WHITE NOISE GENERALIZATION OF THE CLARK-OCONE FORMULA UNDER CHANGE OF MEASURE

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Abstract. We prove white noise generalization of the Clark-Ocone formula under change of measure by using white noise analysis on Malliavin calculus. In this paper, it is shown that for any random variable $F \in L^2(P)$

$$F(\omega) = E_Q[F] + \int_0^T E_Q[(D_tF - F\int_t^T D_tu(s)d\hat{W}(s)) \mid \mathcal{F}_t]\, d\hat{W}(t),$$

where E_Q is the expectation under white noise measure Q, $\hat{W}(t)$ is the 1-dimensional Brownian motion constructed on the white noise probability space $(\Omega, \mathcal{B}, Q)$ and $D_tF(\omega)$ is the (Hida) Malliavin derivative. The important point to note here is in this settlement F should not belong to stochastic Sobolev space, $\mathbb{D}_{1,2}$ which is subset of $L^2(P)$ that leads this formula more useful in applications to finance. Moreover, the replicating portfolio for digital option, whose payoff function $\chi_{[K,\infty)}W(T) \notin \mathbb{D}_{1,2}$, is calculated by using generalized Clark-Ocone formula under change of measure.

1. Introduction

Gaussian white noise theory was first introduced by Hida [5]. Afterwards, it is developed by him and other researchers that becomes a powerful tool in mathematical physics (see [6] and the references therein). After that Holden et al [7] emphasized this theory with stochastic partial differential equations (SPDEs) driven by Brownian motion. The first contribution to white noise theory to finance comes from the joint work of Aase et al to prove the generalization of Clark-Ocone formula [1]. By this theorem it is a natural and intrinsic way of computing the replicating portfolio of call option in Black & Scholes type market. They proved that

$$F(\omega) = E[F] + \int_0^T E[D_tF \mid \mathcal{F}_t] \diamond \dot{W}(t)dt. \tag{1.1}$$

Here $E[F]$ denotes the generalized expectation, $D_tF(\omega) = \frac{dF}{d\omega}$ is the (generalized) Malliavin derivative, $\diamond$ is the Wick product and $\dot{W}(t)$ is 1-dimensional Gaussian white noise. This formula holds for all $F \in \mathcal{G}^* \supset L^2(P)$, where $\mathcal{G}^*$ is a space of stochastic distributions and P is the white noise probability measure.

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Actually, the original Clark-Ocone formula was proved by Ocone in 1984 to give an explicit representation to the integrand in Itô integral representation theorem in the context of analysis on the Wiener space $\Omega = C_0([0,T])$, the space of real continuous functions on $[0,T]$ starting at zero [13]. He showed that

$$F(\omega) = E[F] + \int_0^T E[D_t F \mid \mathcal{F}_t] dW(t), \tag{1.2}$$

where D_t is the Malliavin derivative and $W(t)$ is one dimensional Brownian motion. In case of changing measure by Girsanov theorem, Karatzas & Ocone 1991 [9] argued that under some assumptions the Clark-Ocone formula under change of measure is

$$F(\omega) = E_Q[F] + \int_0^T E_Q[(D_t F - F \int_t^T D_t u(s) d\hat{W}(s)) \mid \mathcal{F}_t)]\, d\hat{W}(t), \tag{1.3}$$

where D_t is the Malliavin derivative and $d\hat{W}(t)$ is one dimensional Brownian motion under probability measure Q. However, the main drawback with this setting is that the Malliavin derivative only exists for $F \in \mathbb{D}_{1,2}$. The problem is that this settlement excludes many interesting applications in finance, such as digital option. The purpose of this paper is to represent a new proof of the Clark-Ocone formula under change of measure in the white noise setting.

The first three chapters of this paper constitute sufficient preparation to prove Clark-Ocone formula under change of measure for $F \in L^2(P)$. In the section 2 and 3 we summarize without proofs the relevant material on white noise analysis and special functions respectively. Many versions of those results have already been provided and known by many theorists but we think that it is better to have a unified approach based on white noise theory. In the section 4, we introduce the notion of chaos expansion and define two different kinds of chaos expansions where they coincide somehow. Section 5 contains a brief summary of Hida stochastic test function, distribution space and the Malliavin derivative on this space. In the section 6, we develop the theory of Clark-Ocone formula under change of measure on the analysis of white noise settlement. In the last section, we apply our result to compute the replicating portfolio for Digital option in Black and Scholes type market.

2. White noise analysis

In this section we will set up notation and terminology in Gaussian white noise theory. Let $S = S(\mathbb{R})$ be the Schwartz space of rapidly decreasing smooth functions $\phi \in C^\infty(\mathbb{R})$ such that

$$\| \phi \|_{k,n} = \sup_{x\in\mathbb{R}} \mid \phi^{(j)}(x) x^n \mid < \infty \quad \text{for all } j \leq k,\ n \leq N.$$

The space $S(\mathbb{R})$ equipped with the family of seminorms $\| \ . \ \|_{k,N}$ constitutes a Frechet space. The dual of $S(\mathbb{R})$ is denoted by $S'(\mathbb{R})$ and called the space of tempered distributions equipped with the weak-star topology. If $\omega \in S'(\mathbb{R})$ and $\phi \in S$ we set

$$\omega(\phi) =< \omega, \phi >$$

that denotes the action of the linear functional ω on the test function ϕ.

Theorem 2.1. ***(Bochner-Minlos)***
Let $g : S(\mathbb{R}) \to \mathbb{R}$ be a given function. Then there exists a probability measure P on $\Omega := S'(\mathbb{R})$ s.t.

$$\int_{S'(\mathbb{R})} e^{i<\omega,\phi>} dP(\omega) = g(\phi) \ ; \quad \phi \in S(\mathbb{R}) \tag{2.1}$$

if and only if

- *g(0)=1*
- *g is positive definite*
- *g is continuous in Frechet topology.*

Note that if we take $g(\phi) = e^{-\frac{1}{2}\|\phi\|^2_{L^2(\mathbb{R})}}$ which is positive definite function then by Bochner-Minlos theorem there exists probability measure P on $\Omega = S'(\mathbb{R})$ such as the following equality holds:

$$\int_{S'(\mathbb{R})} e^{i<\omega,\phi>} dP(\omega) = e^{-\frac{1}{2}\|\phi\|^2_{L^2(\mathbb{R})}}. \tag{2.2}$$

Moreover, this probability measure is called *white noise probability measure* and it is defined on the set $\mathcal{B}$ of Borel sets of Ω. The triple $(\Omega = S'(\mathbb{R}), \mathcal{B}, P)$ is called the white noise probability space. From the settlement (2.2) and by using the Taylor expansion we can easily prove that

$$E[< \omega, \phi >] := \int_{S'(\mathbb{R})} < \omega, \phi > dP(\omega) = 0 \tag{2.3}$$

$$E[< \omega, \phi >^2] := \int_{S'(\mathbb{R})} < \omega, \phi >^2 dP(\omega) = \| \phi \|^2_{L^2(\mathbb{R})} . \tag{2.4}$$

Hence, we can extend the definition of $< \omega, \phi >$ from $\phi \in S(\mathbb{R})$ to $\phi \in L^2(\mathbb{R})$ by using the properties above and fact that $S(\mathbb{R})$ is dense in $L^2(\mathbb{R})$ and $L^2(P)$ is complete. Therefore, it is natural to define

$$\tilde{W}(t, \omega) = \tilde{W}(t) =< \omega, \chi_{[0,t]}(.) >$$

where $\chi_{[0,t]}$ belongs to $L^2(\mathbb{R})$ for all t and defined as follows:

$$\chi_{[0,t]}(s) = \begin{cases} 1 & \text{if} \quad s \in [0,t] \ \text{ or } s \in [t,0] (t < 0) \\ 0 & \text{otherwise} \end{cases}$$

Then, $\tilde{W}(t)$ is a Gaussian process with mean 0 and variance t. By the Kolmogorov continuity theorem [11] it has continuous version, denoted by $W(t) = W(t, \omega)$ which is a Brownian motion that we will work on this special construction. Note that since $W(t, \omega)$, $t \in \mathbb{R}$, $\omega \in \Omega = S'(\mathbb{R})$ is constructed with this special way, ω can be regarded as a tempered distribution.

