v_i(x), v_i'(x) - 1, v_i(x) - v_{1-i}(x - g_{i,1-i})\right] = 0, \quad x > 0, \ i = 0, 1.$$

Proposition : Regularity of the value functions

The functions v_i, $i = 0, 1$, are of class C^1 on $(0, \infty)$ and C^2 on $\mathcal{C}_i \cup \mathcal{D}_i$.

Qualitative results on the switching regions

- Model and problem formulation
- Dynamic programming system and properties of v_i
- $\Rightarrow$ Qualitative results on the switching regions

Benchmark 0

We consider the value of the firm without investment/disinvestment $i = 0$:

$$\hat{V}_0(x) = \sup_{Z \in \mathcal{Z}} \mathbb{E}\left[\int_0^{T_0^-} e^{-\rho t} dZ_t\right],$$

▷ $\hat{V}_0$ the value function of a singular problem

▷ characterized as the unique solution on $(0, \infty)$ with growth linear condition to $(V.I)_0$:

$$\min\left[\rho\hat{V}_0 - \mathcal{L}_0\hat{V}_0 \ , \ \hat{V}_0' - 1\right] = 0, \quad x > 0, \tag{1}$$

$$\hat{V}_O(0) = 0. \tag{2}$$

Benchmark 0

Explicit solution : Jeanblanc and Shiryaev (95) or Radner and Shepp (96)

▷ $\hat{V}_O$ explicit expression

▷ **Optimal cash reserve process** : the reflected diffusion process at threshold $\hat{x}_0$

▷ **Optimal dividend process** : the local time at this boundary.

Benchmark 1

We consider the value of the firm without investment/disinvestment $i = 1$:

$$\hat{V}_1(x) = \sup_{Z \in \mathcal{Z}} \mathbb{E}\left[\int_1^{T_1^-} e^{-\rho t} dZ_t + e^{-\rho T_1} L\right],$$

▷ $\hat{V}_1$ the value function of a singular problem

▷ characterized as the unique viscosity solution on $(0, \infty)$ with growth linear conditions to $(V.I)_1$:

$$\min\left[\rho \hat{V}_1 - \mathcal{L}_1 \hat{V}_1 \ , \ \hat{V}_1' - 1\right] = 0, \quad x > 0, \tag{3}$$

$$\hat{V}_1(0) = L. \tag{4}$$

where $L = \hat{V}_0((1 - \lambda)g)$.

Benchmark 1

▷ **Explicit solution** : see Goguslavskaya (03)

• If $L \geq \frac{\mu_1}{\rho}$, then :

$$\hat{V}_1(x) = x + L, \quad x \geq 0.$$

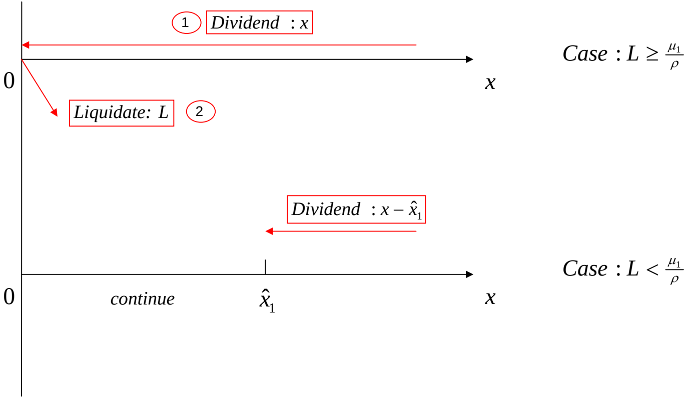
▷ The optimal strategy is to distribute the initial cash reserve immediately, and so to liquidate the firm at $X_t = 0$ and receiving L.

• If $L < \frac{\mu_1}{\rho}$, then

▷ the optimal cash reserve process : the reflected diffusion process at a threshold $\hat{x}_1$

▷ the optimal dividend process : the local time at this boundary.

