Second General AMAMEF Conference, Bedlewo 2007

On $\sigma-$additive robust representation of convex risk measures for unbounded financial positions in the presence of uncertainty about the market model

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[1] This research was supported by the Deutsche Forschungsgemeinschaft through the SFB 649 „Economic Risk“

0 Introduction

- $\mathfrak{X}$ vector space of available financial positions (e.g. class of stochastic processes)
- **static** risk assessment by a (generalized) risk measure $\rho : \mathfrak{X} \to \mathbb{R}$

  liquid derivatives as financial instruments (cf. Frittelli/Scandolo (2006))
- robust representation of convex ρ : [convex analysis]

  $$\rho(X) = \sup_{\Lambda \in \mathfrak{X}^*}(-\Lambda(X) - \beta(\Lambda)) = \max_{\Lambda \in \mathfrak{X}^*}(-\Lambda(X) - \beta(\Lambda))$$

  $\mathfrak{X}^*$ space of linear forms on $\mathfrak{X}$, β penalty function

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subject of the talk:

- $\sigma-$**additive robust representation of** ρ :

  $$\rho(X) = \sup_{\Lambda \in \mathfrak{X}^*(\sigma)} (-\Lambda(X) - \beta(\Lambda))$$

- **strong** $\sigma-$**additive robust representation of** ρ :

  $$\rho(X) = \max_{\Lambda \in \mathfrak{X}^*(\sigma)} (-\Lambda(X) - \beta(\Lambda))$$

  $\mathfrak{X}^*(\sigma) := \{\Lambda \in \mathfrak{X}^* \mid \Lambda$ representable by a ($\sigma -$ additive) probability measure$\}$

- full solutions for (genuine) risk measures if $\mathfrak{X} \widehat{=} L_p(\mathrm{P})$ $(p \in [1, \infty])$, P reference measure:
  - p real:

    always strong $\sigma-$additive robust representation (e.g. Ruszczynski/Shapiro (2006))
  - $p = \infty$:
    * so-called **Fatou property** is necessary and sufficient condition for $\sigma-$additive robust representation (cf. Delbaen (2002), Föllmer/Schied (2004))
    * necessary and sufficient condition for strong $\sigma-$additive robust representation in Jouini/Schachermayer/Touzi (2006)

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- **representation under uncertainty** about the reference measure:
  - Fatou property is not sufficient in general (Delbaen (2002))
  - sufficient criteria for **bounded** one-period positions concerning
    - strong $\sigma-$additive robust representation in Föllmer/Schied (2004), Krätschmer (2005)
    - $\sigma-$additive robust representation in metric space context in Krätschmer (2005)

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extensions:

* **unbounded** positions
* non topological framework
* **necessary and sufficient** condition for strong $\sigma-$additive robust representation
* when remains the Fatou property sufficient for $\sigma-$additive robust representation?

Contents

1 Robust representation of convex risk measures

$$\Omega \neq \varnothing \text{ (e.g. } \Omega = \underbrace{\widetilde{\Omega}}_{\text{state space}} \times \underbrace{\mathbb{T}}_{\text{time set}} \text{)}$$

$\geq$ pointwise partial order on $\mathbb{R}^{\Omega}$

$\mathfrak{X} \subseteq \mathbb{R}^{\Omega}$ vector space of available financial positions

$\mathbb{R} \subseteq \mathfrak{C} \subseteq \mathfrak{X}$ vector space of financial instruments

$\pi : \mathfrak{C} \to \mathbb{R}$, positive linear price functional, $\pi(1) = 1$

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risk measure $\rho : \mathfrak{X} \to \mathbb{R}$ w.r.t π :

- monotonicity : $\rho(X) \leq \rho(Y)$ for $X \geq Y$
- translation invariance w.r.t. π : $\rho(X + Y) = \rho(X) - \pi(Y)$ for $Y \in \mathfrak{C}$

$\to$ ρ as capital requirement

$$\rho(X) = \inf\{\pi(Y) \mid \rho(X + Y) \leq 0\} \text{ for } X \in \mathfrak{X}$$

$\beta_\rho : \mathfrak{X}^* \to \mathbb{R} \cup \{\infty\},\ \Lambda \mapsto \rho^*(-\Lambda),\ \rho^*$ Fenchel-Legrendre transform of ρ