From now we assume that Brownian motion $W(t)$, $t \in \mathbb{R}$ is constructed on the white noise probability space.

3. Review on special functions

The scope of this section is to recall some facts about special functions especially about Hermite polynomials and functions. By using them in the next section we will define an orthogonal basis for $L^2(P)$. Hermite polynomials plays a crucial role in probability theory. The family of these polynomials constitute an orthogonal basis for $L^2(\mathbb{R}, \mu(dx))$ if the measure $\mu = \frac{1}{\sqrt{2\pi}} e^{\frac{x^2}{2}}$. Hermite polynomials $h_n(x)$ are defined by as follows:

$$h_n(x) = (-1)^n e^{\frac{1}{2}x^2} \frac{d^n}{dx^n}(e^{-\frac{1}{2}x^2}),\ x \in \mathbb{R}, n = 0, 1, 2, \ldots \tag{3.1}$$

Thus the first Hermite polynomials are

$$h_0(x) = 1, h_1(x) = x, h_2(x) = x^2 - 1, h_3(x) = x^3 - 3x, \ldots$$

Moreover,

$$h_n^{'}(x) = nh_{n-1}(x),\ n \geq 1 \tag{3.2}$$

$$h_{n+1}(x) = xh_n(x) - nh_{n-1}(x),\ n \geq 1 \tag{3.3}$$

are commonly used important relations between Hermite polynomials. Moreover, there is a practical formula proved by Itô [8] for the computation of iterated Itô integrals when the integrand is the tensor power of a given function. Then, for the tensor power of $g \in L^2([0, T])$ we have

$$n! \int_0^T \int_0^{t_n} \cdots \int_0^{t_2} g(t_1)g(t_2)\cdots g(t_n)\, dW(t_1)\cdots dW(t_n) = \| g \|^n\ h_n(\frac{\int_0^T g(t)dW(t)}{\| g \|_{L^2([0,T])}}). \tag{3.4}$$

Note that we represent iterated Itô integrals on the interval $[0, T]$ satisfying $0 \leq t_1 \leq t_2 \leq \cdots \leq t_n \leq T$ by J_n,

$$J_n(g) := \int_0^T \int_0^{t_n} \cdots \int_0^{t_2} g(t_1, t_2, \cdots, t_n)\, dW(t_1)\cdots dW(t_n)$$

and if g is symmetric function (i.e. $g \in \hat{L}^2([0, T]^n)$) then

$$n!J_n(g) = I_n(g)$$

where indeed I_n can be represented as follows:

$$I_n(g) = \int_0^T \int_0^T \cdots \int_0^T g(t_1, t_2, t_3, \cdots, t_n)dW(t_1)\cdots dW(t_n). \tag{3.5}$$

Hermite functions, $\{e_k\}_{k\geq 1}$, can be described in terms of Hermite polynomials as follows:

$$e_k(x) = \pi^{-1/4}((k-1)!)^{-\frac{1}{2}} e^{-\frac{1}{2}x^2} h_{k-1}(\sqrt{2}x),\ k = 1, 2, 3, \ldots \tag{3.6}$$

where e_k is the k'th Hermite function. Moreover, note that $\{e_k\}_{k\geq 1}$ constitutes an orthonormal basis for $L^2(\mathbb{R})$ and $e_k \in S(\mathbb{R})$ for all k. They will be used in the following section for constructing the orthogonal basis for $L^2(P)$.

4. Chaos expansions

Chaos expansion aims to represent a random variable in $L^2(P)$ in terms of fundamental unique functions. In this section we will present two different kinds of Wiener-Itô chaos expansion and we will state the relation between them.

Theorem 4.1. ***(The Wiener-Itô chaos expansion I)***
Let $F(\omega)$ be an F_T-measurable random variable in $L^2(P)$, where

$$L^2(P) = \{F : \Omega \to \mathbb{R}\, s.t. \parallel X \parallel^2_{L^2(P)} := \int_\Omega F^2(\omega)dP(\omega) < \infty\}.$$

Then there exists unique sequence $\{f_n\}_{n=0}^{\infty}$ of functions $f_n \in \hat{L}^2([0,T]^n)$ such that

$$F = \sum_{n=0}^{\infty} I_n(f_n), \tag{4.1}$$

where the convergence is in $L^2(P)$. Moreover, there exist the following isometry;

$$\parallel F \parallel^2_{L^2(P)} = E[F^2] = \sum_{n=0}^{\infty} n! \parallel f_n \parallel^2_{L^2([0,T]^n)}$$

Proof. The proof of this representation can be found in Di Nunno et al [12]. □

The above representation theorem is in terms of iterated Itô integrals. Let us now construct a new representation for the random variables in $L^2(P)$ and deal with the relations between these two representations. Define

$$\theta_k(\omega) := < \omega, e_k > = \int_{\mathbb{R}} e_k(x)dW(x). \quad \omega \in \Omega \tag{4.2}$$

Let $\mathcal{I}$ be the set of all finite multi-indices $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)$, where $\alpha_m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ m=1,2,... and $\mid \alpha \mid = \alpha_1 + \alpha_2 + \ldots + \alpha_m$. Define

$$H_\alpha := \prod_{j=1}^{m} h_{\alpha_j}(\theta_j(\omega)), \quad \omega \in \Omega. \tag{4.3}$$

The family $\{H_\alpha\}_{\alpha \in \mathcal{I}}$ is an orthogonal sequence that constitutes basis for the Hilbert space $L^2(P)$.

Theorem 4.2. ***(The Wiener-Itô chaos expansion II)***
For all $F \in L^2(P)$ there exists unique constants $c_\alpha \in \mathbb{R}$ such that

$$F = \sum_{\alpha \in \mathcal{I}} c_\alpha H_\alpha. \tag{4.4}$$

Moreover, there exists the following equality

$$\| F \|_{L^2(P)}^2 = \sum_{\alpha \in \mathcal{I}} \alpha!\, c_\alpha^2, \tag{4.5}$$

where $\alpha! = \alpha_1!\alpha_2! \dots \alpha_m!$ *for* $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_m)$.

Note that by the result of Itô [8] we have

$$I_n(e^{\hat{\otimes}\alpha}) = \prod_{i=1}^{n} h_{\alpha_i}(\theta_i) = H_\alpha \tag{4.6}$$

where I is iterated integral defined in (3.5), θ is defined in the equation (4.2) and H_α is defined in the equation (4.3).
Therefore, the connection between these two expansions is if

$$f_n = \sum_{\alpha \in \mathcal{I}:|\alpha|=n} c_\alpha e^{\hat{\otimes}\alpha}, \quad n = 0, 1, 2, \dots, \tag{4.7}$$

where

$$e^{\hat{\otimes}\alpha} = e_1^{\otimes\alpha_1} \hat{\otimes} e_2^{\otimes\alpha_2} \hat{\otimes} \dots \hat{\otimes} e_m^{\otimes\alpha_m}$$

and $\mid \alpha \mid = \sum_{i=1}^m \alpha_i$ then

$$F = \sum_{n=0}^{\infty} I_n(f_n) = \sum_{\alpha \in \mathcal{I}, |\alpha|=n} c_\alpha H_\alpha$$

for some suitable constants c_α. In the above equation $\otimes$ and $\hat{\otimes}$ stands for the tensor product and symmetrized tensor product, respectively. For example, if f , g and h are real functions on $\mathbb{R}$ then

$$(f \otimes g \otimes h)(x_1, x_2, x_3) = f(x_1)g(x_2)h(x_3)$$

and

$$(f \hat{\otimes} g \hat{\otimes} h)(x_1, x_2, x_3) = \frac{1}{6} \sum_{\sigma} f(x_{\sigma_1})g(x_{\sigma_2})h(x_{\sigma_3}),$$

where the sum is taken over all permutations of σ of $1, 2, 3$.