Benchmark 1



Benchmark 1: a singular control problem

Main results and description of the solution

We have to mainly distinguish two separate cases depending on the model parameters :

- Case 1 : $\hat{V}_0((1-\lambda)g) \geq \frac{\mu_1}{\rho}$
- Case 2 : $\hat{V}_0((1-\lambda)g) < \frac{\mu_1}{\rho}$

Case 1 : $\hat{V}_0((1-\lambda)g) \geq \frac{\mu_1}{\rho}$

Theorem

Suppose that $\hat{V}_0((1-\lambda)g) \geq \frac{\mu_1}{\rho}$.

Then, we have $v_0(x) = \hat{V}_0(x)$ *and* $v_1(x) = \hat{V}_0(x+(1-\lambda)g) = x+(1-\lambda)g - \hat{x}_0 + \frac{\mu_0}{\rho}$.

$\triangleright$ *It is optimal to never switch from regime* 0 *to regime* 1.

$\triangleright$ *In regime* 1, *it is optimal to distribute all the surplus as dividends and to switch to regime* 0.

Case 2 : $\hat{V}_0((1-\lambda)g) < \frac{\mu_1}{\rho}$

1.) Subcase : $\frac{\mu_1 - \mu_0}{\rho} \leq \hat{x}_1 + g - \hat{x}_0$.

Theorem

Suppose that

$$\hat{V}_0((1-\lambda)g) < \frac{\mu_1}{\rho} \leq \frac{\mu_0}{\rho} + \hat{x}_1 + g - \hat{x}_0. \quad (5)$$

Then

$$v_0 = \hat{V}_0,$$
$$v_1 = \hat{V}_1.$$

It is never optimal, once in regime $i = 0$*, to switch to regime* $i = 1$.

In regime 1*, it is optimal to switch to regime* 0 *at the threshold* $x = 0$.

Case 2 : $\hat{V}_0((1-\lambda)g) < \frac{\mu_1}{\rho}$

2.) Subcase : $\frac{\mu_1 - \mu_0}{\rho} \geq \hat{x}_1 + g - \hat{x}_0$.

We reformulate as a coupled pure stopping time and pure singular problem.

First step : introduction of a pure singular problem

We first notice that $\mathcal{S}_1 = \{0\}$, so that v_1 is the unique solution, with the boundary data $v_1(0) = v_0((1-\lambda)g)$, to

$$\min\left[\rho v_1 - \mathcal{L}_1 v_1, v_1' - 1\right] \quad = \quad 0, \quad x > 0.$$

Therefore, v_1 is the firm value problem in regime $i = 1$ with liquidation value $v_0((1-\lambda)g)$:

$$v_1(x) \quad = \quad \sup_{Z \in \mathcal{Z}} \mathbb{E}\left[\int_0^{T_1^-} e^{-\rho t} dZ_t + e^{-\rho T_1} v_0((1-\lambda)g)\right]. \tag{6}$$

The form of v_1 is described by Benchmark 1, but at this stage the liquidation value $v_0((1-\lambda)g)$ is still unknown.

Case 2 : $\hat{V}_0((1-\lambda)g) < \frac{\mu_1}{\rho}$

Second step : introduction of a pure stopping time problem

We introduce the pure stopping time problem

$$\bar{v}_0(x) = \sup_{\tau \in \mathcal{T}} \mathbb{E}\left[e^{-\rho(\tau \wedge T_0)} \max\left(\hat{V}_0(R^{x,0}_{\tau \wedge T_0}), v_1(R^{x,0}_{\tau \wedge T_0} - g)\right)\right], \quad (7)$$

where $\mathcal{T}$ denotes the set of stopping times valued in $[0, \infty]$.

Theorem

Suppose that

$$\hat{V}_0((1-\lambda)g) < \frac{\mu_1}{\rho} \text{ and } \frac{\mu_1 - \mu_0}{\rho} > \hat{x}_1 + g - \hat{x}_0. \quad (8)$$

Then, we have

$$v_0 = \bar{v}_0$$

and v_1 *given by* (6).