$\mathfrak{X}^* \mathrel{\widehat{=}}$ set of linear forms on $\mathfrak{X}$

- **basic representation result**:

$$\rho \text{ convex} \iff \rho(X) = \max_{\Lambda \in \mathfrak{X}^*} \big(-\Lambda(X) - \beta_\rho(\Lambda)\big) \text{ for all } X \in \mathfrak{X}$$

every Λ from domain of β_ρ has to be a positive linear form extending π

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- **reduction to bounded positions:** ρ convex, $\mathfrak{X}$ vector lattice

$$\rho(X) = \sup_{Q \in \mathcal{M}} \left(-E_Q[X] - \beta_\rho(E_Q)\right) \text{ for all bounded } X$$

$$\Leftrightarrow \quad \rho(X) = \sup_{Q \in \mathcal{M}} \left(-E_Q[X] - \beta_\rho(E_Q)\right) \text{ for all } X$$

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every Λ from domain of β_ρ is the asymmetric Choquet integral w.r.t. some (finitely additive) probability content on $\sigma(\mathfrak{X})$

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$\Updownarrow$ Greco's representation theorem

$\lim_{n \to \infty} \rho(-\lambda(X - n)^+) = \rho(0)$ for $\lambda > 0, \; X \in \mathfrak{X}, X \geq 0$ cutting condition

2 Robust representation of convex risk measures by inner regular probability measures

$\mathfrak{L} \subseteq \mathfrak{X}$ vector lattices, $\sigma(\mathfrak{L}) = \sigma(\mathfrak{X})$

- **Theorem 2.1:**

  sufficient conditions for $\sigma-$additive robust representation (on $\sigma(\mathfrak{X})$):

(2.1) $\lim_{n\to\infty} \rho(-\lambda(X-n)^+) = \rho(0)$ for $\lambda > 0,\ X \in \mathfrak{X}, X \geq 0$

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$E := \{\sup_n Y_n \mid Y_n \in \mathfrak{L}, Y_n \geq 0$ bounded$\}$

(2.3) $\rho(X_n) \searrow \rho(X)$ for $X_n \in \mathfrak{L}$ bounded $\nearrow X \in \mathfrak{L}$ bounded

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    $\mathcal{S} := \{\bigcap_{n=1}^{\infty} X_n^{-1}([x_n, \infty[) \mid X_n \in \mathfrak{L}$ nonnegative, $x_n > 0\}$ (inner Daniell Stone)

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  - $\mathfrak{L} = \mathfrak{X} \longrightarrow$ (2.2), (2.4) redundant

- **Examples for $\mathfrak{X}, \mathfrak{L}$:**

  - $\mathfrak{X} \widehat{=}$ space of bounded Borel-measurable mapping w.r.t. some metrizable topology on Ω

    $\mathfrak{L} \widehat{=}$ space of bounded continuous mappings

    $$\downarrow$$

    $E\ =$ set of nonnegative bounded lower semicontinuous mappings

    [cf. Krätschmer (2005)]

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    $E \;=$ set of nonnegative bounded lower semicontinuous mappings

    [cf. Krätschmer (2005)]

  - $\Omega = \widetilde{\Omega} \times [0, T]$ $(\mathcal{F}_t)_{t \in [0,T]}$ filtration of $\sigma-$algebras on $\widetilde{\Omega}$

    $\mathfrak{X} \widehat{=}$ space of cadlag processes

    $\mathfrak{L} \widehat{=}$ vector space spanned by stochastic intervals $[S_1, S_2[$

3 Strong $\sigma-$additive robust representation of convex risk measures

$\mathfrak{X}$ vector lattice

$\rho : \mathfrak{X} \to \mathbb{R}$ convex risk measure w.r.t. π

- **Theorem 3.1:**

  Let $\mathfrak{X}$ contain all at most countable convex combinations of $\{X_n \mid n \in \mathbb{N}\}$ whenever $X_n \searrow 0$. Then

$$\begin{aligned} & \rho \text{ admits strong } \sigma - \text{additive robust representation (on } \sigma(\mathfrak{X})) && (1) \\ \Leftrightarrow \quad & \rho(X_n) \searrow \rho(X) \text{ for } X_n \nearrow X && (2) \\ \Leftrightarrow \quad & \Lambda \text{ representable by a probability measure if } \beta_\rho(\Lambda) < \infty && (3) \end{aligned}$$

- **Remarks:**

  – continuity property (2) may be simplified (cf. paper)