5. Hida Stochastic test function and distribution space

This section contains a brief summary of (Hida)Malliavin derivative based on the representations and mention about some useful properties of it. There exists a useful analogy between Schwartz space, $S(\mathbb{R})$, and tempered distribution space, $S'(\mathbb{R})$, between Hida stochastic test function space and distribution space on the white noise probability space. We will denote Hida test functions (or functionals) space and Hida distributions space with (S) and $(S)^*$ respectively. For more information about these spaces one can see [6] and [7].
Definition 5.1.

(i) $f = \sum_{\alpha\in\mathcal{I}} a_\alpha H_\alpha \in L^2(P)$ belongs to the Hida test function Hilbert space $(S)_k$ for k $\in \{1, 2, \ldots\}$ if

$$\| f \|_k^2 := \sum_{\alpha\in\mathcal{I}} \alpha!\, a_\alpha^2\, (2\mathbb{N})^{\alpha k} < \infty$$

where

$$(2\mathbb{N})^\alpha = \prod_{i=1}^{m} (2i)^{\alpha_i}, \text{ for } \alpha = (\alpha_1, \ldots, \alpha_m) \in \mathcal{I}. \tag{5.1}$$

Then define *Hida space of stochastic test functions* (S) as

$$(S) = \bigcap_{k=1}^{\infty} (S)_k, \tag{5.2}$$

with the projective topology (i.e. as n goes to infinity, $f_n \to f$ in (S) iff $\| f_n - f \|_k \to 0$, for all $k \in \mathbb{N}$).

(ii) Similarly, let $(S)_{-q}$, $q = 1, 2, \ldots$, be the set of all expansions $F = \sum_{\alpha\in\mathcal{I}} b_\alpha H_\alpha$ such that

$$\| F \|_{-q}^2 := \sum_{\alpha\in\mathcal{I}} b_\alpha^2\, \alpha!\, (2\mathbb{N})^{-\alpha q} < \infty,$$

where $(2\mathbb{N})^\alpha$ is defined in the equation (5.1). Then define *Hida space of stochastic distributions* (S) as

$$(S)^* = \bigcup_{q=1}^{\infty} (S)_{-q} \tag{5.3}$$

with the inductive topology (i.e. as n goes to infinity, $F_n \to F$ in $(S)^*$ iff there exists q such that $\| F_n - F \|_{-q} \to 0$).

Note that $(S)^*$ is the dual of (S) and one can define the action of $F = \sum_{\alpha\in\mathcal{I}} b_\alpha H_\alpha \in (S)^*$ on $f = \sum_{\alpha\in\mathcal{I}} a_\alpha H_\alpha \in L^2(P) \in (S)$ as follows:

$$< F, f >= F(f) = \sum_{\alpha\in\mathcal{I}} a_\alpha\, b_\alpha \alpha!$$

From the definition of these spaces we can easily extract the following inclusions.

$$(S) \subset (S)_k \subset L^2(P) \subset (S)_{-q} \subset (S)^*, \quad \text{for all} \quad k, q. \tag{5.4}$$

It is convenient and natural way to define Wick product on the space $(S)^*$.

Definition 5.2. If $X = \sum_\alpha a_\alpha H_\alpha \in (S)^*$ and $Y = \sum_\beta b_\beta H_\beta \in (S)^*$ then the Wick product $X \diamond Y$ of X and Y is defined by

$$X \diamond Y := \sum_{\alpha, \beta} a_\alpha\, b_\beta\, H_{\alpha+\beta}$$

5.1. **The (Hida) Malliavin Derivative.**
Although the Wiener space is natural space to work on when dealing with Brownian motion, this approach has some disadvantages. However, if we work on $\Omega = S'(\mathbb{R})$ instead and follow the theory of Hida [4] then we obtain a complete agreement between the directional derivative and the corresponding stochastic gradient(or the Hida-Malliavin derivative). The scope is this part is to define the (Hida) Malliavin derivative and some auxiliary theorems on the white noise probability space $(\Omega = S'(\mathbb{R}), \mathcal{B}, P)$.

Definition 5.3.

(i) Let $F \in L^2(P)$ be random variable and let $\gamma \in L^2(\mathbb{R}) \subset \Omega$ be a deterministic function. Then the directional (or Gateaux) derivative of F in the direction of γ is defined by

$$D_\gamma F(\omega) = \lim_{\varepsilon \to 0} \frac{F(\omega + \varepsilon\gamma) - F(\omega)}{\varepsilon} \tag{5.5}$$

if the limit exists in $(S)^*$.

(ii) Suppose there exists a function $\psi : \mathbb{R} \to (S)^*$ such that

$$D_\gamma F(\omega) = \int_{\mathbb{R}} \psi(t)\gamma(t)dt \quad forall \gamma \in L^2(\mathbb{R}), \tag{5.6}$$

then we say that F is (Hida-)Malliavin differentiable and we put

$$D_t F = \psi(t) \quad t \in \mathbb{R}.$$

$D_t F$ is called the Hida-Malliavin derivative or the stochastic gradient of F at t [12].

Theorem 5.1. ***Chain Rule***
Suppose $F \in L^2(P)$ is Malliavin differentiable. Let $\Phi \in \mathcal{C}^1(\mathbb{R})$ and assume that $D_t F \in L^2(P)$ and $\Phi'(F) D_t F \in L^2(\lambda \times P)$ where λ is the Lebesgue measure on $\mathbb{R}$. Then $\Phi(F)$ is Malliavin differentiable and

$$D_t(\Phi(F)) = \Phi'(F) D_t F \tag{5.7}$$

Proof. In this proof we will use the definition of directional derivative of F in the direction $\gamma \in L^2(\mathbb{R})$.

$$\begin{aligned} D_\gamma(\Phi(F)) &= \lim_{\varepsilon \to 0} \frac{\Phi(F(\omega + \varepsilon\gamma) - \Phi(F(\omega)))}{\varepsilon} \\ &= \lim_{\varepsilon \to 0} \frac{\Phi(F(\omega) + \varepsilon D_\gamma F) - \Phi(F(\omega))}{\varepsilon} \\ &= \Phi'(F) \int_{\mathbb{R}} D_t F \, \gamma(t) \, dt. \end{aligned}$$

Writing explicitly the left hand side we have the following result;

$$\int_{\mathbb{R}} D_t(\Phi(F)) \, \gamma(t) \, dt = \int_{\mathbb{R}} \Phi'(F) \, D_t F \, \gamma(t) \, dt,$$

which proves that $\Phi(F)$ is also (Hida)-Malliavin differentiable and chain rule holds. □

More generally, if $F_1, F_2, \ldots, F_m$ are Malliavin differentiable, $\{D_t F_i\}_{i\in\{1,2,\ldots,m\}} \in L^2(P)$, $\{\Phi^i(F) D_t F_i\}_{i\in\{1,2,\ldots,m\}} \in L^2(\lambda \times P)$ and $\Phi : \mathbb{R}^m \to \mathbb{R} \in \mathcal{C}^1$, then

$$D_t(\Phi(F_1, F_2, \ldots, F_m)) = \sum_{i=1}^{m} \frac{\partial \Phi}{\partial F_i}(F_1, F_2, \ldots, F_m) \cdot D_t F_i$$

Example 5.1. *Let* $F(\omega) =< \omega, f >= \int_{\mathbb{R}} f(t) dW(t)$*, for some* $f \in L^2(\mathbb{R})$*. Then*

$$\begin{aligned} D_\gamma F(\omega) &= \lim_{\varepsilon \to 0} \frac{< \omega + \varepsilon\gamma, f > - < \omega, f >}{\varepsilon} \\ &= < \gamma, f > \end{aligned} \tag{5.8}$$

Hence, F *is Malliavin differentiable and* $D_t F(\omega) = f(t)$*, i.e.,*

$$D_t\left(\int_{\mathbb{R}} f(s) dW(s)\right) = f(t). \tag{5.9}$$

Remark.

(i) Note that as a consequence of general chain rule we can easily write product rule, i.e.,

$$D_t(F_1 F_2) = F_1 D_t F_2 + F_2 D_t F_1 \tag{5.10}$$

(ii) Another result of chain rule is

$$\begin{aligned} D_t(h_n(< \omega, f >)) &= h_n^{'}(< \omega, f >) f(t) \\ &= n h_{n-1}(< \omega, f >) f(t), \end{aligned}$$

where $\{h_k(x)\}$ are the Hermite polynomials.