Case 2 : $\hat{V}_0((1-\lambda)g) < \frac{\mu_1}{\rho}$

Remark : The below representation of pure optimal singular and stopping problems for v_1 and v_0 is coupled and not easily computable.

$$\begin{aligned} v_1(x) &= \sup_{Z \in \mathcal{Z}} \mathbb{E}\left[\int_0^{T_1^-} e^{-\rho t} dZ_t + e^{-\rho T_1} v_0((1-\lambda)g)\right], \\ v_0(x) &= \sup_{\tau \in \mathcal{T}} \mathbb{E}\left[e^{-\rho(\tau \wedge T_0)} \max\left(\hat{V}_0(R^{x,0}_{\tau \wedge T_0}), v_1(R^{x,0}_{\tau \wedge T_0} - g)\right)\right]. \end{aligned}$$

Decoupling the coupled representation

We decouple this representation by considering the sequence of pure optimal stopping and singular control problems,

starting from $\hat{V}_1^{(0)} = \hat{V}_1$ and $\hat{V}_0^{(0)} = \hat{V}_0$:

$$\hat{V}_0^{(k)}(x) = \sup_{\tau \in \mathcal{T}} \mathbb{E}\left[e^{-\rho(\tau \wedge T_0)} \max\left(\hat{V}_0(R^{x,0}_{\tau\wedge T_0}), \hat{V}_1^{(k-1)}(R^{x,0}_{\tau\wedge T_0} - g)\right)\right], \quad k \geq 1,$$

$$\hat{V}_1^{(k)}(x) = \sup_{Z \in \mathcal{Z}} \mathbb{E}\left[\int_0^{T_1^-} e^{-\rho t} dZ_t + e^{-\rho T_1} \hat{V}_0^{(k)}((1-\lambda)g)\right], \quad k \geq 1.$$

Decoupling the coupled representation

The next result shows the convergence of this procedure.

Proposition

Under the conditions

$$\hat{V}_0((1-\lambda)g) \; < \; \frac{\mu_1}{\rho} \text{ and } \frac{\mu_1 - \mu_0}{\rho} \; > \; \hat{x}_1 + g - \hat{x}_0. \tag{9}$$

we have for all $x > 0$:

$$\lim_{k\to\infty} \hat{V}_0^{(k)}(x) \; = \; v_0(x), \qquad \lim_{k\to\infty} \hat{V}_1^{(k)}(x) \; = \; v_1(x).$$

Decoupling the coupled representation

According to previous Proposition, the value functions can be constructed recursively starting from $(\hat{V}_0, \hat{V}_1)$.

Two cases have to be considered :

Case A : $\hat{V}_0^{(1)}((1-\lambda)g) = \hat{V}_0((1-\lambda)g)$.

Then we have

- $(\hat{V}_0^{(k)})_k$ is constant for $k \geq 1$
- $(\hat{V}_1^{(k)})_k$ is constant for $k \geq 0$.

▷ Therefore, $v_0 = \hat{V}_0^{(1)}$ and $v_1 = \hat{V}_1$.

Decoupling the coupled representation

Case B : $\hat{V}_0^{(1)}((1-\lambda)g) > \hat{V}_0((1-\lambda)g)$.

We introduce the sequence

$$\begin{aligned}\hat{\theta}_0^{(k)}(x) &= \sup_{\tau \in \mathcal{T}} \mathbb{E}\left[e^{-\rho(\tau \wedge T_0)}\hat{\theta}_1^{(k-1)}(R^{x,0}_{\tau \wedge T_0} - g)\right], \quad k \geq 1, \\ \hat{\theta}_1^{(k)}(x) &= \sup_{Z \in \mathcal{Z}} \mathbb{E}\left[\int_0^{T_1^-} e^{-\rho t} dZ_t + e^{-\rho T_1}\hat{\theta}_0^{(k)}((1-\lambda)g)\right], \quad k \geq 1.\end{aligned}$$

starting from $\hat{\theta}_1^{(0)} = \hat{V}_1$ and $\hat{\theta}_0^{(0)} = \hat{V}_0$.