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    [$(2) \Rightarrow (3)$ Föllmer/Schied (2004), $(3) \Rightarrow (2)$ Krätschmer (2005)]

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    [$(2) \Rightarrow (3)$ Föllmer/Schied (2004), $(3) \Rightarrow (2)$ Krätschmer (2005)]

  – for proof of $(1) \Rightarrow (2)$ use Simons' Lemma $\leftarrow$ assumptions on $\mathfrak{X}$

- **Proof of Theorem 3.1:**

  $(2) \overset{2.1}{\Rightarrow} (3) \overset{\text{trivial}}{\Rightarrow} (1)$

  $\underline{(1) \Rightarrow (2)}$: (sketch)

  Let $X_n \nearrow X$

  one may rewrite for some real c

$$\rho(X) = \max_{\beta\rho(\Lambda)\le c} f(\Lambda) \overset{(1)}{=} \max_{\beta\rho(E_\mathrm{P})\le c} f(E_\mathrm{P})$$

$$\rho(X_n) = \max_{\beta\rho(\Lambda)\le c} f_n(\Lambda) \overset{(1)}{=} \max_{\beta\rho(E_\mathrm{P})\le c} f_n(E_\mathrm{P})$$

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$\rightarrow$ $\{E_{\mathrm{P}} \mid \beta_\rho(E_{\mathrm{P}}) \leq c\}$ boundary for $f, f_n (n \in \mathbb{N})$

$\rightarrow$ suggests application of Simons' lemma

$\downarrow$ assumption on $\mathfrak{X}$

$$\lim_{n\to\infty} \rho(X_n) = \rho(X)$$

4 $\sigma-$additive robust representation of convex risk measures and Fatou properties

$\rho : \mathfrak{X} \to \mathbb{R}$ convex risk measure w.r.t. π

$\mathfrak{X} \subseteq B(\Omega) \widehat{=}$ space of bounded real-valued mappings on Ω

$\tau_p \widehat{=}$ pointwise topology on $B(\Omega)$

- Fatou property:

$$\liminf_n \rho(X_n) \geq \rho(X) \text{ whenever } (X_n)_n \text{ uniformly bounded, } X_n \to X \text{ pointwise}$$

necessary, but not sufficient in general (cf. Delbaen (2002))

$\to$ impose additional assumptions on τ_p

$\to$ Krein-Smulian argument

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- **Theorem 4.1:**

  Suppose

(4.1) $B_r := \{X \in \mathfrak{X} \cap B(\Omega) \mid \|X\|_\infty \leq r\}$ $(r > 0)$ $\tau_p -$ closed

(4.2) every $Z \in cl^{\tau_p}(\{X \in B_r \mid \rho(X) \leq 0\})$ is pointwise limit of a sequence $(X_n)_n$ with $\rho(X_n) \leq 0$ and $\|X_n\| \leq r$ $(r > 0)$

ρ admits a $\sigma-$additive robust representation (on $\sigma(\mathfrak{X})$) $\Leftrightarrow$ ρ satisfies Fatou property.

- **Remarks:**
  - Theorem 4.1 immediately applicable if Ω at most countable and $\mathfrak{X}$ sequentially closed w.r.t. τ_p (e.g. if $\mathfrak{X} \widehat{=}$ space of bounded X, measurable w.r.t. a $\sigma-$algebra on Ω)
  - under (4.1) alone a nonsequential counterpart of Fatou property is sufficient, and it is even necessary if (4.1) is further strengthened (cf. paper)
  - both Fatou properties are equivalent under (4.2)
  - nonsequential Fatou propery breaks down for $\rho(X) = -E_{\mathrm{P}}[X]$ whenever P is an absolutely continuous distribution on $\mathbb{R}$
  - (4.2) is very restrictive (cf. theory of angelic spaces)

5 Final Remarks

- representation results for $\mathfrak{X} = L_p(\mathrm{P})\ p \in [1, \infty],\ \mathrm{P}$ reference measure, may be retained (cf. paper)

- representation result w.r.t. inner regular probability measures may be further generalized (cf. ealier version of paper: technical measure theory, not very accessible)

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- In face of uncertainty about the market model the Fatou properties (sequential and nonsequential) seem to be unsatisfactory for very general spaces of financial positions
- **summary:**
  - strong $\sigma-$additive robust representation is solved
  - a more general criterion for $\sigma-$additive robust representation
  - a class of spaces of financial positions where Fatou property is necessary and sufficient for $\sigma-$additive robust representation