Lemma 5.2. *Let* $F = I_n(f_n) \in L^2(P)$ *for some* $f_n \in \hat{L}^2([0,T]^n)$*. Then*

$$D_t F = n I_{n-1}(f_n(., t)), \tag{5.11}$$

where $I_{n-1}(f_n(., t))$ *stands for the (n-1)-iterated Itô integral which is defined with respect to the* $n-1$ *first variables* $t_1, t_2, \ldots, t_{n-1}$ *of* $f_n(t_1, t_2, \ldots, t_{n-1}, t)$*.*

Proof. Here we will give the sketch of the proof. Let us firstly assume $f_n = f^{\otimes n}$ for some $f \in L^2([0,T])$. Then by Itô [8],

$$I_n(f_n) = \| f \|^n h_n\left(\frac{< \omega, f >}{\| f \|}\right).$$

Hence by Remark (ii)

$$\begin{aligned} D_t I_n(f_n) &= D_t\left(\| f \|^n h_n\left(\frac{< \omega, f >}{\| f \|}\right)\right) \\ &= n \| f \|^{n-1} h_{n-1}\left(\frac{< \omega, f >}{\| f \|}\right) \frac{f(t)}{\| f \|} \end{aligned}$$

$$= n \parallel f \parallel^{n-1} h_{n-1}(\frac{<\omega, f>}{\parallel f \parallel}) f(t)$$
$$= nI_{n-1}(f_n(.,t)) \tag{5.12}$$

Secondly, suppose f_n has the form

$$f_n = e_1^{\hat{\otimes}\alpha_1} \hat{\otimes} e_2^{\hat{\otimes}\alpha_2} \hat{\otimes} \ldots \hat{\otimes} e_k^{\hat{\otimes}\alpha_k}, \tag{5.13}$$

where $\mid \alpha \mid = \alpha_1 + \alpha_2 + \ldots + \alpha_k = n$ for $\alpha = (\alpha_1, \ldots, \alpha_k)$, $\hat{\otimes}$ denotes the symmetrized tensor product and $\{e_k\}_{k\geq 1}$ is the family of hermite functions which are orthonormal basis of $L^2(\mathbb{R})$. By Itô formula [8],

$$I_n(f_n) = h_{\alpha_1}(<\omega, e_1>) \cdots h_{\alpha_k}(<\omega, e_k>)$$

then the equality (5.11) holds again by using chain rule. Since any function $f_n \in \hat{L}^2([0,T]^n)$ can be approximated in $L^2([0,T]^n)$ by linear combinations of basis of the form given by (5.13) the proof is completed. □

Hence, Hida-Malliavin derivative defined above coincides with the Malliavin derivative on the stochastic Sobolev space, $\mathbb{D}_{1,2}$ with Wiener space settlement for this case. In the next section we will prove the Hida-Malliavin derivative for $F = \sum_{n=0}^{\infty} I_n(f_n) \in L^2(P)$ by constructing the spaces $\mathcal{G}$ and $\mathcal{G}^*$.
Moreover, if we represent F in terms of Wiener-Itô chaos expansion II, i.e.
$F(\omega) = \sum_{\alpha \in \mathcal{I}} c_\alpha H_\alpha \in (S)^*$ then

$$D_t H_\alpha = \mid \alpha \mid I_{|\alpha|-1}(e^{\hat{\otimes}\alpha}(.,t)) \tag{5.14}$$

or

$$\begin{aligned} D_t H_\alpha &= D_t(\prod_{i=1}^{m} h_{\alpha_i}(<\omega, e_i>)) \\ &= D_t(h_{\alpha_1}(<\omega, e_1>) h_{\alpha_2}(<\omega, e_2>) \ldots h_{\alpha_m}(<\omega, e_m>)) \\ &= \sum_{k=1}^{m} \prod_{i=1, j\neq k} \alpha_k h_{\alpha_k - 1}(<\omega, e_k>) e_k(t) h_{\alpha_i}(<\omega, e_i>) \\ &= \sum_{k=1}^{m} \alpha_k e_k(t) H_{\alpha - \varepsilon^k}, \end{aligned} \tag{5.15}$$

where $\alpha - \varepsilon^{(k)} = (\alpha_1, \alpha_2, \ldots, \alpha_k - 1, \ldots, \alpha_m)$ and e_k is the kth Hermite function.

Lemma 5.3.

(i) *Let $G \in (S)^*$. Then $D_t G \in (S)^*$ for a.a. $t \in \mathbb{R}$.*

(ii) *Suppose $G, G_n \in (S)^*$ for all $n \in \mathbb{N}$ and*

$$G_n \to G \ \ in \ \ (S)^*.$$

Then there exist a subsequence $\{G_{n_k}\}_{k\geq 1}$ such that

$$D_t G_{n_k} \to D_t G \ \ in \ \ (S)^*,$$

for almost all $t > 0$.

Proof. **(i)** Let $G = \sum_{\alpha\in\mathcal{I}} c_\alpha H_\alpha \in (S)^*$. Then there exists a $q < \infty$ such that

$$\begin{aligned} \| G \|^2_{(S)_{-q}} &:= \sum_{\alpha\in\mathcal{I}} c_\alpha^2 \alpha!(2\mathbb{N})^{-\alpha q} \\ &= \sum_{n=0}^{\infty} \sum_{|\alpha|=n} c_\alpha^2 \alpha!(2\mathbb{N})^{-\alpha q} < \infty. \end{aligned}$$

By using equation (5.14), the Hida-Malliavin derivative of G is as follows:

$$\begin{aligned} D_t G(\omega) &= \sum_{\alpha\in\mathcal{I}} c_\alpha D_t(H_\alpha(\omega)) \\ &= \sum_{\alpha\in\mathcal{I}} c_\alpha \sum_i \alpha_i H_{\alpha-\varepsilon^{(i)}}(\omega) e_i(t) \\ &= \sum_\beta (\sum_i c_{\beta+\varepsilon^{(i)}}(\beta_i+1)e_i(t))H_\beta(\omega) \\ &:= \sum_\beta g_\beta(t) H_\beta(\omega), \end{aligned}$$

where $g_\beta(\omega)(t) = \sum_i c_{\beta+\varepsilon^{(i)}}(\beta_i+1)e_i(t)$.
We want to prove that

$$\| D_t G \|^2_{(S)_{-q-1}} := \sum_{n=0}^{\infty} (\sum_{|\beta|=n} g_\beta^2 \beta!)(2\mathbb{N})^{-\beta(q+1)} < \infty \text{ for a.a. t.}$$

Note that,

$$\int_{\mathbb{R}} g_\beta^2(t)dt = \sum_\beta c^2_{\beta+\varepsilon^{(i)}}(\beta_i+1)^2.$$

and

$$\begin{aligned} (2\mathbb{N})^{-\beta} &= \prod_i (2\cdot i)^{-\beta_i} \\ &\leq \prod_i e^{-\beta_i(\log 2)} = e^{-|\tilde{\beta}|} \end{aligned}$$

where $\tilde{\beta}_i = (\log 2)\, \beta_i$ for all $i \in \mathcal{I}$. Hence,

$$\begin{aligned} \int_{\mathbb{R}} \| D_t G \|^2_{(S)_{-q-1}} dt &= \sum_\beta (\sum_i c^2_{\beta+\varepsilon^{(i)}}(\beta_i+1)^2)\beta!(2\mathbb{N})^{-\beta(q+1)} \\ &= \sum_\beta (\beta_i+1)(2\mathbb{N})^{-\beta(q+1)} \sum_{\alpha,\, |\alpha|=|\beta|+1} c_\alpha^2\, \alpha! \\ &< \sum_n \sum_{|\tilde{\beta}|=n} (n+1)e^{-n} \sum_{|\alpha|=(\log 2)^{-1}n+1} c_\alpha^2\, \alpha!\, (2\mathbb{N})^{-\alpha q} \end{aligned}$$

Using the fact that $(n+1)e^{-n} \leq 1$ for all n, we get

$$\int_{\mathbb{R}} \| D_t G \|^2_{(S)_{-q-1}} dt \;<\; \sum_n (\sum_{|\alpha|=(\log 2)^{-1}n+1} c_\alpha^2 \, \alpha!) \, (2\mathbb{N})^{-\alpha q}$$
$$\leq \;\| G \|_{(S)_{-q}} < \infty \tag{5.16}$$

Therefore, $D_t G \in (S)_{-q-1} \subset (S)^*$ for a.a. t.