Decoupling the coupled representation

Convergence of $(\hat{\theta}_0^{(k)}, \hat{\theta}_1^{(k)})$

$(\hat{\theta}_0^{(k)}, \hat{\theta}_1^{(k)})$ converges to $(\hat{\theta}_0^{(\infty)}, \hat{\theta}_1^{(\infty)})$ solution of the system of VI :

$$\min\left(\rho\hat{\theta}_0^{\infty} - \mathcal{L}_0\hat{\theta}_0^{\infty}, \hat{\theta}_0^{\infty} - \hat{\theta}_1^{\infty}(. - g)\right) = 0,$$

$$\min\left(\rho\hat{\theta}_1^{\infty} - \mathcal{L}_1\hat{\theta}_1^{\infty}, (\hat{\theta}_1^{\infty})' - 1\right) = 0,$$

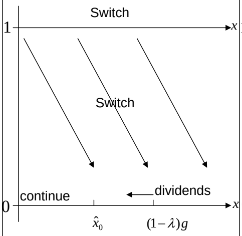
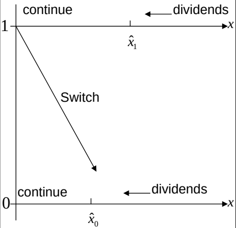
with initial conditions $\hat{\theta}_1^{\infty}(0) = \hat{\theta}_0^{\infty}((1-\lambda)g)$ and $\hat{\theta}_0^{\infty}(0) = 0$.

$\hat{\theta}_0^{\infty}$: accumulate cash reserve in order to invest/no dividend payment.
we distinguish two cases :

- $\hat{\theta}_0^{\infty}(\hat{x}_0) > \hat{V}_0(\hat{x}_0) \Rightarrow$ no dividend distribution in order to invest.
- $\hat{\theta}_0^{\infty}(\hat{x}_0) \leq \hat{V}_0(\hat{x}_0) \Rightarrow$ Ignore the strategy $\hat{\theta}_0^{\infty}$.

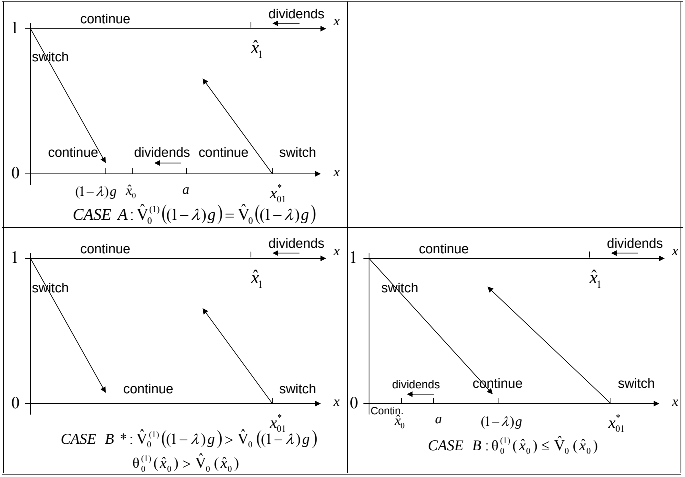
Results and solution description

Synthetic table 1

$\frac{\mu_1}{\rho} \le \hat{V}_0((1-\lambda)g)$	$\hat{V}_0((1-\lambda)g) < \frac{\mu_1}{\rho} \le \frac{\mu_0}{\rho} + \hat{x}_1 + g - \hat{x}_0$	$\frac{\mu_1}{\rho} > \max\left(\hat{V}_0((1-\lambda)g), \frac{\mu_0}{\rho} + \hat{x}_1 + g - \hat{x}_0\right)$
$v_0(x) = \hat{V}_0(x)$ $v_1(x) = x + (1-\lambda)g - \hat{x}_0 + \frac{\mu_0}{\rho}$	$v_0(x) = \hat{V}_0(x)$ $v_1(x) = \hat{V}_1(x)$	$v_0(x) = \hat{V}_0^{\infty}(x)$ $v_1(x) = \hat{V}_1^{\infty}(x)$
		See figure 1

Results and solution description

Figure 1



Thank you for your attention !