(ii) To prove this part, it sufficies to prove that if $G_n \to 0$ in $(S)_{-q}$, then there exist a subsequence $\{G_{n_k}\}_{k\geq 1}$ such that $D_t G_{n_k} \to 0$ in $(S)^*$ as k goes to infty, for a.a. t. We have proved that

$$\int_{\mathbb{R}} \| D_t G_n \|^2_{(S)_{-q-1}} dt \leq \| G_n \|^2_{(S)_{-q}} \to 0. \tag{5.17}$$

Therefore,

$$\| D_t G_n \|_{(S)_{-q-1}} \to 0 \quad \text{in} \quad L^2(\mathbb{R}).$$

So, there exists a subsequence $\{\| D_t G_{n_k} \|_{(S)_{-q-1}}\}_{k\geq 1}$ such that $\| D_t G_{n_k} \|_{(S)_{-q-1}} \to 0$ for a.a. t as $k \to \infty$. □

6. The Clark-Ocone formula under change of measure

In this section, we will prove the extended Clark-Ocone formula under change of measure by using the white noise theory. We will work on the space $\mathcal{G}^* \supset L^2(P)$, the space of stochastic distributions. The advantage of using $\mathcal{G}^*$ lies in the fact that it is convenient space to work with the conditional expectation. Moreover, $D_t F$ is not well defined in the classical sense, because we have not assumed that $F \in \mathbb{D}_{1,2}$. Therefore, $D_t F$ is assumed to be an element of $(S)^*$, where $E[D_t F \mid \mathcal{F}_t] \in L^2([0,T] \times P)$.

6.1. Skorohod Integral.

Definition 6.1. Let $u(t)$ be an $\mathcal{F}_T-$measurable stochastic process such that $||u(t)||_{L^2(P)} < \infty$ for all t, i.e. $u(t) \in L^2(P)$ and have the following chaos expansion,

$$u(t) = \sum_{n=0}^{\infty} I_n(f_n(.,t))$$

where $\sum_{n=0}^{\infty}(n+1)! \, \| f_n(.,t) \|_{L^2([0,T]^{n+1})} < \infty$ and $f_n(.,t)$ is a symmetric function with respect to first n variables. Then the Skorohod integral of $u(t)$ with respect to $W(t)$, is defined by

$$\delta(u) := \int_0^T u(t) \delta W(t) = \sum_{n=0}^{\infty} I_{n+1}(\tilde{f}_n), \tag{6.1}$$

where $\tilde{f}_n$ are the symmetric functions derived from $f_n(.,t)$ for $n = 1, 2, \ldots$.
Note that

$$E[(\int_0^T u(t)\delta W(t))^2] \;=\; E[(\sum_{n=0} I_{n+1}(\tilde{f}_n))^2]$$

$$= \sum_{n=0}^{\infty} E[I_{n+1}^2(\tilde{f}_n)]$$

$$= \sum_{n=0}^{\infty} (n+1)! \parallel \tilde{f}_n \parallel^2_{L^2([0,T]^{n+1})} < \infty. \tag{6.2}$$

Hence $\int_0^T u(t)\delta W(t) \in L^2(P)$ when $u \in L^2(P)$.

6.2. **The construction of the spaces $\mathcal{G}$ and $\mathcal{G}^*$.**

Definition 6.2. (i) Let $\lambda \in \mathbb{R}$. The space $\mathcal{G}_\lambda$ consists of all formal expansions

$$F = \sum_{n=0}^{\infty} I_n(f_n)$$

such that

$$\parallel F \parallel^2_{\mathcal{G}_\lambda} := \sum_{n=0}^{\infty} n! \parallel f_n \parallel^2_{L^2(\mathbb{R}^n)} e^{2\lambda n} < \infty. \tag{6.3}$$

(ii) Define

$$\mathcal{G} = \bigcap_{\lambda \in \mathbb{R}} \mathcal{G}_\lambda$$

equip with the projective topology and

$$\mathcal{G}^* = \bigcup_{\lambda \in \mathbb{R}} \mathcal{G}_\lambda$$

equip with inductive topology. Then $\mathcal{G}^*$ is the $\mathcal{G}$, and the action of $Y = \sum_{n \geq \infty} I_n(f_n) \in \mathcal{G}^*$ on $X = \sum_{m \geq \infty} I_m(g_m) \in \mathcal{G}$ is

$$< Y, X >_{\mathcal{G},\mathcal{G}^*} = \sum_{n=0}^{\infty} n!(f_n, g_n)_{L^2(\mathbb{R}^n)}. \tag{6.4}$$

Lemma 6.1.

(i) *Suppose $F \in \mathcal{G}^*$. Then $D_t F \in \mathcal{G}^*$ for a.a. $t \in \mathbb{R}$.*
(ii) *Suppose F, $F_n \in \mathcal{G}^*$ for all $n \in \mathbb{N}$ and*

$$F_n \to F \text{ in } \mathcal{G}^*.$$

Then there exists a subsequence $\{F_{n_k}\}_{k \geq 1}$ such that

$$D_t F_{n_k} \to D_t F \text{ in } \mathcal{G}^*, \tag{6.5}$$

for almost all $t > 0$.

Proof. The proof and further details can be found in [1]. □

It is also possible to express $\mathcal{G}_\lambda$ in terms of Hermite expansions. For more information see [1]. Also, note that

$$(S) \subset \mathcal{G} \subset L^2(P) \subset \mathcal{G}^* \subset (S)^*$$

Theorem 6.2. *Let* $F = \sum_{n=0}^{\infty} I_n(f_n) \in L^2(P)$. *Then*

$$D_t F = \sum_{n=1}^{\infty} n I_{n-1}(f_n(.,t)). \tag{6.6}$$

Proof. Let $F \in L^2(P)$. Define $F_m = \sum_{n=0}^{m} I_n(f_n)$. Then $F_m \to F$ in $L^2(P)$ which implies $F_m \to F$ in $\mathcal{G}^*$. By Lemma 6.1., there exists a subsequence F_{m_k} such that $D_t F_{m_k} \to D_t F$ in $\mathcal{G}^*$, i.e.,

$$\sum_{n=0}^{m_k} n I_{n-1}(f_n(.,t)) \to D_t F$$

in $\mathcal{G}^*$. We want to prove that

$$\sum_{n=0}^{m_k} n I_{n-1}(f_n(.,t)) \to \sum_{n=0}^{\infty} n I_{n-1}(f_n(.,t))$$

in $\mathcal{G}^*$ as m_k goes to infinity. Consider

$$\| \sum_{m_{k+1}}^{\infty} n I_{n-1}(f_n(.,t)) \|_{\mathcal{G}_{-q-1}}^2 = \sum_{m_{k+1}}^{\infty} (n-1)! n^2 \| f_n(.,t) \|_{L^2(\mathbb{R}^{n-1})}^2 e^{-2q(n-1)}. \tag{6.7}$$

Then taking the integral of both sides,

$$\begin{aligned} \int_{\mathbb{R}} \| \sum_{m_{k+1}}^{\infty} n I_{n-1}(f_n(.,t)) \|_{\mathcal{G}_{-q}}^2 \, dt &= \sum_{m_{k+1}}^{\infty} n n! \| f_n(.,t) \|_{L^2(\mathbb{R}^n)}^2 e^{-2q(n-1)} \\ &\leq K \sum_{m_{k+1}}^{\infty} n! \| f_n \|_{L^2(\mathbb{R}^n)}^2 \\ &= K \| F \|_{L^2(P)} < \infty \end{aligned} \tag{6.8}$$

where K is a constant. □

6.3. **Conditinal Expectation in $\mathcal{G}^*$.**

Definition 6.3. Let $F = \sum_{n=0}^{\infty} I_n(f_n) \in \mathcal{G}^*$. Then the conditional expectation of F with respect to filtration $\mathcal{F}_t$ is defined by

$$E[F \mid F_t] = \sum_{n=0}^{\infty} I_n(f_n \chi_{[0,t]^{\otimes n}}). \tag{6.9}$$

Note that this coincides with the usual expectation if $F \in L^2(P)$. Moreover, since

$$\| E[F \mid \mathcal{F}_t] \|_{\mathcal{G}_\lambda} \leq \| G \|_{\mathcal{G}_\lambda}, \text{ for all } \lambda \in \mathbb{R}$$

then

$$E[F \mid \mathcal{F}_t] \in \mathcal{G}^* \tag{6.10}$$

Proposition 6.3. *If* $F \in \mathcal{G}^*$ *then* $E[F \mid \mathcal{F}_t] \in \mathcal{G}^*$ *and*

$$D_s E[F \mid \mathcal{F}_t] = E[D_s F \mid \mathcal{F}_t] \chi_{[0,t]}(s). \tag{6.11}$$

Proof. We begin by proving this equality for $F = I_n(f_n) \in \mathcal{G}^*$. By using the preceding definition and Lemma 5.2, we have

$$\begin{aligned} D_s E[F \mid \mathcal{F}_t] &= D_s E[I_n(f_n) \mid \mathcal{F}_t] \\ &= D_s(I_n(f_n \chi_{[0,t]}^{\otimes n})) \\ &= n I_{n-1}(f_n(.,s)\chi_{[0,t]}^{\otimes(n-1)}(.))\chi_{[0,t]}(s) \\ &= E[D_s F \mid \mathcal{F}_t]\chi_{[0,t]}(s) \qquad (6.12) \end{aligned}$$

Next, assume $F = \sum_{n=0}^{\infty} I_n(f_n) \in \mathcal{G}^*$. Then by definition, $E[F \mid \mathcal{F}_t] = \sum_{n=0}^{\infty} I_n(f_n \chi_{[0,t]}^{\otimes n})$. Consider,

$$\begin{aligned} \| E[F \mid \mathcal{F}_t] \|_{\mathcal{G}_{-q}}^2 &= \sum_{n=0}^{\infty} n! \| f_n \chi_{[0,t]}^{\otimes n} \|_{L^2(\mathbb{R}^n)}^2 e^{-2qn} \\ &< \sum_{n=0}^{\infty} n! \| f_n \|_{L^2(\mathbb{R}^n)}^2 \\ &= \| F \|_{\mathcal{G}_{-q}}^2 < \infty \qquad (6.13) \end{aligned}$$

Therefore, if $E[F \mid \mathcal{F}_t] \in \mathcal{G}^*$ and

$$\begin{aligned} D_s(E[F \mid \mathcal{F}_t]) &= \sum_{n=1}^{\infty} n I_{n-1}(f_n(.,s)\chi_{[0,t]}^{\otimes(n-1)})\chi_{[0,t]}(s) \\ &= E[D_s F \mid \mathcal{F}_t]\chi_{[0,t]}(s). \qquad (6.14) \end{aligned}$$

□

Corollary 6.4. *Let $u = u(t) \in L^2(P)$, $t \in [0,T]$, be an $\mathbb{F}$-adapted stochastic process. Then*
(i) *$D_s u(t)$ is $\mathbb{F}$ adapted for all s,*
(ii) *$D_s u(t) = 0$ for $s > t$.*

Proof. The proof is easily derived by using the previous corollary. □

Theorem 6.5. ***Fundamental theorem of stochastic calculus:***
Let $u \in L^2(P)$ be a stochastic process satisfying the following conditions:
(i) $E[\int_0^T u(s)^2 ds] < \infty$
(ii) *$s \to D_t u(s)$ is Skorohod integrable in $(S)^*$, for all $t \in [0,T]$*
(iii) $E[\int_0^T (\int_0^T D_t(u(s))\delta W(s))^2 dt] < \infty$

then $\delta(u) = \int_0^T u(s)\delta W(s) \in L^2(P)$ and

$$D_t(\int_0^T u(s)\delta W(s)) = \int_0^T D_t u(s)\delta W(s) + u(t). \qquad (6.15)$$

Proof. Let us firstly assume that $u(s) = I_n(f_n(.,s))$, where $f_n(.,s) \in L^2([0,T]^{n+1})$ and $f_n(.,s)$ is symmetric with respect to the first n variables. Then by definition, the skorohod integral of $u(s)$ is

$$\delta(u) = I_{n+1}(\tilde{f}_n(t_1, t_2, \cdots, t_{n+1})),$$

where

$$\tilde{f}_n(t_1, t_2, \cdots, t_{n+1}) = \frac{1}{n+1}[f_n(\cdot, t_1) + \ldots + f_n(\cdot, t_{n+1})]. \tag{6.16}$$

Then,

$$\begin{aligned} D_t\delta(u) &= (n+1)I_n(\tilde{f}_n(\cdot, t)) \\ &= (n+1)I_n(\frac{1}{n+1}[f_n(t, \cdot, t_1) + \ldots + f_n(t, \cdot, t_n) + f_n(\cdot, \cdot, t)]) \\ &= I_n(f_n(t, \cdot, t_1) + \ldots + f_n(t, \cdot, t_n) + f_n(\cdot, \cdot, t)). \end{aligned}$$

On the other hand,

$$\begin{aligned} \delta(D_t u) &= \int_0^T D_t u(s)\delta W(s) \\ &= \int_0^T nI_{n-1}(f_n(\cdot, t, s))\delta W(s) \\ &= nI_n(\hat{f}_n(\cdot, t, \cdot)), \end{aligned}$$

where

$$\hat{f}_n(t_1, \cdots, t_{n-1}, t, t_n) = \frac{1}{n}[f_n(t, \cdot, t_1) + \cdots + f_n(t, \cdot, t_n)].$$

Therefore,

$$\delta(D_t u) = I_n(f_n(t, \cdot, t_1) + \cdots + f_n(t, \cdot, t_n)).$$

Hence, $D_t(\delta(u)) - \delta(D_t(u)) = u(t)$, which completes the first part of the proof. Next, assume $u(s)$ is the infinite summation of iterated integrals, i.e., $u(s) = \sum_{n=0}^{\infty} I_n(f)$. Define $u_m(s) = \sum_{n=0}^{m} I_n(f_n(\cdot, s))$. Then $u_m \to u$ in $L^2(P)$ which implies $u_m \to u$ in $\mathcal{G}^*$. By Lemma, there exists a subsequence $\{u_{m_k}\}_{k\geq 1}$ such that $D_t(u_{m_k}) \to D_t(u)$ in $\mathcal{G}^*$. Since Skorohod integral of random variable in $\mathcal{G}^*$ is in $(S)^*$,(i.e. $\delta(D_t(u_{m_k})) \in (S)^*$) by assumption (ii) and lemma for Skorohod integrals then

$$\int_0^T \| D_t(\delta(u)) - D_t(\delta(u_{m_k})) \|^2_{(S)_{-q}} dt \to 0$$

and

$$\int_0^T \| \delta(D_t u) - \delta(D_t u_{m_k}) \|^2_{(S)_{-q}} dt \to 0.$$

□

Corollary 6.6. *Let u be as in previous theorem and in addition to this assume $u(s)$ is $\mathbb{F}$-adapted then*

$$D_t(\int_0^T u(s)dW(s)) = \int_t^T D_t u(s)dW(s) + u(t) \tag{6.17}$$

Proof. It is an immediate consequence of Corollary 6.3 and Theorem 6.4. □

6.4. **The Clark-Ocone theorem for $L^2(P)$.**

Theorem 6.7. ***(The Clark-Ocone theorem for*** $L^2(P)$***)***
Let λ denote the Lebesgue measure on $\mathbb{R}$. Suppose $F(\omega) \in L^2(P)$ be $\mathcal{F}_T$-measurable. Then

$$(t, \omega) \to E[D_t F \mid \mathcal{F}_t](\omega) \in L^2(\lambda \times P)$$

and

$$F(\omega) = E[F] + \int_0^T E[D_t F \mid \mathcal{F}_t] dW(t) \tag{6.18}$$

Proof. The proof can be found in Aase et al [1]. □

6.5. **The Clark-Ocone theorem under change of measure for $L^2(P)$.**

Theorem 6.8. ***(Girsanov)*** *Let $W(t)$ be a Brownian motion on the white noise probability space $(\Omega, \mathcal{F}, P)$. Let $(u(t))_{0 \le t \le}$ be an adapted measurable process satisfying $\int_0^T u^2(s)ds < \infty$ a.s. and such that $Z(T)$ defined by*

$$Z(T) = \exp\{-\int_0^T u(s)dW(s) - \frac{1}{2}\int_0^T u^2(s)ds\} \tag{6.19}$$

is a martingale. Then under the white noise probability measure Q with density $Z(T)$ relative to P, the process $\hat{W}(t)$ defined by $\hat{W}(t) = W(t) + \int_0^t u(s)ds$ is a Brownian motion under measure Q.

Lemma 6.9. *Let $F \in L^2(P)$ be $\mathcal{F}_T$-measurable, Q and $Z(T)$ is defined as in Girsanov theorem. Then*

$$D_t(Z(T)F) = Z(T)\{D_t F - F[u(t) + \int_t^T D_t u(s) d\hat{W}(s)]\} \tag{6.20}$$

Proof. By the product rule,

$$D_t(Z(T)F) = D_t F Z(T) + D_t Z(t) F.$$

So it remains to find the (Hida)Malliavin derivative of $Z(T)$. By chain rule, Corollary 7.2 and Corollary 7.4,

$$\begin{aligned} D_t Z(T) &= Z(T)\{-D_t(\int_0^T u(s)dW(s)) - \frac{1}{2}D_t(\int_0^T u^2(s))\} \\ &= Z(T)\{-(\int_t^T D_t u(s)dW(s) + u(t)) - \int_0^T u(s)D_t u(s)ds\} \\ &= Z(T)\{-\int_t^T D_t u(s)d\hat{W}(s) - u(t)\} \end{aligned}$$

□

Theorem 6.10. ***(The Clark-Ocone formula under change of measure for*** $L^2(P)$***)*** *Suppose* $F \in L^2(P)$ *is* $\mathcal{F}_T$*-measurable and that*

$$E_Q[| F |] < \infty \tag{6.21}$$

$$E_Q[\int_0^T | D_tF |^2 \, dt] < \infty \tag{6.22}$$

$$E_Q[| F | \int_0^T (\int_0^T D_tu(s)dW(s) + \int_0^T u(s)D_tu(s)ds)^2 dt] < \infty \tag{6.23}$$

Then

$$F(\omega) = E_Q[F] + \int_0^T E_Q[(D_tF - F\int_t^T D_tu(s)d\hat{W}(s)) \mid \mathcal{F}_t]d\hat{W}(t), \tag{6.24}$$

where $\hat{W}(t)$ *is a Brownian motion under white noise probability measure* Q *and* $D_tF \in \mathcal{G}^*$ *is Hida-Malliavin derivative.*

Proof. Let $Y(t) = E_Q[F \mid \mathcal{F}_t]$. Note that

$$E_Q[F \mid \mathcal{F}_0] = E_Q[F]$$

and

$$E_Q[F \mid \mathcal{F}_T] = F.$$

Let us define

$$\lambda(t) = Z^{-1}(t) = \exp\{\int_0^t u(s)dW(s) + \frac{1}{2}\int_0^t u^2(s)ds\}. \tag{6.25}$$

By Itô Formula,

$$d\lambda(t) = \lambda(t)u(s)d\hat{W}(s).$$

By Bayes' formula (Karatzas and Shreve Lemma 3.5.3. [10]) and the settlement of $Y(t)$,

$$\begin{aligned} Y(t) &= E[Z(T)F \mid \mathcal{F}_t]Z^{-1} \\ &= \lambda(t)E[Z(T)F\mathcal{F}_t]. \end{aligned}$$

By Clark-Ocone formula for the random variables in $L^2(P)$ (Aase et al Theorem 3.11. [1]) and Proposition 7.1.,

$$\begin{aligned} E[Z(T)F \mid \mathcal{F}_t] &= E[Z(T)F] + \int_0^t E[D_s(Z(T)F) \mid \mathcal{F}_s]dW(s) \\ &:= U(t). \end{aligned}$$

Hence, $Y(t) = \lambda(t)U(t)$ and by using Bayes' formula we have the following:

$$\begin{aligned} dY(t) &= \lambda(t)dU(t) + U(t)d\lambda(t) + d < \lambda, U >_t \\ &= \{E_Q[D_tF \mid \mathcal{F}_t] - E_Q[Fu(t) \mid \mathcal{F}_t] \\ &\quad - E_Q[F\int_t^T D_tu(s)d\hat{W}(s) \mid \mathcal{F}_t] + u(t)E_Q[F \mid \mathcal{F}_t]\}d\hat{W}(t). \end{aligned}$$

If we integrate both sides on $[0, T]$, the proof will be completed. □

7. Application to finance

In this section we will demonstrate how extended Clark-Ocone theorem under change of measure can be applied in portfolio optimization. The main advantage of this setting is that F need not be in $\mathbb{D}_{1,2}$. Let us assume that we have two possible investments which are a risk free asset, bond, and a risky asset,stock. Moreover, suppose the prices of these two financial instrument follow the following stochastic differential equations under probability measure P;

(i) Risk free asset (Bond)

$$\begin{aligned} dS_0(t) &= \rho(t)S_0(t)dt, \\ S_0(0) &= 1. \end{aligned} \tag{7.1}$$

(ii) Risky asset (Stock)

$$\begin{aligned} dS_1(t) &= \mu(t)S_1(t)dt + \sigma(t)S_1(t)dW(t), \\ S_1(0) &> 0. \end{aligned} \tag{7.2}$$

Here $\rho(t) = \rho(t,\omega)$, $\mu(t) = \mu(t,\omega)$ and $\sigma(t) = \sigma(t,\omega)$, $\omega \in \Omega$ are $\mathcal{F}_t$-measurable processes for all $t \geq 0$ satisfying the following condition,

$$E[\int_0^T \{\mid \rho(t) \mid + \mid \mu(t) \mid + \sigma^2(t)\}dt] < \infty.$$

Moreover, suppose in the economy there exists a contingent-claim. In this paper, we will deal with Digital option which has a payoff function at the maturity

$$F = \chi_{[K,\infty)}(W(T)), \tag{7.3}$$

where K is the exercise price of this contingent-claim. Our aim is to find the replicating portfolio for this option where at the maturity the value of the portfolio is equal to the payoff function. If $\theta(t) = \theta(t,\omega) = (\theta_0(t), \theta_1(t))$, $\omega \in \Omega$, denotes the number of the unites invested at time t in risk free and risky assets respectively, then the value of the portfolio will be

$$dV^\theta(t) = \theta_0(t)S_0(t) + \theta_1(t)S_1(t).$$

For computational purposes it is often convenient to assume the portfolio self-financing, i.e.,

$$dV^\theta = \theta_0(t)dS_0(t) + \theta_1 dS_1(t).$$

Then the value of the portfolio at time t can be represented as follows:

$$dV^\theta(t) = [\rho(t)V^\theta(t) + (\mu(t) - \rho(t))\theta_1(t)S_1(t)]dt + \sigma(t)\theta_1(t)S_1(t)dW(t). \tag{7.4}$$

There are various ways to define market price of risk. It is convenient to define market price of risk, $u(t)$ as follows

$$\mu(t) - \rho(t) = \sigma(t)u(t),$$

or equivalently

$$u(t) = \frac{\mu(t) - \rho(t)}{\sigma(t)}.$$

Then by Girsanov theorem,

$$\hat{W}(t) = W(t) + \int_0^t u(t)dt \tag{7.5}$$

is a Wiener process with respect to the measure Q. Then using equation (7.5) we can rewrite the value of the portfolio in terms of the Wiener process $\hat{W}(t)$,

$$dV^\theta(t) = \rho(t)V^\theta(t)dt + \sigma(t)\theta_1(t)S_1(t)d\hat{W}(t). \tag{7.6}$$

Define discounted value of the portfolio $U^\theta(t)$,

$$e^{-\int_0^T \rho(s)ds}V^\theta(t) = V^\theta(0) + \int_0^T e^{-\int_0^t \rho(s)ds}\sigma(t)\theta_1(t)S_1(t)d\hat{W}(t). \tag{7.7}$$

If we apply Clark-Ocone formula under change of measure for $U^\theta(t)$,

$$U^\theta(T){=}E_Q[U^\theta(T)] + \int_0^T E_Q[(D_tU^\theta(T) - U^\theta(T)\int_t^T D_tu(s)d\hat{W}(s)) \mid \mathcal{F}_t]d\hat{W}(t). \tag{7.8}$$

Comparing the terms between equations (7.7) and (7.8), the number of risky assets in the replicating portfolio can be found as follows:

$$\theta_1(t) = e^{\int_0^t \rho(s)ds}\sigma^{-1}(t)S_1^{-1}(t)E_Q[(D_t(e^{-\int_0^T \rho(s)ds}F) - e^{-\int_0^T \rho(s)ds}F\int_t^T D_tu(s)d\hat{W}(s)) \mid \mathcal{F}_t].$$

In particular, if we choose ρ constant and μ, σ deterministic functions then the equation turns out to be

$$\theta_1(t) = e^{-\rho(T-t)}\sigma^{-1}(t)S_1^{-1}(t)E_Q[D_tF \mid \mathcal{F}_t],$$

where $F = \chi_{[K,\infty)}(W(T))$. In order to calculate $E_Q[D_t(F = \chi_{[K,\infty)}(W(T)))|\mathcal{F}_t]$ we will use Donsker delta function. For more information we refer to Aase et al. [2].

Definition 7.1. Let $Y : \Omega \to \mathbb{R}$ be a random variable which belongs to $\mathcal{G}^*$. Then the continuous function

$$\delta_Y(.) : \mathbb{R} \to (\mathcal{G}^*)$$

is called Donsker delta function of Y if it has the following property,

$$\int_{\mathbb{R}} g(y)\delta_Y(y)dy = g(Y) \quad \text{a.s.}$$

for all measurable function g, $g : \mathbb{R} \to \mathbb{R}$ such that the integral converges.

Theorem 7.1. *Let $\phi : [0,T] \to \mathbb{R}$ and $\alpha : [0,T] \to \mathbb{R}$ be deterministic functions such that $||\phi||_{L^2([0,T])}$ and $||\alpha||_{L^2([0,T])} < \infty$. Define*

$$Y(t) = Y(t,\omega) = \int_0^t \phi(s)d\hat{W}(s) + \int_0^t \phi(s)\alpha(s)ds.$$

Let $g : \mathbb{R} \to \mathbb{R}$ be bounded. Then

$$f(Y(T)) = V_0 + \int_0^T u(t,\omega) \diamond (\alpha(t) + \dot{\hat{W}}_t)dt, \tag{7.9}$$

where

$$V_0 = \int_{\mathbb{R}} \frac{g(y)}{\sqrt{2\pi}\,||\phi||_{L^2([0,T])}} \exp[-\frac{y^2}{2||\phi||_{L^2([0,T])}}]dy,$$

$$u(t,\omega) = \phi(t) \int_{\mathbb{R}} \frac{g(y)}{\sqrt{2\pi}\,||\phi||_{L^2([0,T])}} \exp^{\diamond}[-\frac{(y-Y(t))^{\diamond 2}}{2||\phi||_{L^2([0,T])}}] \diamond \frac{y-Y(t)}{||\phi||_{L^2([0,T])}}dy,$$

$\diamond$ is Wick product and $\dot{\hat{W}}_t$ is the white noise of Wiener process $\hat{W}(t)$ under measure Q.

If we take $g(y) = \chi_{[K,\infty)}(y)$ and $Y(T) = W(T)$ which implies $\phi(t) = 1, \alpha(t) = -u(t)$ where $u(t)$ is defined in equation (7.5) then $u(t,\omega)$ is expressed as follows;

$$u(t,\omega) = \int_K^{\infty} (\sqrt{2\pi})^{-1/2}\, T^{-1/2} \exp^{\diamond}[-\frac{(y-W(t))^{\diamond 2}}{2T}] \diamond \frac{y-W(t)}{T}dy, \tag{7.10}$$

Also note that by using Clark-Ocone formula we can write the payoff of the digital function which is;

$$\chi_{[K,\infty)}(W(T)) = E_Q[\chi_{[K,\infty)}(W(T))] + \int_0^T E_Q[D_t\{\chi_{[K,\infty)}(W(T))\}|\mathcal{F}_t]d\hat{W}(t) \tag{7.11}$$

Substituting equation (7.10) into the equation (7.9) and comparing the terms of the equation (7.11) with (7.9), we obtain the following result:

$$E_Q[D_t\{\chi_{[K,\infty)}(W(T))\}|\mathcal{F}_t] = (2\pi)^{-1/2}\, T^{-1/2} \int_K^{\infty} \exp^{\diamond}[-\frac{(y-W(t))^{\diamond 2}}{2T}] \diamond \frac{y-W(t)}{T}dy. \tag{7.12}$$

By Aase et al (Lemma 3.8, page 362) [2], equation (7.12) equals to

$$E_Q[D_t\{\chi_{[K,\infty)}(W(T))\}|\mathcal{F}_t] = (2\pi)^{-1/2}\,(T-t)^{-1/2} \int_K^{\infty} \exp[-\frac{(y-W(t))^2}{2(T-t)}]\,\frac{y-W(t)}{T-t}dy.$$

Therefore, in the replicating portfolio the number of the risky assets for hedging Digital option should be

$$\theta_1(t) = e^{-\rho(T-t)}(2\pi(T-t))^{-1/2}\sigma^{-1}(t)S_1^{-1}(t) \int_K^{\infty} \exp[-\frac{(y-W(t))^2}{2(T-t)}]\,\frac{y-W(t)}{T-t}dy. \tag{7.13}$$

References

[1] K. Aase, B. Øksendal, N. Privault and J. Ubøe, White noise generalizations of the Clark-Haussmann-Ocone theorem with application to mathematical finance, *Finance and Stochastic*, **4** (2000) 465-496.

[2] K. Aase, B. Øksendal and J. Ubøe, Using the Donsker delta function to compute hedging strategies, *Potential Analysis*, **14** (2001) 351-374.

[3] I. M. Gel'fand and N. Y. Vilenkin, *Generalized functions* **4** Newyork, 1977.

[4] T. Hida, *Brownian Motion*. Springer-Verlag, 1980.

[5] T. Hida, White noise anlysis and its applications, *Proc. Int. Math. Conf., ed. by Chen et al*, North-Holland, Amsterdam, (1982)43-48.

[6] T. Hida, H. Kuo, J. Potthoff and L. Streit, *White Noise, An Infinite Dimensional Approach.* Kluwer, 1993.

[7] H. Holden, B. Øksendal, J. Ubøe and T. S. Zhang, *Stochastic Partial Differential Equations: A Modelling, White Noise Functioanal Approach.* Boston: Birkhäuser, 1996.

[8] K. Itô, Multiple Wiener Integral, *Journal of Math. Soc. Japan*, **3** (1951) 157-169.

[9] I. Karatzas and D. Ocone , A generalized Clark representation formula, with application to optimal portfolios, *Stoch. and Stoch. Rep.***34** (1991) 187-220.

[10] I. Karatzas and S. Shreve, *Brownian Motion and Stochastic Claculus.* Springer-Verlag, 1987.

[11] M. Loève, *Probability Theory II.* Spring er-Verlag, 1978.

[12] G. di Nunno, B. Øksendal and F. Proske, *Malliavin Calculus for Lévy Processes with Applications to Finance.*

[13] D. Ocone, Malliavin calculus and stochastic integral representations of diffusion processes, *Stochastics* **12 (3-4)** (1984) 161-185.

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